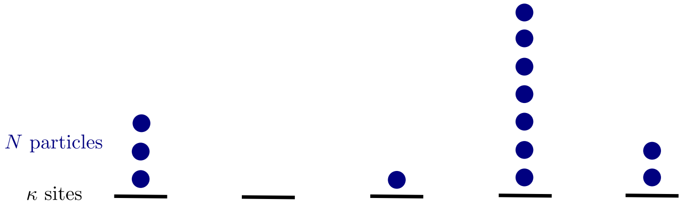
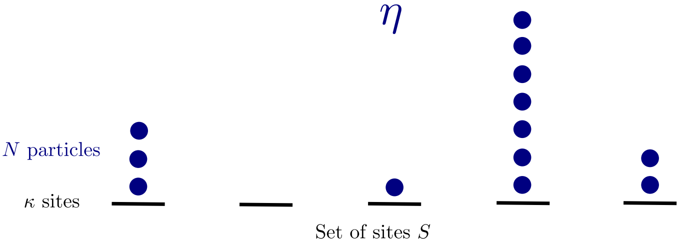
Nucleation for the zero range process on a finite set of sites

Johel Beltrán, PUCP - IMCA

Joint work with
M. Jara and C. Landim (IMPA)



N particles
κ sites


η
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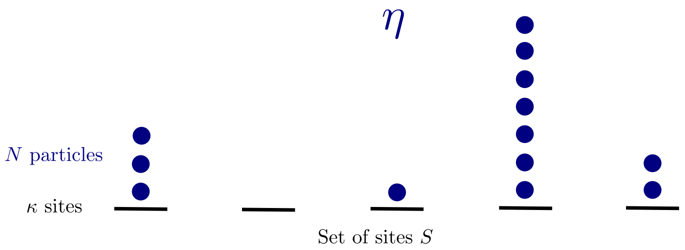

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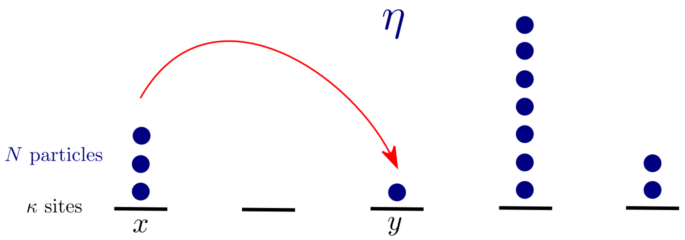

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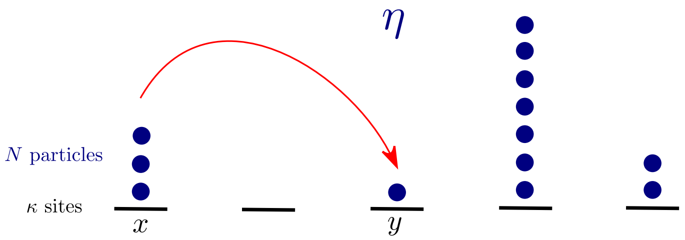

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$$g(\eta(x))r(x,y)$$

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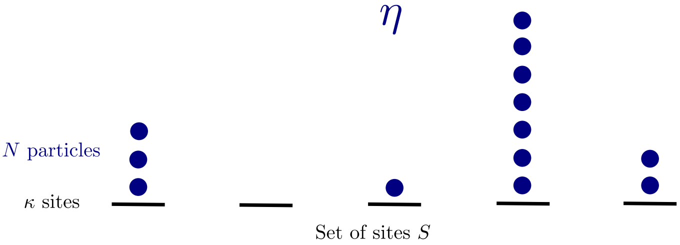
$$g(n) \sim \mathbf{1} + \frac{b}{n}$$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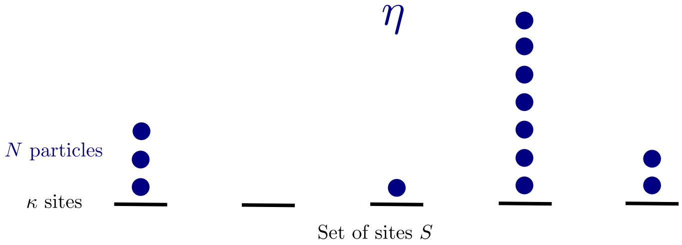

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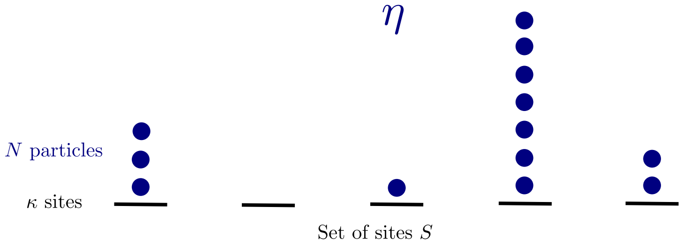

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Conditions on g produce a condensation effect!

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For each $N \geq 1$, let η^N be a configurations of N particles and let $v \in \mathbb{E}_S$ such that $\eta^N/N \to v$.

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$$\{U_t^N : t \geq 0\} \xrightarrow{(Law)} \{U_t : t \geq 0\}$$

where $\{U_t\}_{t \geq 0}$ is a *Markov process* starting at v.

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Then for $0 \le t \le \tau$, $(U_t(x))_{x \in A}$ follows the generator

$$L_A \phi(u) \ = \ \sum_{x,y \in A} \left\{ b \frac{c_A(r,x,y)}{u(y)} \partial_{\mathbf{e}_x} + a_A(r,x,y) \partial_{\mathbf{e}_x} \partial_{\mathbf{e}_y} \right\}$$

where $c_A(r,x,x) < 0$ and $c_A(r,x,y) > 0$ for $x \neq y$.

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and, for each $\epsilon > 0$ define $\Delta(\epsilon) := \{u \in \mathbb{E}_S : \|u - \mathbf{e}_x\| > \epsilon\}$.

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- $\mathbf{E}_{\eta^N}\left[\int_0^T \mathbf{1}_{\Delta(\ell_N/N)}(X_t^N)\, dt\right] \to 0$, for every $T > 0$.
- The trace of $\{X_t^N : t \geq 0\}$ on $\mathbb{E}_S \setminus \Delta(\ell_N/N)$ converges to a *Markov process* $\{X_t : t \geq 0\}$ on $\{\mathbf{e}_x : x \in S\}$.

References

- Tunneling and metastability of continuous time Markov chains; J. B., C. Landim. Journal of Stat. Phys. Vol 140, No 6, pp 1065 – 1114, (2010).
- Metastability of reversible condensed zero range processes on a finite set; J. B., C. Landim. Probability Theory and Related Fields. Vol 152, No 3-4, pp. 781 – 807, (2012).

References

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- Metastability for a non-reversible dynamics: the evolution of the condensate in totally asymmetric zero range processes; C. Landim

Thank you

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