Macroscopic shape and $L^{1/3}$ fluctuations for the level lines of the (2+1)-D SOS model

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Plan

- SOS: a random discrete interface model
- Hard wall and entropic repulsion
- Macroscopic shape
- Level lines as interacting random polymers, and cube-root fluctuations
- Heuristics, difficulties and methods
- Relations with other 1/3's (open discussion...)
- Open problems

(2+1)Dimensional SOS model

Discrete height: $\varphi = \{\varphi_x,\ x \in \mathbb{Z}^2\}$, with $\varphi_x \in \mathbb{Z}$.

Λ square of side L in $\mathbb{Z}^2$ centered at 0.

0 boundary condition: $\varphi_x = 0$ for all $x \in \mathbb{Z}^2 \setminus \Lambda$.

Gibbs measure: $\beta > 0$

$$\pi(\varphi) = \pi_{\beta,L}(\varphi) = \frac{1}{Z_{\beta,L}} \exp\Big(- \beta \sum_{x \sim y} |\varphi_x - \varphi_y| \Big)$$

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Roughening transition:

Low temperature (large β, rigid phase): localization
$\pi_{\beta,L}(\varphi_0^2) \leqslant C_\beta$ (exponential tails, via Peierls argument)

High temperature (small β, rough phase): delocalization
$\pi_{\beta,L}(\varphi_0^2) \approx \log L$ (difficult! see Fröhlich-Spencer CMP 1981).
[One expects convergence to Gaussian Free Field]

$(2+1)$D SOS with a wall: Entropic repulsion

$\varphi_x \in \mathbb{Z}$ and

$$\pi_+(\varphi) = \pi(\varphi \,|\, \varphi_x \;\geqslant\; 0 \;\forall x \in \Lambda)$$

Entropic repulsion heuristics (β large):

- shift heights $h \to h+1$ at energy loss $-4\beta L$ (boundary)
- full downward spikes at x give the gain in entropy $+L^2\, e^{-4\beta h}$.
- surface grows until $4\beta L \approx L^2\, e^{-4\beta h}$ or $h \approx \frac{1}{4\beta} \log L$.

Bricmont, El Mellouki, Fröhlich '82:

$$\frac{1}{|\Lambda|} \sum_{x\in\Lambda} \pi_+[\varphi_x] \in \left[\frac{c_1}{\beta} \log L, \frac{c_2}{\beta} \log L \right]$$

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We will see: most heights are either at height $\lfloor \frac{1}{4\beta} \log L \rfloor$ or $\lfloor \frac{1}{4\beta} \log L \rfloor - 1$, according to the fractional part of $\frac{1}{4\beta} \log L$.

Results 1: typical height

From now on, β is sufficiently large (so that we are deep in the rigid phase) and δ is a small constant, tending to zero if $\beta \to \infty$.

Let $H(L), \alpha(L)$ be the integer/fractional part of $1/(4\beta) \log L$

and $E_h = \{\varphi : \#\{x : \varphi_x = h\} \geq (1 - \delta)L^2\}$.

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Theorem

One has $\lim_{L\to\infty} \pi_+(E_{H(L)-1} \cup E_{H(L)}) = 1$.
Moreover, consider a subsequence L_k *such that* $\alpha(L_k) \to \alpha$.
There exists $\alpha_c(\beta)$ *such that:*

- *if* $\alpha > \alpha_c(\beta)$ *then*

$$\lim_{k\to\infty} \pi_+(E_{H(L_k)}) = 1$$

- *if* $\alpha < \alpha_c(\beta)$ *then*

$$\lim_{k\to\infty} \pi_+(E_{H(L_k)-1}) = 1.$$

Results 2: macroscopic shape

For $j \in \mathbb{Z}$, let $\mathcal{L}_j$ be the collection of closed level lines (loops) at height $H(L_k) - j$, of length $\gg \log L_k$.

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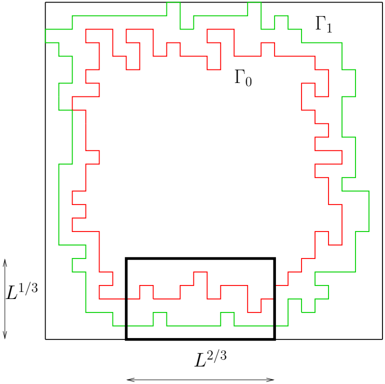
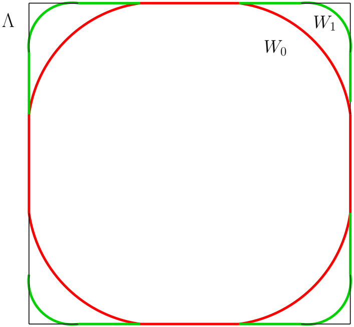
Theorem

There exist deterministic shapes $\mathcal{W}_j, j \geq 0$ *such that the following holds with probability* $\to 1$ *as* $L_k \to \infty$*:*

- *for* $j < 0$*, the collection of loops* $\mathcal{L}_j$ *is empty*
- *assume that* $\alpha > \alpha_c(\beta)$*. Then every* $\mathcal{L}_j, j \geq 0$ *contains a single loop* Γ_j*. Rescaling,* $(1/L_k)\Gamma_j$ *tends to* $\mathcal{W}_j$
- *assume that* $\alpha < \alpha_c(\beta)$*. The same holds, except that* $\mathcal{L}_0$ *is empty.*

Results 2: macroscopic shape





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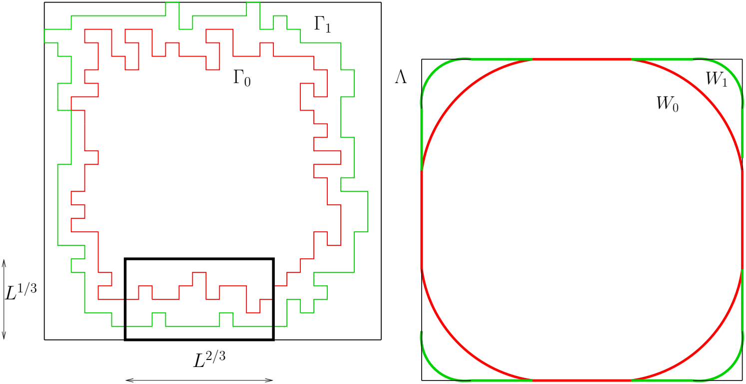
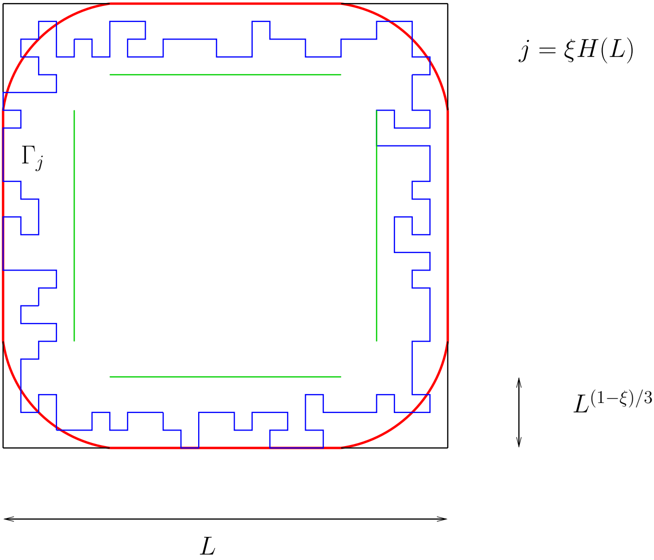


We will see in a moment how the $\mathcal{W}_j$ are related to a Wulff construction. $\partial\mathcal{W}_j$ has both flat and curved portions.

Results 3: Cube-root fluctuations



Results 3: Cube-root fluctuations

Distance from flat part of the boundary
(say $(0,0)$ is midpoint of the bottom side of Λ):

$$\Delta_j(L) = \min\{y \ : \ (0,y) \in \Gamma_j\}$$

Theorem (simplified version)

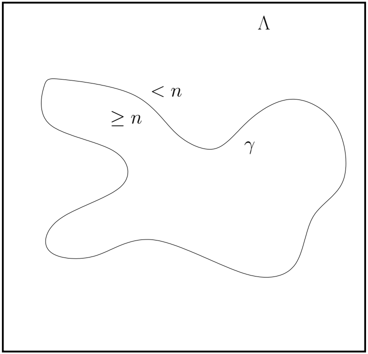
Say that $\alpha > \alpha_c(\beta)$. For every $\varepsilon > 0$, w.h.p.

$$L^{1/3-\varepsilon} < \Delta_0(L) < L^{1/3+\varepsilon}.$$

For $0 \leq \xi < 1$, $j = \lfloor \xi H(L) \rfloor$: For every $\varepsilon > 0$, w.h.p.

$$\Delta_j(L) < L^{(1-\xi)/3+\varepsilon}.$$

The basic fact: probability of a large contour



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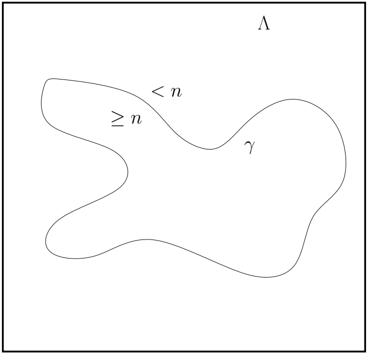

Basic observation:

$$\pi_+(\gamma \text{ is a } \textit{n-contour}) \sim \exp\left[-\beta|\gamma| + A(\gamma)\pi(\varphi_x = n)\right]$$

Link with Ising with small external field

Now, $\pi(\varphi_x = n) \overset{n\to\infty}{\sim} C_\beta \exp(-4\beta n)$.
If $n = H(L) = 1/(4\beta) \log L - \alpha(L)$,

$$\pi_+(\gamma \text{ is a } H(L)\text{-contour}) \sim \exp\left(-\beta|\gamma| + \frac{\lambda}{L} A(\gamma)\right)$$

where $\lambda(\beta, L) = C_\beta e^{4\beta\alpha(L)}$.

It is like Ising with $1/L$ magnetic field (Schonmann, Shlosman '95).

If λ exceeds a critical λ_c, a macroscopic $H(L)$-contour is favorable.
For $(H(L) - j)$-contours, λ is replaced by $\lambda e^{4\beta j}$.

Macroscopic shape and λ_c: Heuristics

To minimize energy, a $H(L)$-contour γ of area u^2 will have a Wulff shape and an energy cost $-uW_\beta = -L\left(\frac{u}{L}\right)W_\beta \approx -4L(\frac{u}{L})\beta$.

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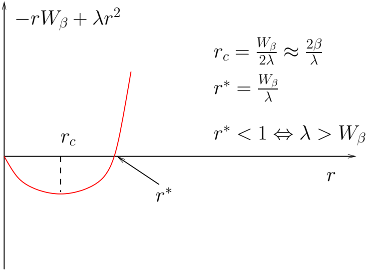
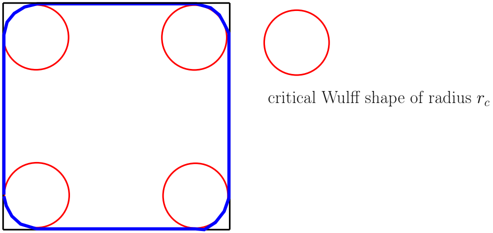
Energy-entropy competition when $r = (u/L)$ is of order 1.



Macroscopic shape and λ_c: Heuristics

The previous argument would give $\lambda_c = W_\beta$.

A more careful analysis reveals that the macroscopic shape $\mathcal{W}_0$ of Γ_0 is



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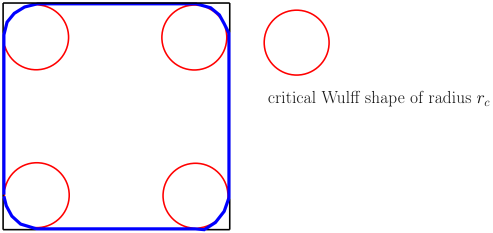

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For $H(L) - j$ contour, $\lambda \to \lambda e^{4\beta j}$ and $r_c \to r_c e^{-4\beta j}$.

Technical difficulty

$$\log P(\Gamma_0, \Gamma_1, \dots) \simeq -\beta|\Gamma_0| - \beta|\Gamma_1| + \dots$$

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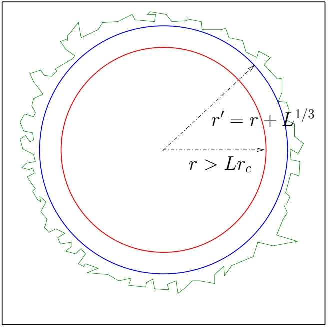
$$+\Phi(\Gamma_0, \Gamma_1, \dots)$$

(interaction term, with no definite sign).

Φ is small with β and decays fast with distance, but still, it is the main technical pain.

Why cube-root fluctuations?

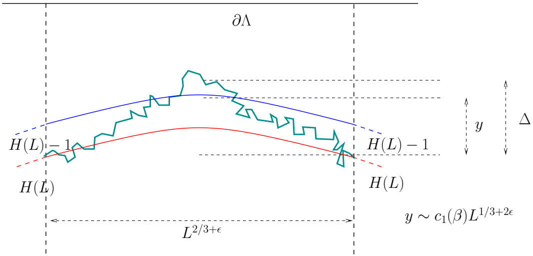
Basic step: conditionally on the $H(L)$-contour containing a Wulff shape of radius $\ell > r_c L$, it contains w.h.p. a Wulff shape of radius $\ell + L^{1/3}$.



Works as long as the blue shape is at distance at least $L^{1/3+\epsilon}$ from the boundary.

Why cube-root fluctuations?

Local analysis of contour



If Δ is with high probability $\gg y$, we win (argument to be repeated all around the red shape).

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Contour γ behaves like random walk with area term

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Distribution of height Δ is approximately

$$P(\Delta) \propto \exp\left[-\frac{\Delta^2}{L^{2/3+\epsilon}\sigma^2(\beta)} + \frac{\lambda}{L}L^{2/3+\epsilon}\Delta\right]$$

Diffusion constant depends on surface tension

Area term

Typical fluctuation of Δ is of order $L^{1/3+\epsilon/2}$, negligible w.r.t. its average $E(\Delta) \sim c_2(\beta)L^{1/3+2\epsilon}$.

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Fact: if $\ell/L > r_c$, then $c_2(\beta) > c_1(\beta)$ so that $\Delta \gg y$.
Entropic repulsion wins against curvature, $H(L)$-contour grows

Why not less than $L^{1/3}$?

For a contour portion of longitudinal size $L^{2/3-\epsilon}$, area term

$$\frac{Area(\gamma)}{L}$$

is of order $1 \Longrightarrow$ negligible.

Transversal fluctuations are normal, of order $\sqrt{L^{2/3-\epsilon}} = L^{1/3-\epsilon/2}$.

Non-trivial technical difficulty: rule out pinning to the boundary $\partial\Lambda$.

Connections with other 1/3's

1. K. Alexander, CMP '01: subcritical FK cluster conditioned to have large area
2. Ferrari-Spohn, AoP '05: Brownian bridge conditioned to stay above a circular/parabolic barrier
3. Velenik, PTRF '04: random walk with penalizing area term.
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In case 4, exact solvability via determinantal representation. In our case, no hope to have exact solution (because of interactions, overhangs, etc).

Open problems

- Order of fluctuations along the curved portions of the limit shapes: $\sqrt{L}$ (as opposed to $L^{1/3}$)?
- Is the fluctuation upper bound $L^{(1-\xi)/3}$ for the line at height $\xi H(L)$ optimal?
- What is limit process of the line ensemble? Connection with Airy line ensembles? (random matrices, TASEP, ...)