Disorder and critical phenomena in pinning models

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Université Paris Diderot and Laboratoire Probabilités et Modèles Aléatoires (LPMA)

January 25, 2012

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- role of IID disorder: Harris criterion
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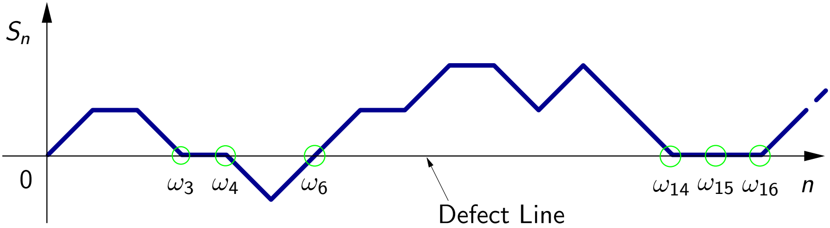
"The Example": $(1+d)$–directed walk models

Symmetric Random Walk $\{S_n\}_n$ with increments in $\{-1, 0, +1\}$ $(d = 1)$



Model ($\beta \geq 0$, $h \in \mathbb{R}$):

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) = \frac{1}{Z_{N,\omega}} \exp\left(\sum_{n=1}^{N} (\beta\omega_n + h)\,\delta_n\right)$$

with $\delta_n = \mathbf{1}_{S_n=0}$. The disorder ω is a IID sequence $\mathcal{N}(0,1)$ of law $\mathbb{P}$.

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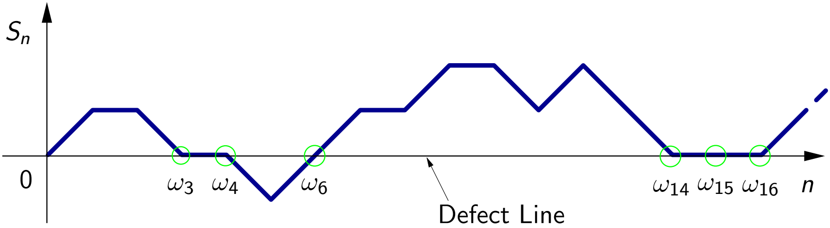

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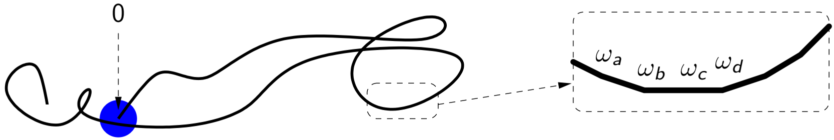
"The Example" again: d–dim. heteropolymer

A polymer chain made up of *charged* monomers, interacting with a potential near a point in space



Many other physical, biological,... systems:

- Two dimensional interfaces near a (rough) wall
- Of course: DNA denaturation (Poland-Scheraga)
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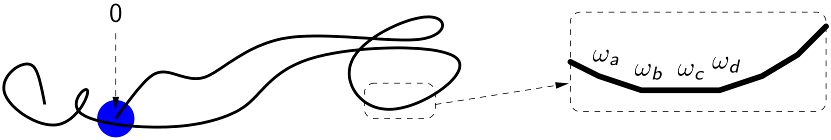

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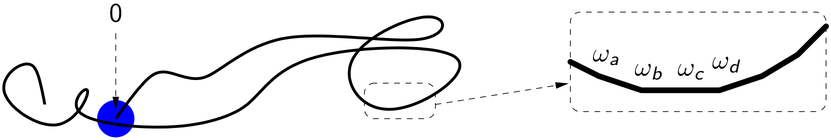

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- many more, but also: exactly solvable if $\beta = 0$

‘Rethinking ‘The Example”

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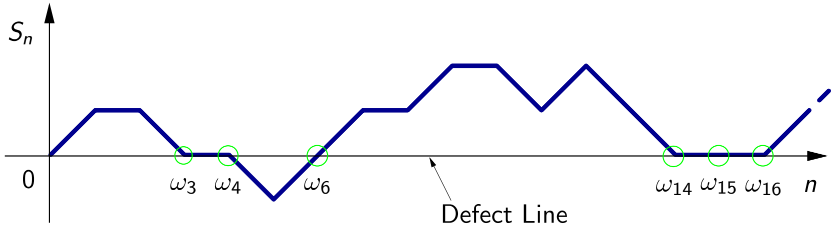

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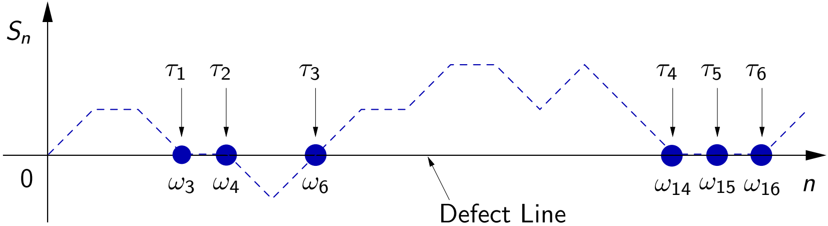

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In the end: renewal based model

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1. $\tau = \{\tau_0, \tau_1, \tau_2, \dots\}$ discrete renewal sequence (that is, $\tau_0 = 0$ and $\{\tau_j - \tau_{j-1}\}_{j\in\mathbb{N}}$ is IID), of law $\mathbf{P}$, s. t.

$$K(n) = \mathbf{P}(\tau_1 = n) \sim C_K/n^{1+\alpha}, \quad (C_K > 0),$$

or even $\mathbf{P}(\tau_1 = n) \sim L(n)/n^{1+\alpha}$, $L(\cdot)$ a slowly varying function, and

$$\sum_{n\in\mathbb{N}} K(n) \le 1.$$

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Obs.: $\alpha = 1/2$ for both $d = 1$ and 3, but $\displaystyle\sum_n K(n) < 1$ if $d = 3$

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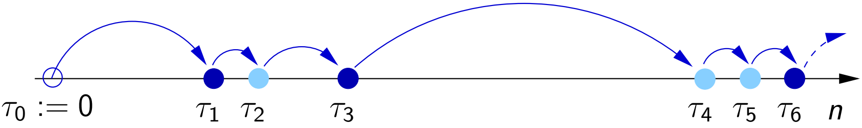
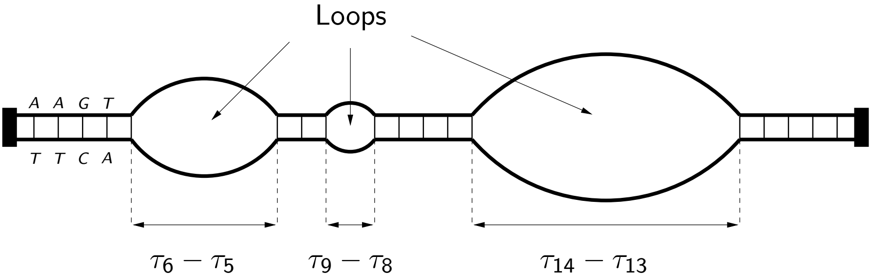


The Poland-Scheraga model



- The two thick lines are the DNA strands. They may be paired, gaining thus energetic contributions that depend on whether the base pair is A-T or G-C.
- There are then sections of unpaired bases (the *loops*) to which an entropy is associated: loops correspond to inter-arrival of length $n \geq 2$.

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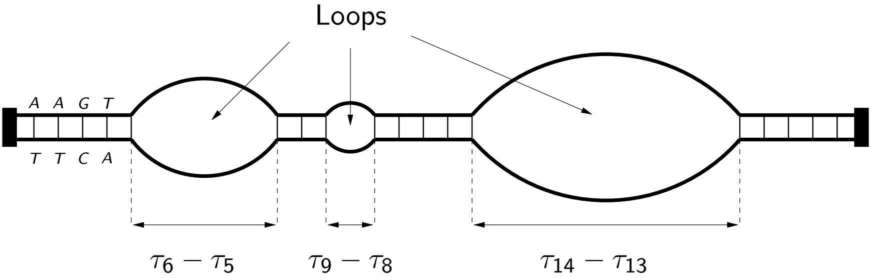

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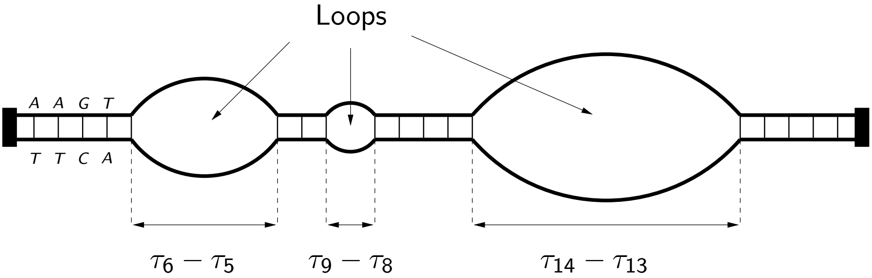

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Annealed, pure and homogeneous models

Observe that

$$\mathbb{E}Z^{c}{}_{N,\omega,\beta,h} = \mathbf{E}\left[\exp\left(\sum_{n=1}^{N}((\beta^2/2)+h)\delta_n\right)\delta_N\right]$$

which is the partition function of a homogeneous model.

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So:

- The annealed (or *pure*) model is just a homogeneous model with pinning potential $h+\beta^2/2$;
- Homogeneous pinning models are *exactly solvable* exhibiting a surprisingly wide spectrum of behaviors (when α varies).

The free energy

Theorem (Existence of the free energy and self-averaging). The limit

$$\lim_{N\to\infty} \frac{1}{N} \log Z_{N,\omega} =: \mathrm{F}(\beta, h)$$

exists $\mathbb{P}(\mathrm{d}\omega)$-a.s. and in $L^1(\mathbb{P})$, $\mathrm{F}(\beta, h)$ is non random and $\mathrm{F}(\beta, h) \geq 0$. Moreover $\mathrm{F}(\cdot,\cdot)$ is convex, $\mathrm{F}(\beta,\cdot)$ is non-decreasing and $\mathrm{F}(0, h) \leq \mathrm{F}(\beta, h) \leq \mathrm{F}(0, h+\beta^2/2) =: \mathrm{F}^a(\beta, h)$.

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- Dichotomy:
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Localization and Delocalization

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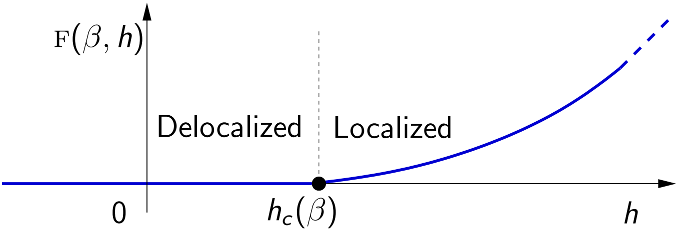

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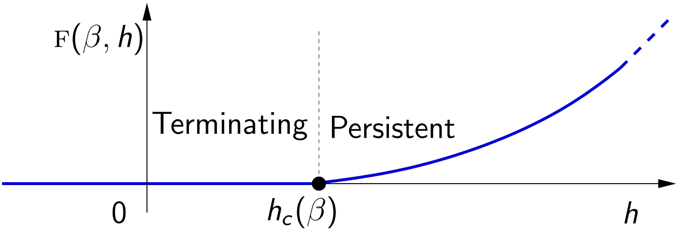
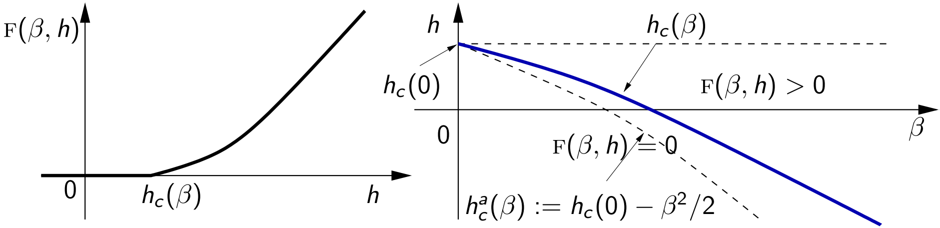


The phase diagram

Critical point/curve: $h_c(\beta) := \max\{h : \mathrm{F}(\beta, h) = 0\}$



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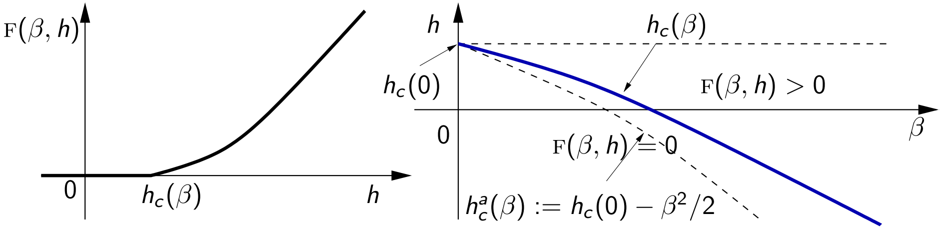

Plenty of questions, but above all:

- Can one compute or estimate $h_c(\beta)$?
- Critical behavior? $\mathrm{F}(\beta, h) \overset{h \searrow h_c(\beta)}{\sim} const.(h - h_c(\beta))^{\nu_q}$

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which directly yields

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with

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...M. Fisher '84. But: Erdos, Pollard, Feller, Garsia, Lamperti... (40's...)

General principles to deal with disorder(?)

Recall the main questions:

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Knowing the critical behavior of the pure system one can decide whether (at small disorder) the critical behavior of pure and disordered systems coincide (the disorder is irrelevant) or differ (the disorder is relevant).

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HC for pinning models [Forgacs et al. (1986), Derrida et al. (1992)]:

- $h_c(\beta) = h_c^a(\beta)$ and $\nu_q = \nu_a$ for β small if $\alpha < 1/2$
- $h_c(\beta) \neq h_c^a(\beta)$ and (probably) $\nu_q \neq \nu_a$ for $\beta > 0$ and $\alpha > 1/2$.

Harris criterion for pinning models: rigorous results

- If $0 \leq \alpha < 1/2$ disorder is irrelevant if β is not too large: there exists $\beta_0 \in (0, \infty]$ such that for $\beta < \beta_0$ we have

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$$\lim_{h \searrow h_c(\beta)} \frac{\log \mathrm{F}(\beta, h)}{\log(h - h_c(\beta))} = 1/\alpha$$

[Alexander 08], [Toninelli 08], [GT09],[Lacoin 10]

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- If $\alpha > 1/2$ disorder is relevant for every $\beta > 0$: $h_c(\beta) > h_c^a(\beta)$ and

$$\nu_q \geq 2 > \nu_a = 1/\alpha \quad \text{smoothing!}$$

Moreover

$$h_c(\beta) - h_c^a(\beta) \approx \begin{cases} \beta^{2\alpha/(2\alpha-1)} & \text{if } \alpha \in (1/2, 1) \\ \beta^2 & \text{if } \alpha > 1 \end{cases}$$

[GT 06], [Derrida GLT 09], [A Zygouras 10]

The marginal case ($\alpha = 1/2$): two parties

- Marginal irrelevance of weak disorder:
  Forgacs, Luck, Nieuwenhuizen, Orland (1986, $1 + 1$-dim. wetting)

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critical point shift of $\approx \exp(-c/\beta^2)$ (more vague about ν_q)

Rigorous: $\nu_q \geq 2$ (poor...)

$$c_\varepsilon \exp(-c/\beta^{2+\varepsilon}) \leq h_c(\beta) - h_c^a(\beta) \leq c \exp(-c/\beta^2)$$

for $\beta \leq \beta_0$ [A08,T08,GLT10,GLT11]

Smoothing inequality

Theorem [GT06, CMP and PRL]

Under assumptions on the disorder, for every $\beta > 0$ there exists C_β such that for every h

$$\mathrm{F}(\beta, h) \le \alpha C_\beta \left(h - h_c(\beta)\right)^2 .$$

Possibly more transparent when written as

$$0 \le \mathrm{F}(\beta, h) - \mathrm{F}(\beta, h_c(\beta)) \le \alpha C_\beta \left(h - h_c(\beta)\right)^2$$

(the result is non trivial only for $h < h_c(\lambda)$). Rephrasing: $\mathrm{F}(\beta, \cdot)$ is $C^{1,1}$ at $h_c(\beta) \Longrightarrow$ the transition is at least of second order (almost third...)

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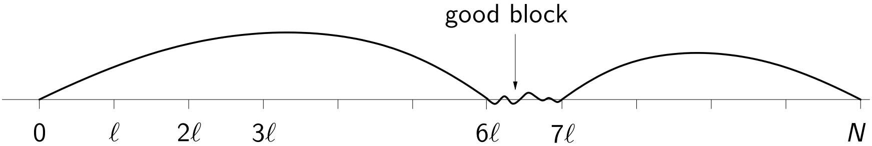
What assumptions?

The disorder is IID and the law of ω_1 either has a strictly positive density (with a finite entropy condition wrt Gaussian) or it has compact support. Generalizes to non IID: e.g. stationary Gaussian with summable covariance. [Berger]

Smoothing argument: the rare stretch strategy



Consider blocks of length ℓ (large, but finite) and choose N to guaranty that with large probability there is at least a (good!) block in which $\log Z^c_{\ell,\theta^{i\ell}\omega,\beta,h_c(\beta)}$ is larger than $\ell\frac{1}{2}\mathrm{F}(\beta, h_c(\beta)+\Delta)$, $\Delta > 0$.

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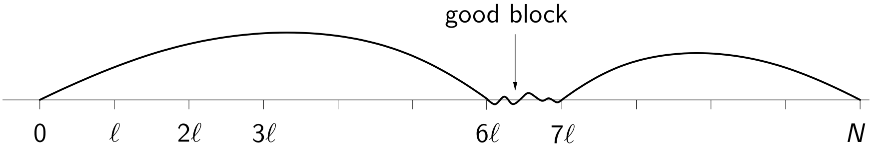

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$$\mathbb{P}\left(\log Z^c_{\ell,\omega,\beta,h_c(\beta)} > \ell\frac{1}{2}\mathrm{F}(\beta, h_c(\beta)+\Delta)\right) \gtrsim \exp\left(-\frac{1}{2}\ell\frac{\Delta^2}{\beta^2}\right)$$

because to have $\log Z^c_{\ell,\omega,\beta,h_c(\beta)} > \ell\frac{1}{2}\mathrm{F}(\beta, h_c(\beta)+\Delta)$ it suffices that the environment looks like $(\omega_1+\Delta/\beta, \ldots, \omega_\ell+\Delta/\beta)$.

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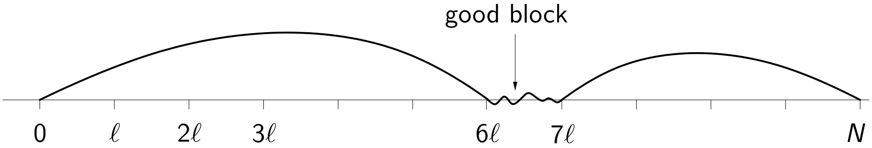

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This requires $N = O\left(\ell \exp\left(\frac{1}{2}\ell\frac{\Delta^2}{\beta^2}\right)\right)$.

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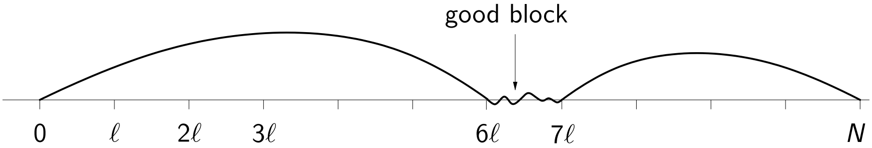

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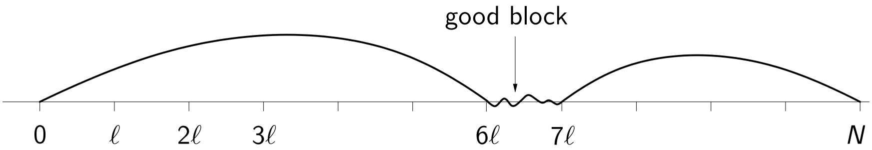
Then make a lower bound on $Z^c_{N,\omega,\beta,h_c(\beta)}$ by considering only the τ trajectories visiting the only the first good block

Smoothing argument: the rare stretch strategy



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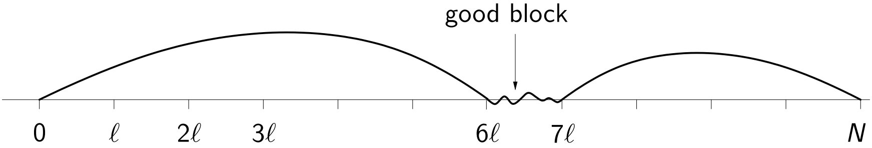

By super-additivity:

$$\frac{1}{N}\mathbb{E}\log Z^{c}_{N,\omega,\beta,h_c(\beta)} \leq \mathrm{F}(\beta, h_c(\beta)) = 0$$

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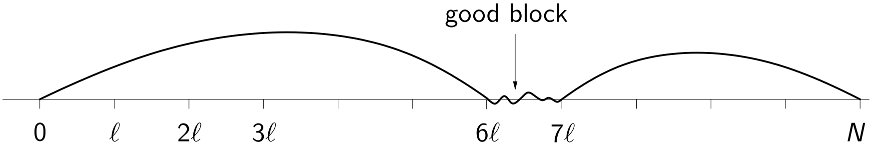

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But (recall $N = O\left(\ell \exp\left(\frac{1}{2}\ell\frac{\Delta^2}{\beta^2}\right)\right)$)

$$0 \ge \mathbb{E}\log Z^c_{N,\omega,\beta,h_c(\beta)} \ge \ell\frac{1}{2}\mathrm{F}(\beta, h_c(\beta)+\Delta) - C\log N$$

$$\ge \ell\left(\frac{1}{2}\mathrm{F}(\beta, h_c(\beta)+\Delta) - \frac{C}{2}\frac{\Delta^2}{\beta^2}\right) + O(\log \ell)$$

and the non-positivity of the blue term is the smoothing inequality.

About smoothing

Other approaches to smoothing:

- The inequality $\nu \geq 2/d$ in [Chayes×2, Fisher, Spencer 86] is about correlation functions and it is valid under *complex assumptions* (conditional result), verified for the Ising model and for the *quenched averaged* correlation length. It is very unclear what this approach yields for pinning models, and above all for $\alpha > 1$.
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Important issue

For $\alpha \geq 1/2$, is $\nu_q > 2$?
This is certainly expected, and observed numerically in [Coluzzi, Yeramian 07] for $\alpha = 1.15$, along with the role of atypical deviations.