Approximation of Markov processes conditioned on non-absorption

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Inhomogeneous Random Systems

1. Quasi-stationary distributions and Yaglom limits

Markov processes with absorption

We consider a Markov process which evolves on a state space $E \cup \{\partial\}$, with absorption at $\partial \notin E$, *i.e.* the process stays in ∂ once it reached it.

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Examples: stochastic models for population sizes.

- Galton-Watson processes on $\mathbb{N}$ (discrete time),
- Birth and death processes on $\mathbb{N}$,
- Stochastic Lotka-Volterra system on $\mathbb{R}_+^{\text{number of types}}$,
- Wright-Fisher processes on [0,1].

Long time behavior of X^∂

Some assumptions

$$\mathbb{P}^\partial_x(\tau_\partial < +\infty) = 1 \text{ and } \mathbb{P}^\partial_x(\tau_\partial > t) > 0,\ \forall t > 0.$$

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We are interested in the distribution of the non-absorbed positions of the process: $\mathbb{P}^\partial_x\left(X^\partial_t \in \cdot | t < \tau_\partial\right)$.

Definition

A quasi-limiting distribution (QLD) is a probability measure α such that $\exists \mu \in \mathcal{M}_1(E)$ satisfying

$$\alpha = \lim_{t\to\infty} \mathbb{P}^\partial_\mu \left(X^\partial_t \in \cdot | t < \tau_\partial\right)$$

Yaglom limits and quasi-stationary distributions

Définitions

A Yaglom limit for X^∂ is a probability measure α on E such that

$$\alpha = \lim_{t\to\infty} \mathbb{P}^\partial_x(X^\partial_t \in .|t < \tau_\partial), \forall x \in E.$$

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A quasi-stationary distribution (QSD) is a probability measure α such that

$$\mathbb{P}_\alpha(X_t^\partial \in .|t < \tau_\partial) = \int_E \mathbb{P}_x(X_t^\partial \in .|t < \tau_\partial) d\alpha(x) = \alpha,\ \forall t \geq 0.$$

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Questions: Existence and uniqueness, value of the QSD, speed of convergence.

General properties

Yaglom limit, QLD and QSD

1. The Yaglom limit is a QLD (definition),
2. Any QSD is a QLD (definition),
3. Any QLD is a QSD.

Briefly: Yaglom limit $\Rightarrow$ QLD $\Leftrightarrow$ QSD.

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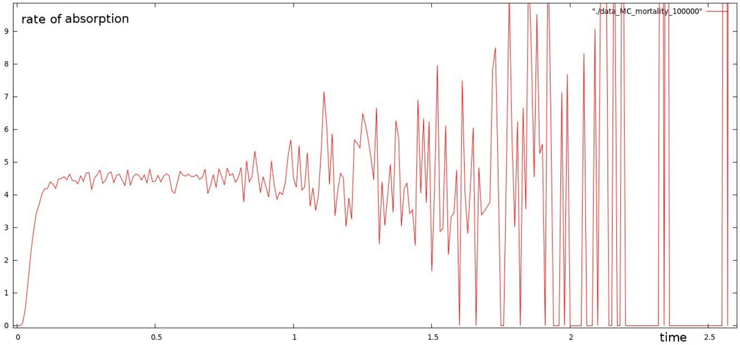
QSD, QLD, absorption rate

1. If α is a QLD, then $\exists \lambda_0 > 0$ such that
   1a. $\mathbb{P}_\alpha(t < \tau_\partial) = e^{-\lambda_0 t}$,
   1b. $\mathbb{E}_\alpha(e^{\theta \tau_\partial}) < +\infty$, $\forall \theta \in [0, \lambda_0[$.
2. If α is a QLD for X starting from μ, then

$$\text{absorption rate}(t) \stackrel{def}{=} \mathbb{P}^\partial_\mu(\tau_\partial \in]t, t+1] | \tau_\partial > t) \xrightarrow[t\to\infty]{} e^{-\lambda_0}.$$

Numerical illustration

Convergence of the absorption rate for a BM in [0,1] absorbed at {0,1}



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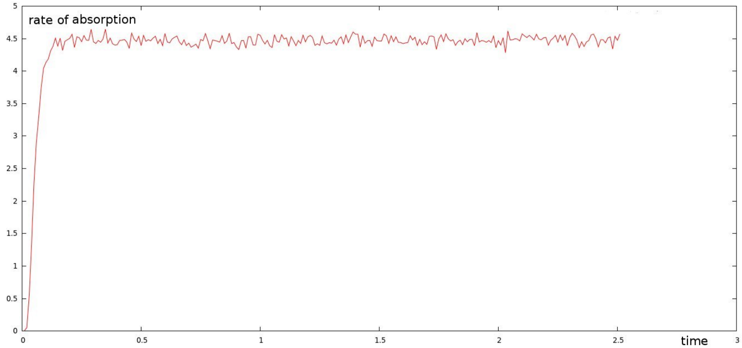

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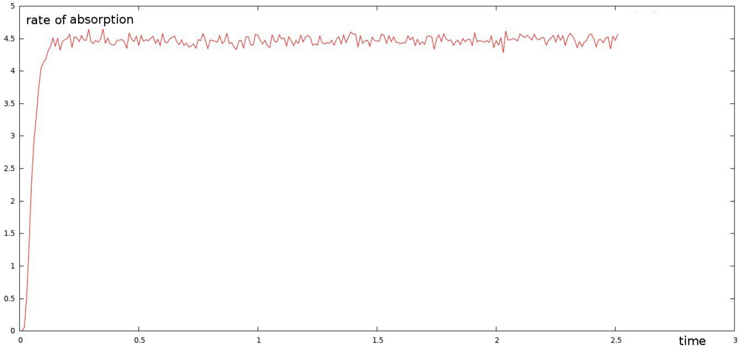

Convergence to the Yaglom limit, then quasi-stationarity (3D Brownian motion with drift).

Criterion for the existence and uniqueness of a QSD

Proposition

If there exist $C,\gamma > 0$ such that, for any μ and ν,

$$\left\|\mathbb{P}_\mu\left(X_t^\partial \in \cdot | t < \tau_\partial\right) - \mathbb{P}_\nu\left(X_t^\partial \in \cdot | t < \tau_\partial\right)\right\|_{TV} \leq Ce^{-\gamma t}, \forall t \geq 0,$$

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Remark 1: Even if the QSD is unique, there is now insurance that this mixing property holds.

Remark 2: The mixing property is also interesting when the semi-group isn't time-homogeneous.

Let E be a bounded and smooth open subset $D \subset \mathbb{R}^d$. Let X^∂ be defined by

$$dX_t^\partial = \sigma(t, X_t^\partial)dB_t + b(t, X_t^\partial)dt,\ X_0^\partial \in D,$$

with time-periodic, Lipschitz and elliptic coefficients.

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Theorem (Del Moral, V. 2011)

If

$$\sum_{k,l} \frac{\partial \phi}{\partial x_k}(z)\frac{\partial \phi}{\partial x_l}(z)[\sigma\sigma^*]_{kl}(t,\epsilon,z) = f(t,\epsilon,z) + g(t,\epsilon,z),$$

where $\phi(z) = d(z,\partial D)$, f_i is positive and of class $C^{1,2}$ and $g(t,\epsilon,z) \leq k_0\phi(z)$,

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Existence and uniqueness results can also be found in Pinsky 1985, Gong, Qian, Zhao 1988, Knobloch, Partzsch 2010.

Let $E = \mathbb{N}^*$ and $\partial = 0$. Let X^∂ be a non-explosive strong Markov process on $E \cup \{\partial\}$, with transition rate matrix $(Q(x,y))_{(x,y)\in\mathbb{N}^2}$.

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Theorem (Martínez, San Martín, V.)

Assume that $P_t(x,y) > 0$, $\forall x,y \in \mathbb{N}^*$ and that there exists a finite subset $K \subset \mathbb{N}^*$ such that

$$\inf_{x\in\mathbb{N}^*\setminus K} \left(Q(x,0) + \sum_{y\in K} Q(y,x) \right) > \sup_{x\in\mathbb{N}^*} Q(x,0).$$

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The existence and uniqueness of the QSD for X^∂ was already known (Ferrari, Marič 2007) in the subcase

$$\sum_{x\in\mathbb{N}^*} \inf_{y\in\mathbb{N}^*} Q(x,y) > \sup_{x\in\mathbb{N}^*} Q(x,0).$$

2. Approximation of conditioned Markov processes

The interacting particle system (Burdzy, Holyst, Ingerman, March 1996)

$(X_t^1,\dots,X_t^N)_{t\geq 0}$ is a càdlàg process:

- Start with N independent copies of X_t^∂,
- Wait for the first killing time (τ_1) the killed particle is sent to the position of one another particle (in E)
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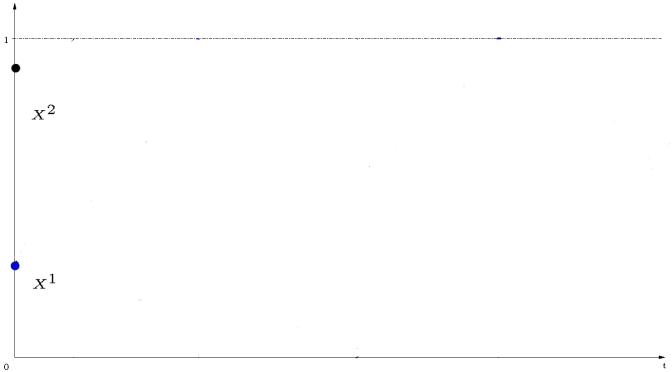
Some notations: sequence of killing times $0 < \tau_1 < \tau_2 < \dots$
A_t^N denotes the number of killings up to time t.

$$\mu_t^N = \frac{1}{N}\sum_{i=1}^{N}\delta_{X_t^i}.$$

Illustration with 2 particles and $E =]0,1[$



$A_t^N = 0$

Illustration with 2 particles and $E =]0,1[$

$$A_t^N = 1$$

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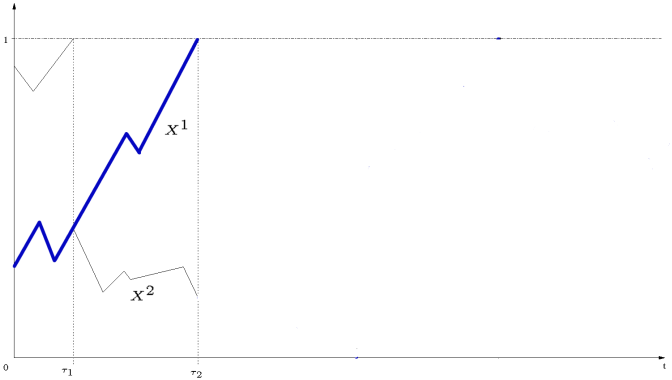
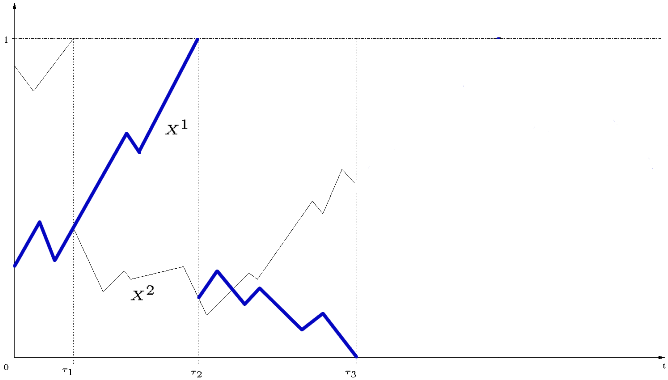


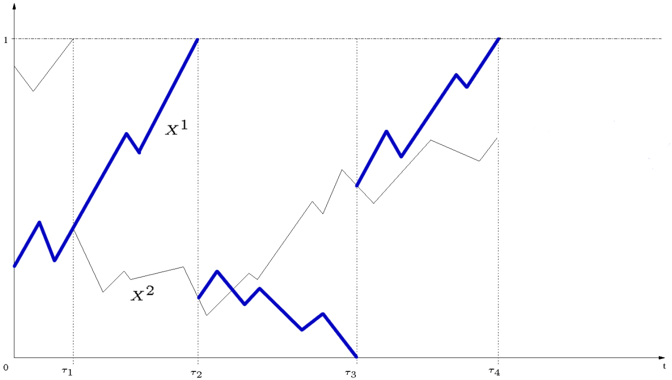
$A_t^N = 2$

Illustration with 2 particles and $E =]0,1[$



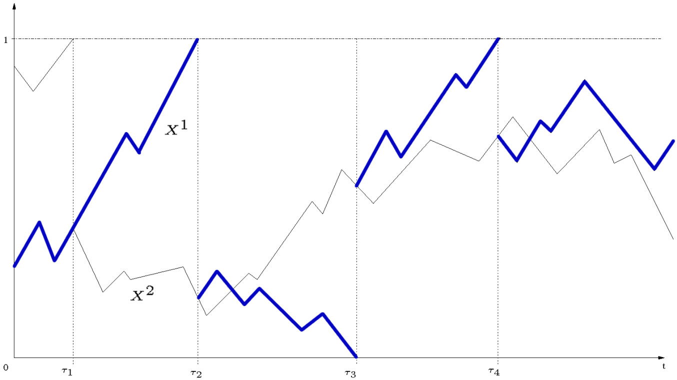
$A_t^N = 3$

Illustration with 2 particles and $E =]0,1[$



$A_t^N = 4$

Illustration with 2 particles and $E =]0,1[$



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Theorem (V 2011)

Assume that $\mu_0^N \xrightarrow[N\to\infty]{} \mu_0 \in \mathcal{M}_1(E)$, that $P_{\mu_0}(t < \tau_\partial) > 0$ and that

$$A_t^N < +\infty, \forall N \geq 2 \text{ almost surely,}$$

then

$$\mu_t^N(dx) \xrightarrow[N\to\infty]{law} \mathbb{P}_x^\partial \left(X_t^\partial \in dx | t < \tau_\partial \right).$$

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$$\mu_t^N(dx) \xrightarrow[N\to\infty]{law} \mathbb{P}_x^\partial\left(X_t^\partial \in dx | t < \tau_\partial\right).$$

Moreover we have, for all measurable function f and all time $T > 0$,

$$\mathbb{E}\left(\left|\mu_T^N(f) - \mathbb{E}_{\mu_0^N}\left(f(X_T^\partial)| T < \tau_\partial\right)\right|\right)$$

$$\leq \frac{4\|f\|_\infty}{\sqrt{N}}\sqrt{\mathbb{E}\left(\frac{1}{\mathbb{P}^\partial_{\mu_0^N}(T < \tau_\partial)^2}\right)}.$$

About $A_t^N < +\infty$ almost surely for all $N \geq 2$...

- Soft obstacle setting with a bounded rate of killing: trivial
- Hard obstacle setting: not always true and difficult (see Burdzy, Holyst, March (2000) Löbus (2007), Bienek, Burdzy, Finch (2009), V. (2010), Grigorescu, Kang (2011), V. (2011))

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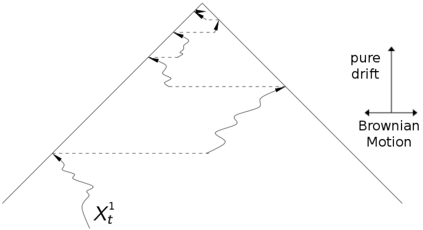
A counter example:



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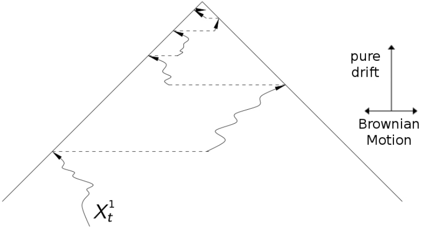

A more interesting counter-example (Bienek, Burdzy, Pal, 2011): $dX_t^\partial = dB_t - \frac{1}{X_t^\partial}dt$.

$E = Env \times D$, with $Env \subset \mathbb{R}^d$ (environment), $D \subset \mathbb{R}^{d'}$ (position).
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Theorem (V. 2011)

If ∂D is of class C^2, s,m,σ,μ are uniformly bounded and

$$\sum_{k,l} \frac{\partial\phi}{\partial x_k}(z)\frac{\partial\phi}{\partial x_l}(z)[\sigma\sigma^*]_{kl}(t,\epsilon,z) = f(t,\epsilon,z) + g(t,\epsilon,z),$$

where $\phi(z) = d(z,\partial D)$, f_i is positive and of class $C^{1,2}$ and $g(t,\epsilon,z) \leq k_0\phi(z)$, then

$$A_t^N < +\infty \text{ almost surely.}$$

Proof (1) attainability of (0,0) by a semi-martingale

Let (Y_t^1, Y_t^2) be a couple of semi-martingale such that

$$dY_t^i = dM_t^i + b_t^i dt + Y_t^i - Y_{t\text{-}}^i,$$

with $Y_t^i - Y_{t\text{-}}^i > 0$ (only positive jumps).

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Proposition (V 2011)

If $\langle M \rangle_t^i = \pi_t^i dt$, where π^i is a *good* semi-martingale, then

$$P(T_{(0,0)} < +\infty) = 1.$$

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To verify "π^i is a *good* semi-martingale" is a simple application of Itô's formula when Y^1 and Y^2 are C^2 functions of Itô's diffusion processes. In our case, $Y^1 = \phi(X^1)$ and $Y^2 = \phi(X^2)$.

Proof (2) $A_t^N = +\infty \Rightarrow$ attainability of (0,0)

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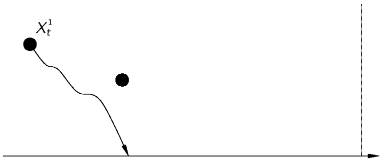

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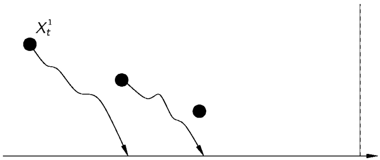

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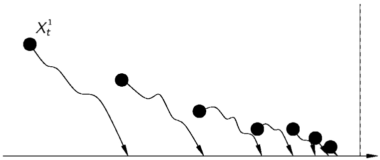

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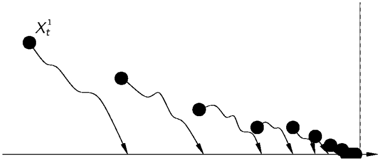

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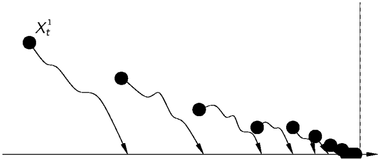

We deduce that

$$\lim_{n\to\infty} \left(\phi(X^1_{\tau_n}), \phi(X^2_{\tau_n})\right) = (0,0)$$

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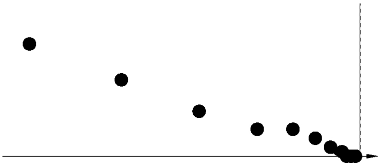

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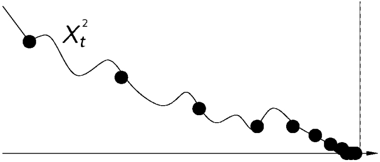

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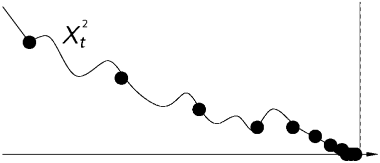

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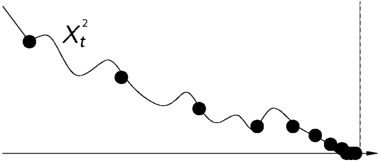

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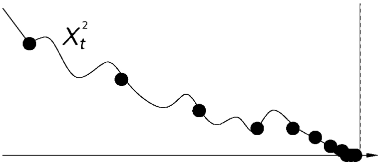

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By our assumptions, (Y^1,Y^2) cannot reach (0,0).

Uniform convergence in time

Let E be a bounded and smooth open subset $D \subset \mathbb{R}^d$. Let X^∂ be defined by $dX_t^\partial = \sigma(t,X_t^\partial)dB_t + b(t,X_t^\partial)dt$, $X_0^\partial \in D$, with time-periodic, Lipschitz and elliptic coefficients.

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Theorem (Del Moral, V. 2011)

If

$$\sum_{k,l} \frac{\partial \phi}{\partial x_k}(z)\frac{\partial \phi}{\partial x_l}(z)[\sigma\sigma^*]_{kl}(t,\epsilon,z) = f(t,\epsilon,z) + g(t,\epsilon,z),$$

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In particular, if X^∂ is time homogeneous then

$$\lim_{N\to\infty}\lim_{t\to\infty} \mu^N_t = \alpha \text{ (the unique QSD of } X^\partial).$$

Let $E = \mathbb{N}^*$ and $\partial = 0$. Let X^∂ be a non-explosive strong Markov process on $E \cup \{\partial\}$, with transition rate matrix $(Q(x,y))_{(x,y)\in\mathbb{N}^2}$.

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Under the condition $\sum_{x\in\mathbb{N}^*} \inf_{y\in\mathbb{N}^*} Q(x,y) > \sup_{x\in\mathbb{N}^*} Q(x,0)$, Ferrari and Marič showed firstly that $\lim_{N\to\infty} \lim_{t\to\infty} \mu_t^N$ exists and identified it as a QSD (see also Asselah, Ferrari, Groisman 2011).

3. Application: study of a multi-type diffusion process

3 types competitive Lotka-Volterra system

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with $\gamma_i = 1$, $c_{ii} = 10$, $c_{ij} = 0.5$, $\forall\ i \neq j \in \{1, 2, 3\}$, and
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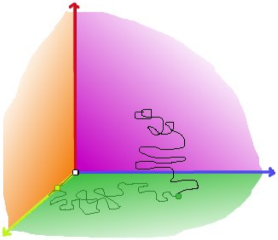
Theorem (Cattiaux, Méléard 2010, Méléard, V. 2011)

For any initial distribution μ which charges $\mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}_+^*$, we have

$$\lim_{t\to\infty} \mathbb{P}_\mu\left(X_t \in \cdot | t < \tau_\partial\right) = \alpha_1 \otimes \delta_0 \otimes \delta_0,$$

where α_1 is the unique QSD for the one-dimensional dynamic
$dZ_t = \sqrt{\gamma_1 Z_t} dB_t + \left(r_1 Z_t - c_{11}(Z_t)^2\right) dt$.

Evolution of the number of types:



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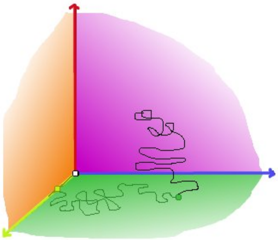

- The process starts with three types {1,2,3},
- Then one (and only one) type disappears. It remains two types {1,2}, {1,3} or {2,3}.
- Then an other type disappears. It remains one type 1, 2 or 3.

Natural questions answered by numerical approximation

1. At a fixed time $t > 0$, what is the probability to have three types, two types or one type ? Which types ?
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Theorem (V. 2011)

We have

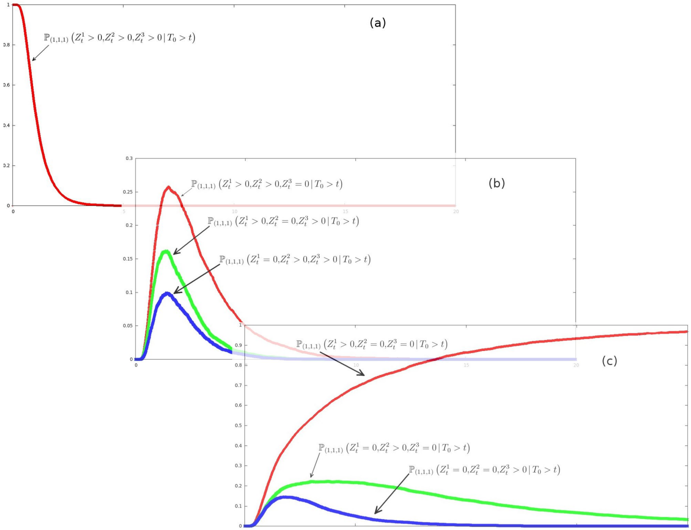
$$\mathbb{P}^{\epsilon}_{(z_1,z_2,z_3)}(X_t^{\partial} \in \cdot | t < \tau_{\partial}) \xrightarrow[\epsilon \to 0]{} \mathbb{P}_{(z_1,z_2,z_3)}(X_t^{\partial} \in \cdot | t < \tau_{\partial})$$

and

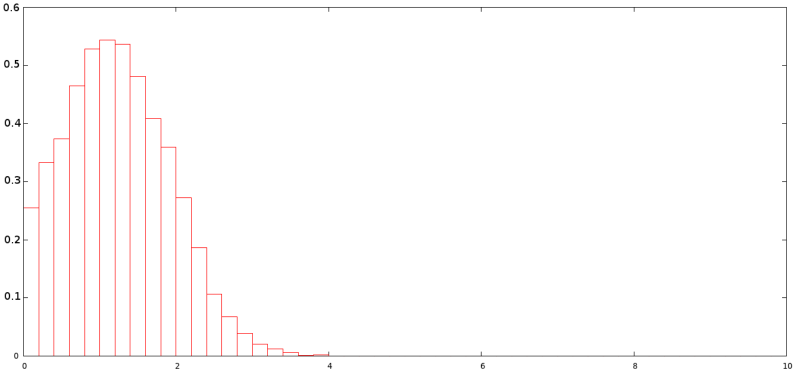
$$\alpha_{\epsilon} \xrightarrow[\epsilon \to 0]{} \alpha$$

.

1. Probability of presence of each types:



2. Approximation of α_1:



Thank you!