Integrable probability on interlacing particles

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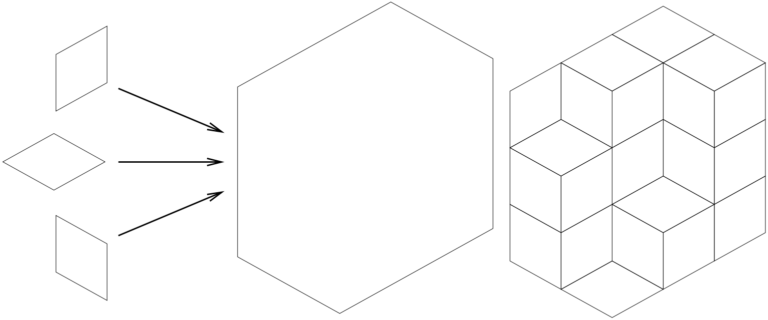
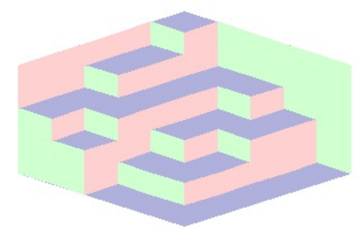
Lozenge tilings: our basic model

Consider an equi-angular hexagon of side lengths a, b, c, a, b, c drawn on the regular triangular lattice.
We are interested in tilings by rhombi with angles $\pi/3$ and $2\pi/3$ and side lengths 1. Each rhombus ("lozenge") is a union of two elementary triangles.





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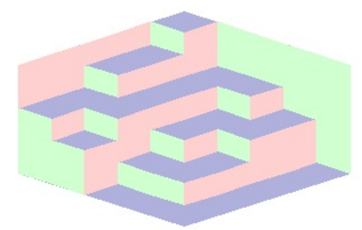
Our topic: models related to *random lozenge tilings*, including KPZ–class growth processes and random matrices.



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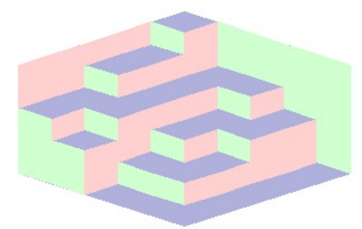

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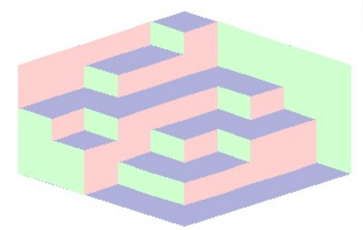

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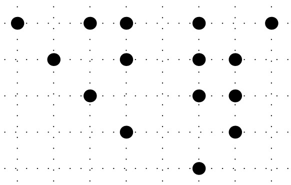
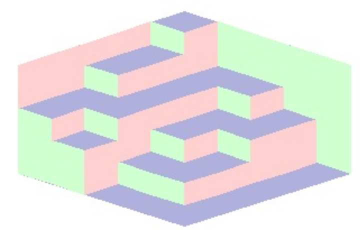
- State space can be typically identified with particle configurations subject to interlacing conditions.





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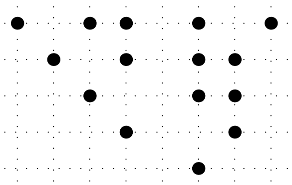

- Probability measures are "integrable" in the sense that many distributions and expectations of observables admit explicit formulas.

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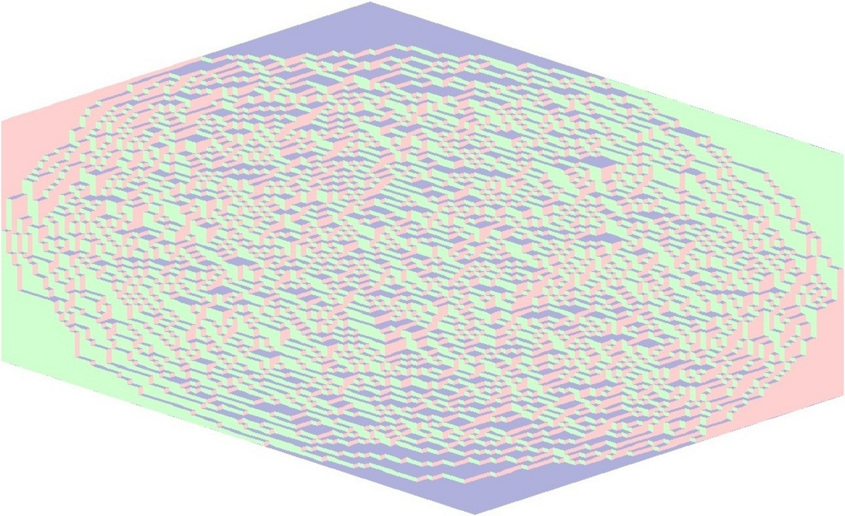
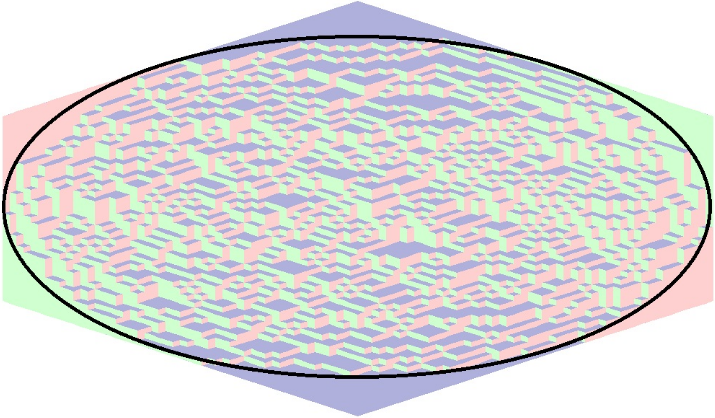


3 types of lozenges — 3 colors

Lozenge tilings: frozen regions

(Cohn–Larsen–Propp, 1998) Tiling is asymptotically *frozen* outside inscribed ellipse



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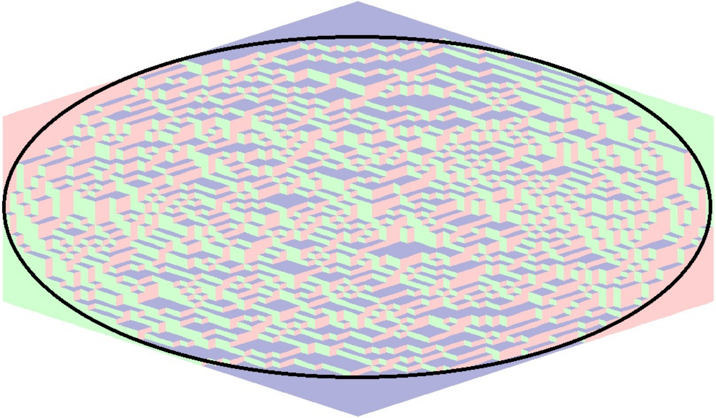

How can this fact be proved?

Lozenge tilings: frozen regions

Method 1. (Cohn–Larsen–Propp, Kenyon-Okounkov-Sheffield) "Limit shape" as the side lengths of the hexagon $a, b, c \to \infty$ can be obtained as a solution of certain *variational problem*. Analyzing it via Euler–Lagrange equation one proves, in particular, the existence of frozen regions.

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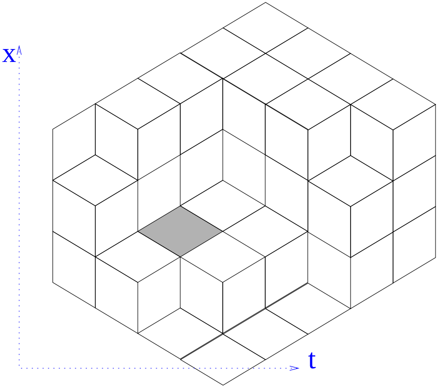
(Kenyon-Okounkov) Tiling of a large *polygon* is asymptotically frozen outside inscribed algebraic curve of minimal degree.

Disadvantage. While solving the problem for the hexagon, this does not give the answer in explicit form for complicated domains.

Lozenge tilings: frozen regions

Method 2. ("Integrable") Consider *horizontal* lozenges of a tiling (they form an interlacing particle configuration).



nth correlation function:

$$\rho_n((x_1, t_1) \dots (x_n, t_n)) = \mathrm{Prob}\{\text{horiz. lozenges in given } n \text{ positions}\}$$

Lozenge tilings: frozen regions

Method 2. ("Integrable")

Proposition. Correlation functions for the uniform measure on tilings of a hexagon have determinantal form

$$\rho_n = \det_{i,j=1,\dots,n} \{K(t_i, x_i; t_j, x_j)\},$$

where *correlation kernel* K does not depend on n.

Thus, horizontal lozenges form a *determinantal point process.*

Lozenge tilings: frozen regions

Method 2A. (Johannson–Nordenstam; Gorin) The correlation kernel K describing positions of horizontal lozenges in uniformly random lozenge tilings of a hexagon can be expressed through Hahn (hypergeometric) orthogonal polynomials.

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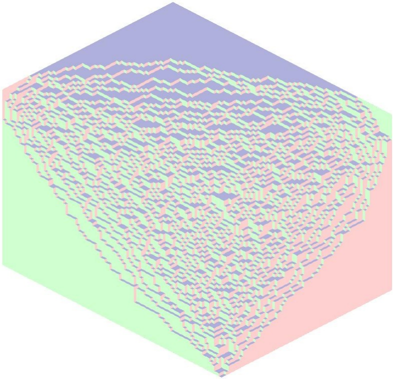
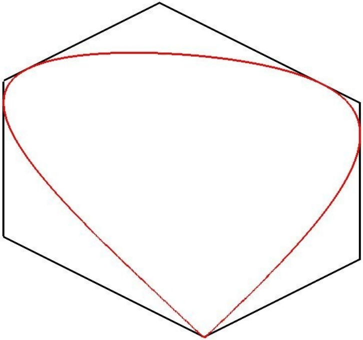
Advantage. This method can be generalized to more complicated measures on tilings of hexagons. Still there are explicit formulas for the boundaries of frozen regions (Borodin–Gorin–Rains–2009).

Lozenge tilings: frozen regions

Method 2A: Non–uniform measures — one example.





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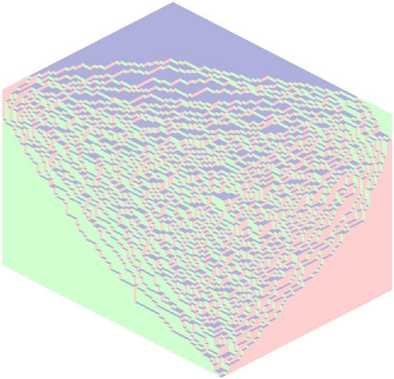

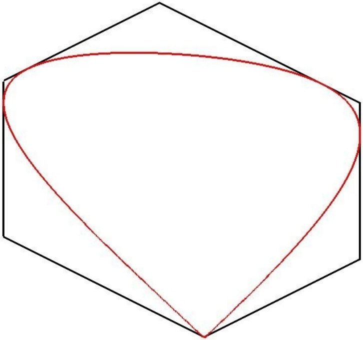

Disadvantage. Breaks down if we change the domain.
General feature. We can have rich models and answers, but methods are not resistent to small fluctuations.

Lozenge tilings: frozen regions

Method 2B. (Petrov–2012) The correlation kernel K describing positions of horizontal lozenges in uniformly random lozenge tilings of a hexagon can be expressed as a *double contour integral* of elementary functions.

Remark. In the context of Schur processes related to random tilings, double contour representations for correlation kernels were found in (Okounkov–Reshetikhin-2001).

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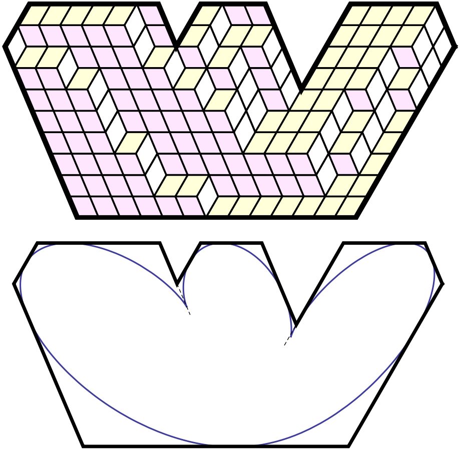
Advantage. This method can be generalized to a class of polygonal domains with horizontal boundaries on two straight lines. Still there are explicit formulas (rational parameterization) for the boundaries of frozen regions (Petrov–2012).

Lozenge tilings: frozen regions

Method 2B: Class of polygonal domains — one example.



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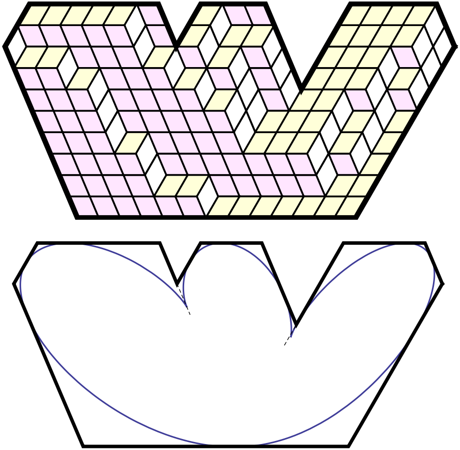

Disadvantage. Breaks down if we change either class of domains or measure.

Lozenge tilings: appearance of $N^{1/3}$

What are the fluctuations of the boundary of the frozen region of $Na \times Nb \times Nc$ hexagon?

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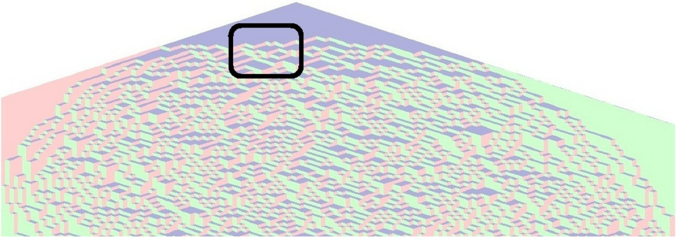
(Baik-Kriecherbauer-McLaughlin-Miller; Petrov) For the uniformly random tilings the right scaling is $N^{1/3}$ in normal direction and $N^{2/3}$ in tangent direction to the frozen boundary.



After rescaling the limit fluctuations of the frozen boundary are governed by the Tracy-Widom distribution F_2.

The whole $2D$ picture is governed by the *extended Airy kernel.*

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Baik-Kriecherbauer-McLaughlin-Miller: a partial result via orthogonal polynomials.

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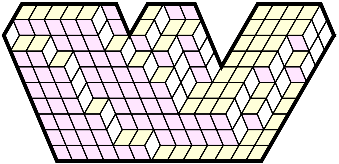
Baik-Kriecherbauer-McLaughlin-Miller: a partial result via orthogonal polynomials.

Petrov: complete result via the steepest descent analysis of the double contour integrals.

The proof again extends to a class of polygonal domains with horizontal boundaries on two straight lines.



Question. Is this $N^{1/3}$ the same as that for TASEP and, more generally, KPZ–class growth models?

From lozenge tilings to TASEP

Relation to KPZ is obtained through the construction of a growth model for tilings.

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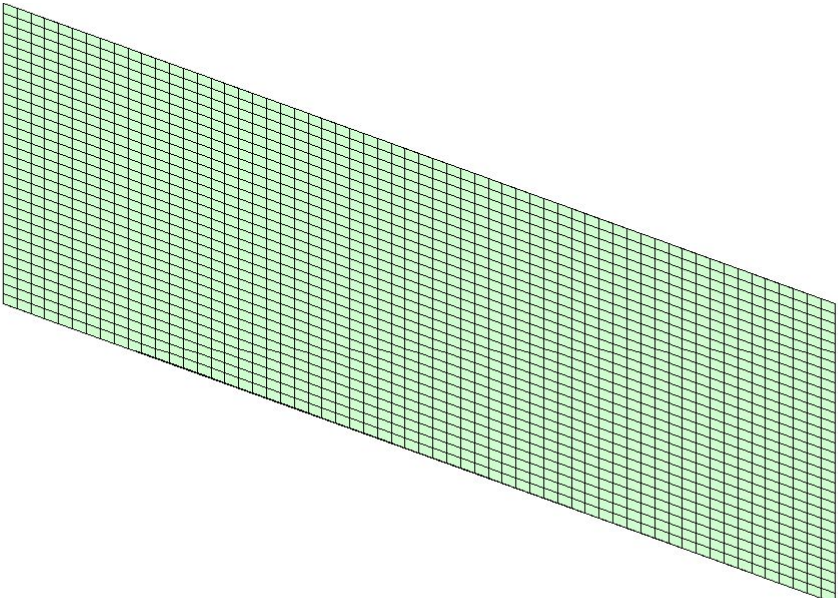
Relation to KPZ is obtained through the construction of a growth model for tilings.

Theorem. (Borodin-Gorin-2009) There exists a simple discrete time Markov chain which relates uniformly distributed tilings of hexagons of various sizes. Elementary step of this chain changes the size of hexagon from $a \times b \times c$ to $a \times (b-1) \times (c+1)$. Algorithmically, one step involves generating some independent one-dimensional random variables.

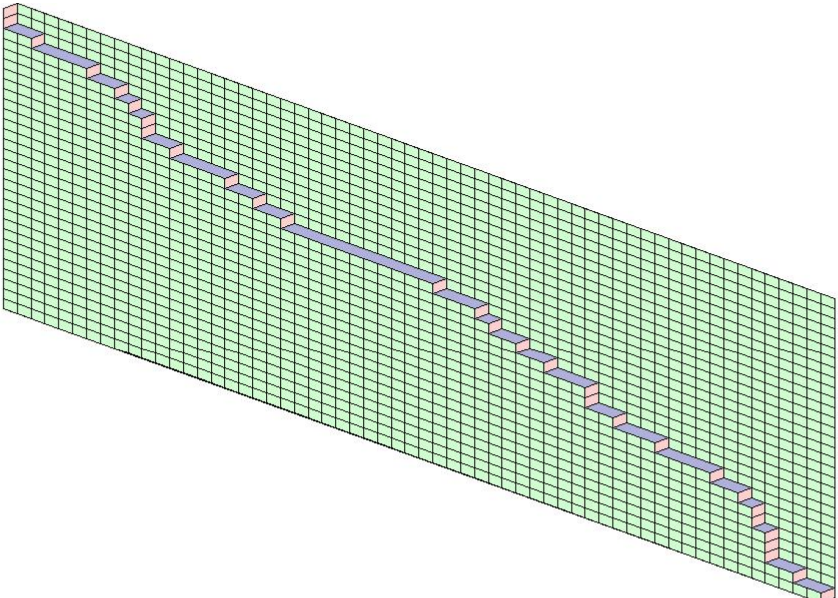
Observation. There is exactly one way to tile $a \times b \times 0$ hexagon.

Remark. The construction can be generalized to more complicated *elliptic* weights on tilings of hexagon (Borodin–Gorin–Rains-2009; Betea–2011)

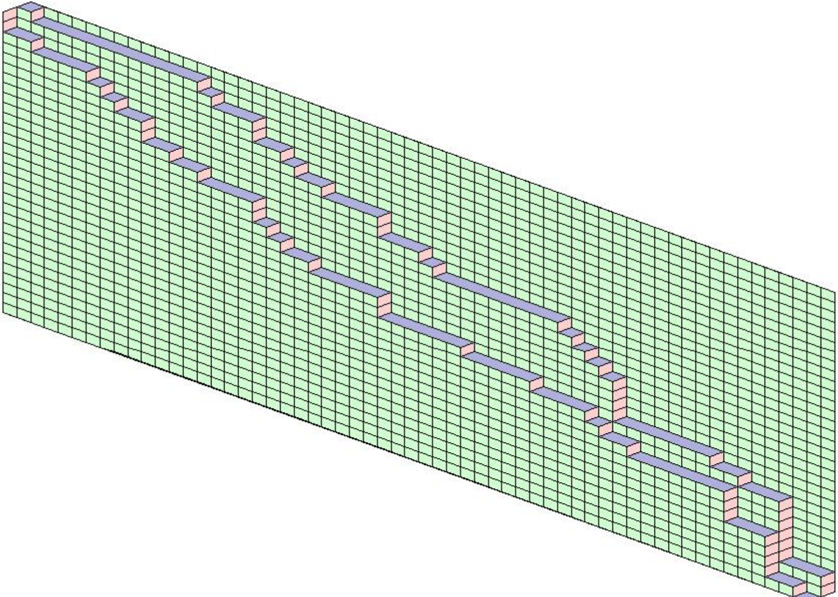
$30 \times 60 \times 0$



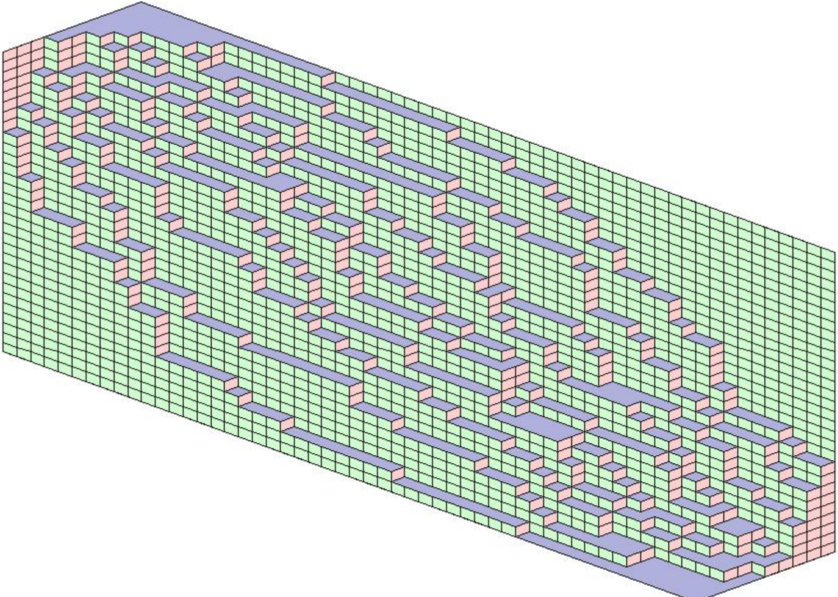
$30 \times 59 \times 1$



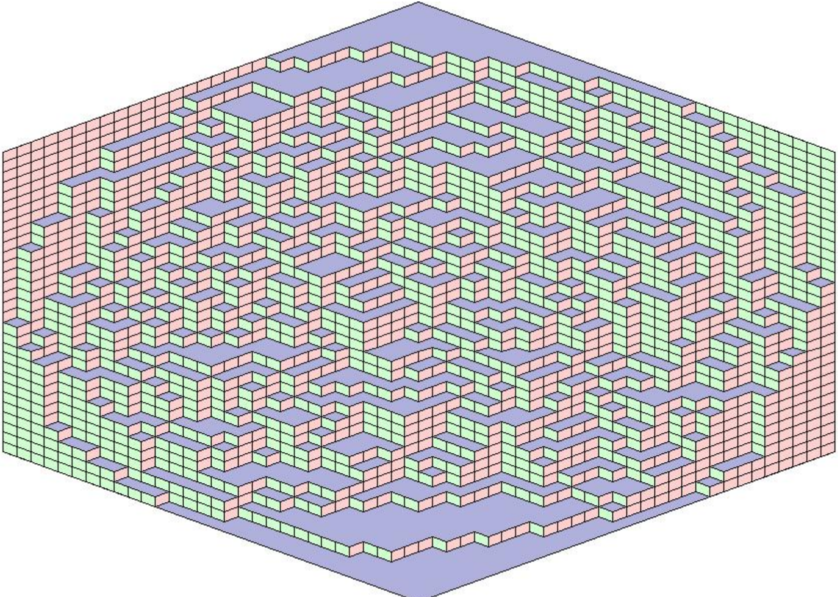
$30 \times 58 \times 2$



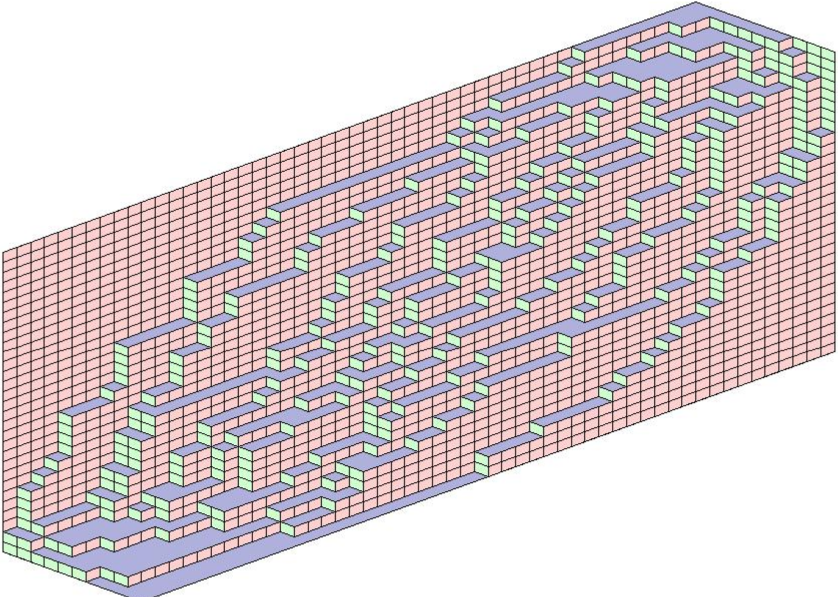
$30 \times 50 \times 10$



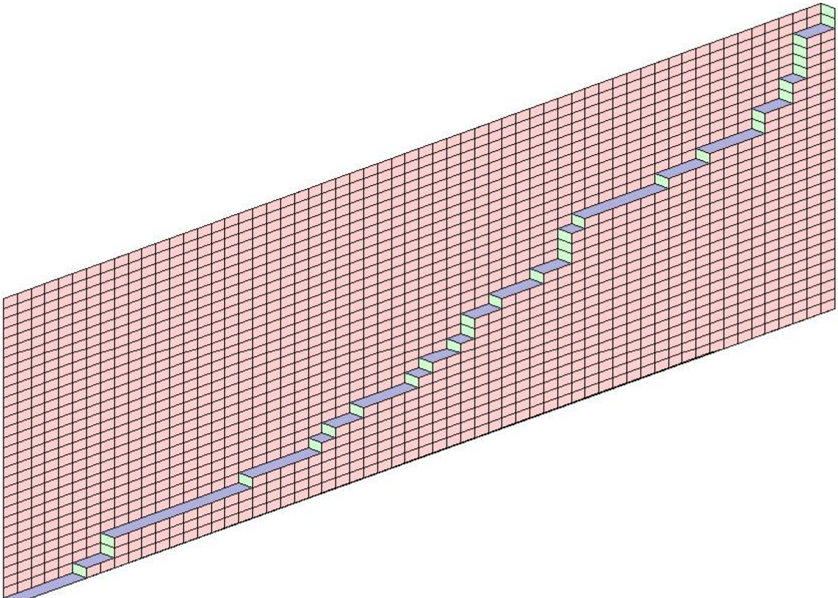
$30 \times 30 \times 30$



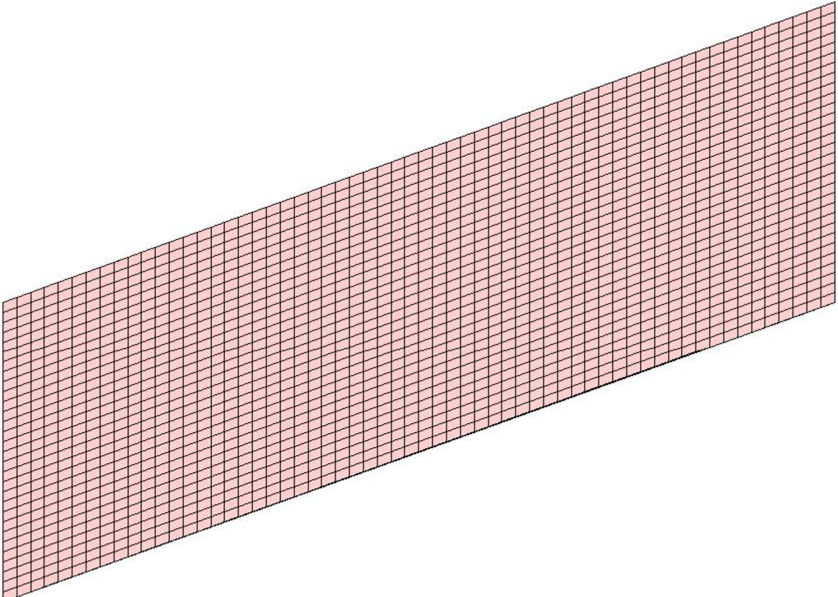
$30 \times 10 \times 50$



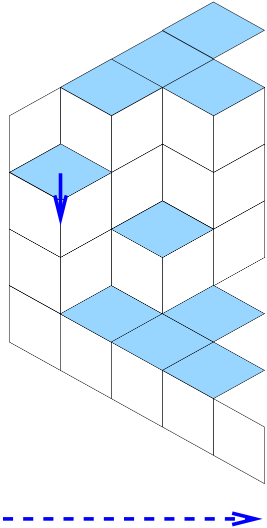
$30 \times 1 \times 59$



$30 \times 0 \times 60$

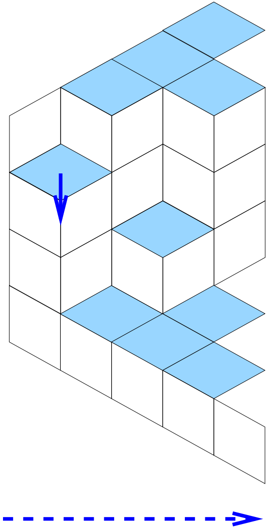


From lozenge tilings to TASEP



Description of Markov chain in terms of (interlacing) horizontal lozenges

From lozenge tilings to TASEP



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- Left to right sequential update

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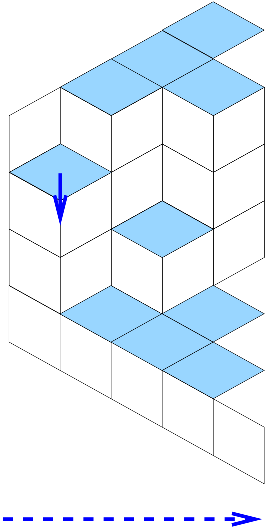

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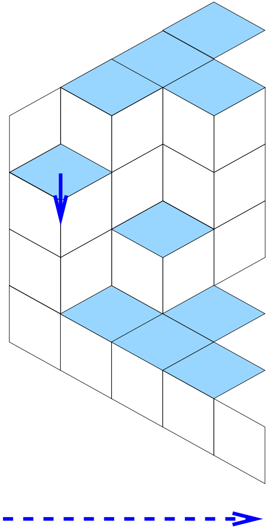

Description of Markov chain in terms of (interlacing) horizontal lozenges

- Left to right sequential update
- Each lozenge jumps down according to (explicit, position-dependent) hypergeometric–type distribution
- Interlacing preserved through block/push interaction

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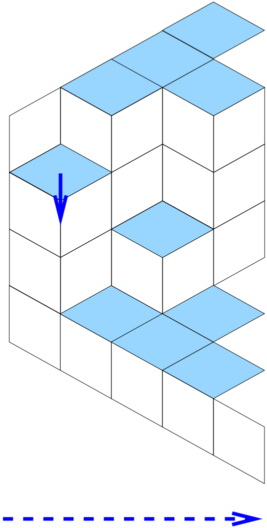

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- Left to right sequential update
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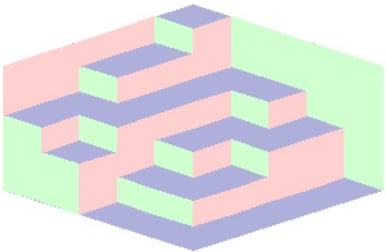
- Interlacing preserved through block/push interaction

Works for more complicated measures, but only for hexagon.

From lozenge tilings to TASEP

This Markov chain is simplified in the limit transition $a, b, c \to \infty$, $ab/c \to t$ (time).



(leftmost horizontal lozenges)

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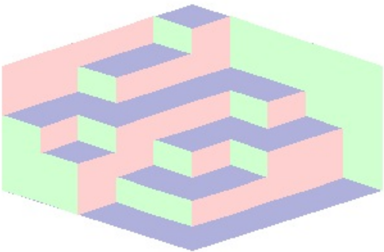

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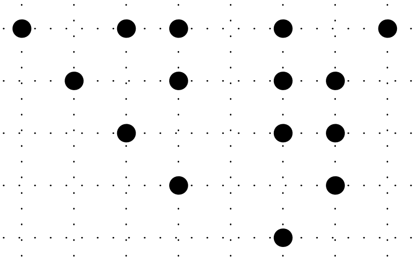
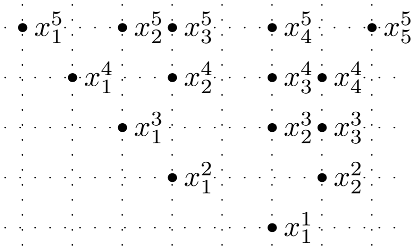


(rotated picture)

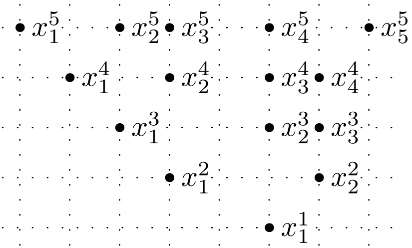
From lozenge tilings to TASEP



In the limit we get a continuous time dynamics $Y(t)$ on particle configurations with integer coordinates subject to interlacing conditions $x_{i-1}^{j} < x_{i-1}^{j-1} \leq x_i^j$.

Each particle has an exponential clock of rate 1. All clocks are independent. When the clock rings, the particle attempts to jump to the right.

From lozenge tilings to TASEP



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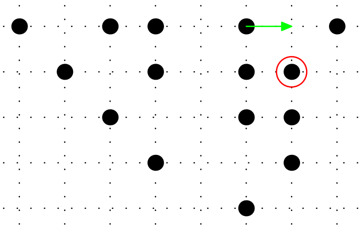
The interlacing conditions are preserved by the rule “if higher, then lighter”.

From lozenge tilings to TASEP

Block:



From lozenge tilings to TASEP

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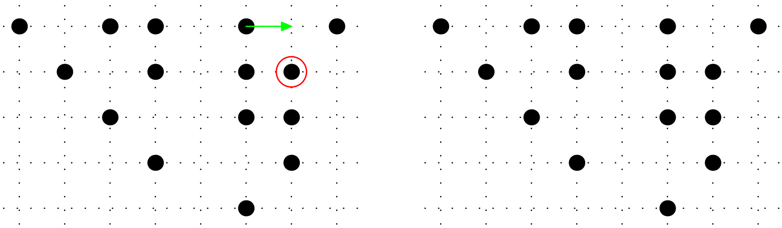

From lozenge tilings to TASEP

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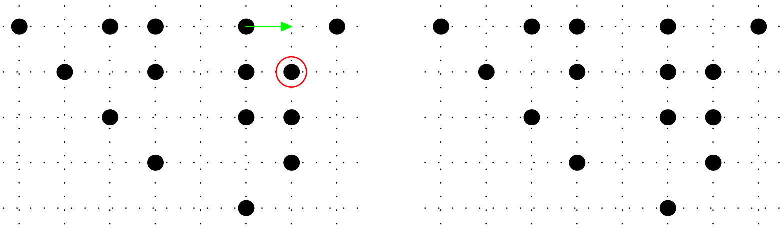
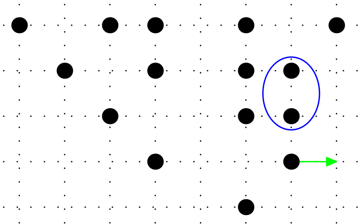
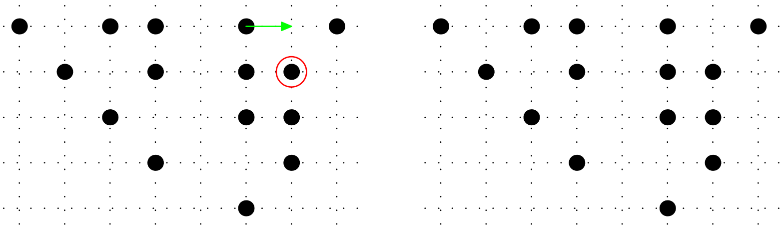


Push:

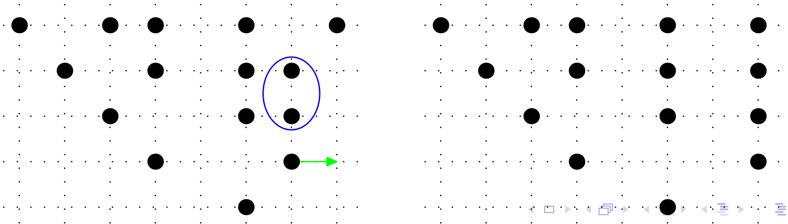


From lozenge tilings to TASEP

Block:



Push:



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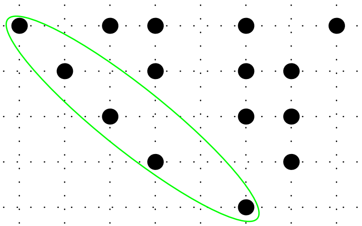

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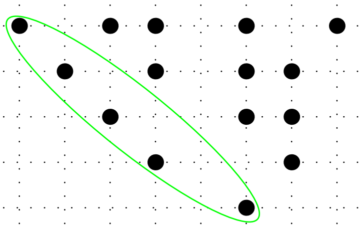

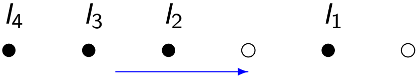

Totally Asymmetric Simple Exclusion Process.

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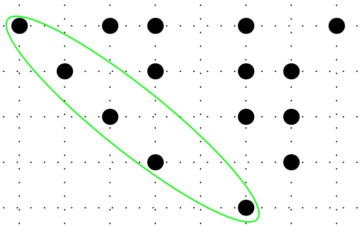

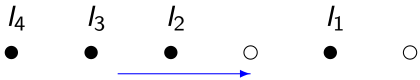

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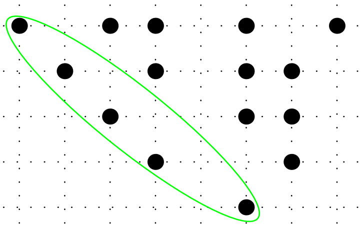

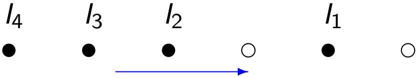

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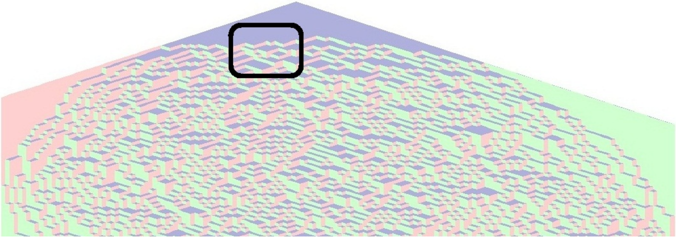
Both TASEP and PushASEP are known to belong to the KPZ universality class of growth models.

Lozenge tilings: appearance of $N^{1/3}$

What are the fluctuations of the boundary of the frozen region of $Na \times Nb \times Nc$ hexagon?

(Baik-Kriecherbauer-McLaughlin-Miller; Petrov) For the uniformly random tilings the right scaling is $N^{1/3}$ in normal direction and $N^{2/3}$ in tangent direction to the frozen boundary.



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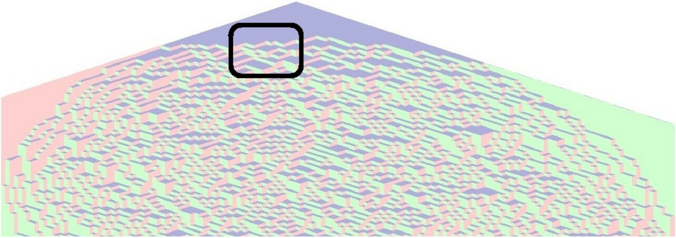

After rescaling the limit fluctuations of the frozen boundary are governed by the Tracy-Widom distribution F_2.
Question. Is this $N^{1/3}$ the same as that for eigenvalues of random matrices?

$N^{1/3}$ in random matrices

Gaussian Unitary Ensemble of rank N is the distribution on the set of $N \times N$ *Hermitian* matrices with density

$$\rho(X) \sim \exp\big(-\mathrm{Trace}(X^2)/2\big).$$

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All eigenvalues are real. *GUE–eigenvalues* density is (Weyl, 20-30s)

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For $N = 1$ we get a standard Gaussian.
Wigner (1955): spacing between eigenvalues model the spacings between the lines in the "spectrum" of a heavy atom.

$N^{1/3}$ in random matrices

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

Let x_i^k be ith eigenvalue of top–left $k \times k$ corner of GUE.
Interlacing condition:

$$x_{i-1}^j \leq x_{i-1}^{j-1} \leq x_i^j$$

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$$\begin{array}{ccccccc} x_1^4 & & x_2^4 & & x_3^4 & & x_4^4 \\ & x_1^3 & & x_2^3 & & x_3^3 & \\ & & x_1^2 & & x_2^2 & & \\ & & & x_1^1 & & & \end{array}$$

The joint distribution of x_i^j is known as GUE–minors process (although, correct name would be GUE–corners)
Given $x_1^N, \dots, x_N^N$, the distribution of x_i^j, $j < N$ is *uniform* on the polytope defined by interlacing conditions (Baryshnikov, 2001)

$N^{1/3}$ in random matrices

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$N^{1/6}(x_N^N - 2N^{1/2})$ converges as $N \to \infty$ towards the distribution F_2.

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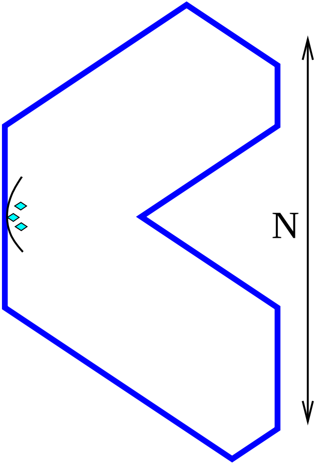
How can a lozenge tiling become a random matrix?

From lozenge tilings to random matrices



General conjecture. For the domain of linear size N near the point where frozen boundary is tangent to the boundary of the domain, the fluctuations are of order $\sqrt{N}$. After rescaling, the distribution of position of one type of lozenges converges to GUE–corners process.

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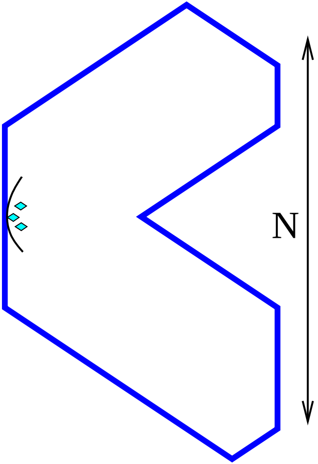

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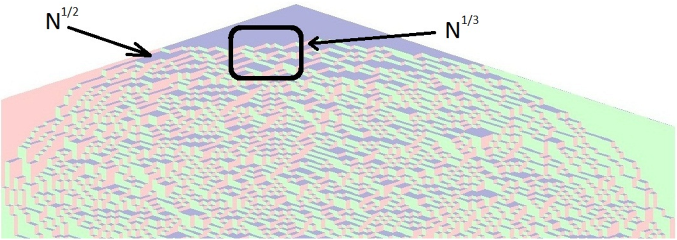
Stated by Okounkov and Reshetikhin in 2006. They also gave an *informal* argument explaining why this should be true.

From lozenge tilings to random matrices



Away from the boundary the fluctuations are of order $N^{1/3}$, with limit governed by Tracy–Widom distribution $F_2(s)$.
Near the boundary fluctuations of the frozen curve of lozenge tiling are of order $N^{1/2}$ with Gaussian limit.
GUE–corners process glues $N^{1/2}$ and $N^{1/3}$ classes together.

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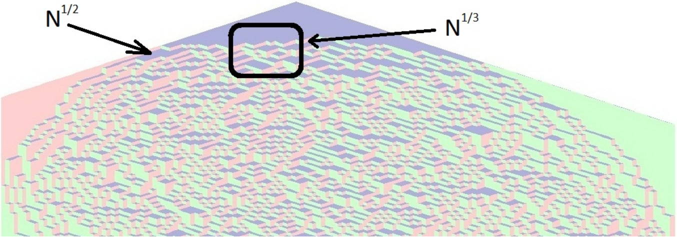

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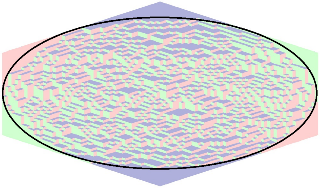
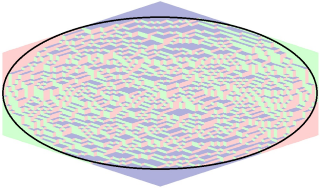
The conjecture itself is now proved in several cases.

From lozenge tilings to random matrices



Theorem. (Johansson–Nordenstam, 2006; Nordenstam, 2009) For the hexagon the fluctuations near the point where inscribed ellipse touches the boundary are of order $\sqrt{N}$ and after rescaling the point process formed by positions of one type of lozenges converges to GUE–corners process.

From lozenge tilings to random matrices



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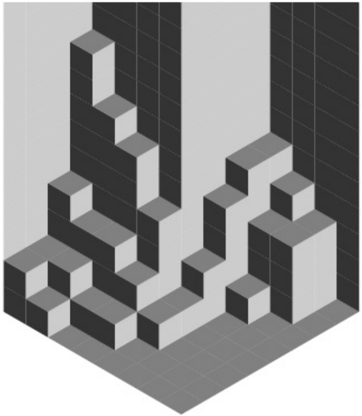
Method: Computation based on Lindström-Gessel-Viennot formula for the number of non-intersecting paths + certain determinant evaluations. The asymptotics itself is a shadow of the convergence of Hahn (hypergeometric) orthogonal polynomials toward Hermite orthogonal polynomials.

From lozenge tilings to random matrices



(picture from Okounkov–Reshetikhin, 2006)

Theorem.
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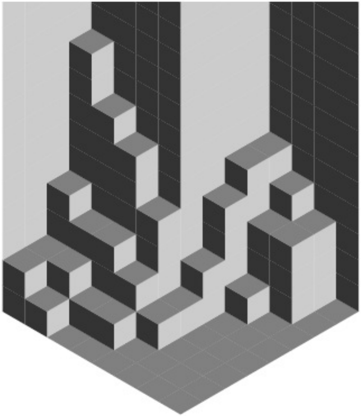

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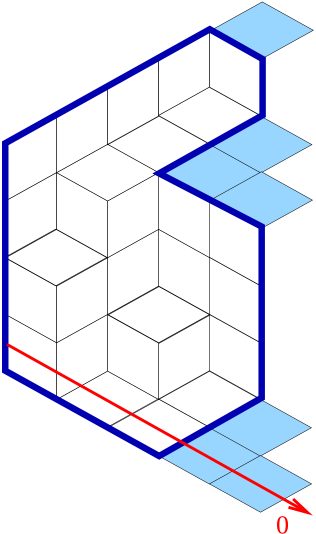
Method: Determinantal point processes + formalism of Schur processes which leads to double contour integral representation of the correlation kernel + steepest descent analysis of double contour integrals.

From lozenge tilings to random matrices

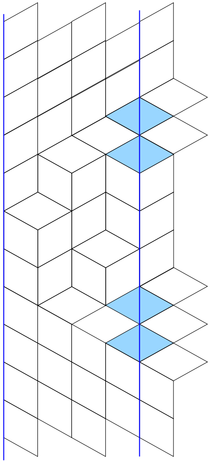
Theorem. (Gorin–Panova, 2013) GUE-convergence conjecture holds for the following class of domains.



Domain $\Omega_{N,y(N)}$ is parameterized by its width N and positions $y(N)_1 < y(N)_2 < \cdots < y(N)_N$ of N horizontal lozenges at the right boundary. Tiling this domain is the same as tiling a certain polygon.

Here $N = 5$ and $y(5) = (0 < 1 < 5 < 6 < 8)$.

From lozenge tilings to random matrices



Method.
How to compute the distribution of k horizontal lozenges $\lambda_1^k, \ldots, \lambda_k^k$ on kth vertical line in uniformly random tiling of $\Omega_{y(N),N}$?

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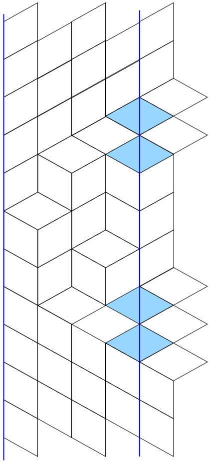

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The distribution itself is complicated. But there are formulas for certain generating functions.

From lozenge tilings to random matrices

Given $\lambda_1 < \lambda_2 < \cdots < \lambda_k$ the corresponding Schur function is a symmetric polynomial in $x_1, \ldots, x_k$ given by (note a bit non-standard notation)

$$s_\lambda(x_1, \ldots, x_j) = \frac{\det \left[x_i^{\lambda_j} \right]_{i,j=1}^{k}}{\prod_{i<j}(x_j - x_i)}$$

Proposition. Let $\lambda^{(k)} = \lambda_1^k, \ldots, \lambda_k^k$ encode k horizontal lozenges on kth vertical line in uniformly random tiling of $\Omega_{y(N),N}$.

$$\mathbb{E}\left(\frac{s_{\lambda^{(k)}}(x_1, \ldots, x_k)}{s_{\lambda^{(k)}}(1, \ldots, 1)} \right) = \frac{s_{y(N)}(x_1, \ldots, x_k, 1, \ldots, 1)}{s_{y(N)}(\underbrace{1, \ldots, 1}_{N})}.$$

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Now the limit theorem boils down to the (still non-trivial!) study of asymptotics of Schur functions as the number of variables tends to infinity, which we do.

Integrable probability on interlacing particles

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One of the reasons for the existence of rich structures: Interlacing particle configurations are *Gelfand–Tsetlin patterns* from representation theory.
(E.g. they parameterize bases of irreducible representations of unitary groups.)