Consensus and Synchronization Problems in Networks with Time Delays

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Diffusive dynamics

- Coupled linear systems

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)), \qquad i = 1, \ldots, n.$$

$x_i(t) \in \mathbb{R}$ is the state unit i at time t
$a_{ij} \geq 0$ is the strength of the influence of unit j on i.

In vector form, with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$,

$$\dot{x} = -Lx(t).$$

Graphs, $\mathcal{G} = \mathcal{G}(V, E)$

- V : Vertex set $\{1, 2, \ldots, n\}$, E : Edge set
- Neighborhood relation: $i \sim j$ if $(i, j) \in E$
- (Weighted) adjacency matrix $A = [a_{ij}]$

$$a_{ij} = \begin{cases} 1 \text{ (or } >0) & \text{if} \quad i \sim j \\ 0 & \text{otherwise} \end{cases}$$

- Vertex degree $d_i = \sum_j a_{ij}$.
- Degree matrix $D = \text{diag}\{d_1, d_2, \ldots, d_n\}$
- Laplacian: $D - A$. Normalized Laplacian: $\bar{L} = I - D^{-1}A$

Applications

- Consensus problem in continuous time

$$\dot{x}_i(t) = \varepsilon \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)),$$

or discrete time

$$x_i(t+1) = x_i(t) + \varepsilon \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t))$$

i.e. $\dot{x} = \varepsilon L x(t)$ or $x(t+1) = (I - \varepsilon L)x(t)$,
where $x_i(t)$ is the "opinion" of person i at time t,
$\varepsilon \geq 0$ is the coupling strength.

The system is said to reach *consensus* if for any set of initial conditions there exists some $c \in \mathbb{R}$ such that $\lim_{t\to\infty} x_i(t) = c$ for all i. The number c is then called the *consensus value*.

- Traffic flow: Car following model

$$\dot{x}_i(t) = \sum_{j=1}^n a_{ij} h(x_j(t) - x_i(t)), \qquad x_i = \text{speed of vehicle } i.$$

$h : \mathbb{R} \to \mathbb{R}$ is monotone increasing, differentiable, $h(0) = 0$.

Desired behavior: All vehicles travel at a common speed regardless of variations in road and traffic conditions.

Linear variational equation about the desired state yields $\dot{x} = -\varepsilon L x$, with $\varepsilon = h'(0)$.

- Coupled phase oscillators

$$\dot{\theta}_i(t) = \omega + \sum a_{ij} h(\theta_j - \theta_i), \quad \theta_i \in S^1,\ i = 1, \ldots, N$$

  where $h : S^1 \to \mathbb{R}$, differentiable.

Synchronous solution: $\theta_i(t) = \Omega t \ \forall i$, for some common frequency Ω. Linear variational equation around the synchronous state yields $\dot{x} = -\varepsilon L x$.

- Random walks on graphs:

$$P[x(t+1) = j | x(t) = i] = \frac{1}{d_i} a_{ij}$$

  If $p = (p_1, \ldots, p_n)$ is a probability distribution on the vertices of a graph, then

$$\begin{aligned} p(t+1) &= p(t) \cdot D^{-1} A \\ &= p(t) \cdot (I - \bar{L}) \end{aligned}$$

Spectrum of the graph Laplacian

For graphs with nonnegative weights, let $\{\lambda_1, \ldots, \lambda_n\}$ denote the Laplacian eigenvalues. Then,

- $\text{Re}(\lambda_i) \geq 0 \ \forall i$.
- $\lambda_1 = 0$ is always an eigenvalue; with (right) eigenvector $(1, 1, \ldots, 1)^\top$.
- $\lambda_1 = 0$ is a simple eigenvalue iff the graph has a spanning tree.

Definition. A graph is said to have a spanning tree if there exists a vertex, called the root, such that for each other vertex j there exists at least one directed path from the root to vertex j.

Laplacian dynamics under delays

- Undelayed system

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} \Big(x_j(t) - x_i(t)\Big)$$

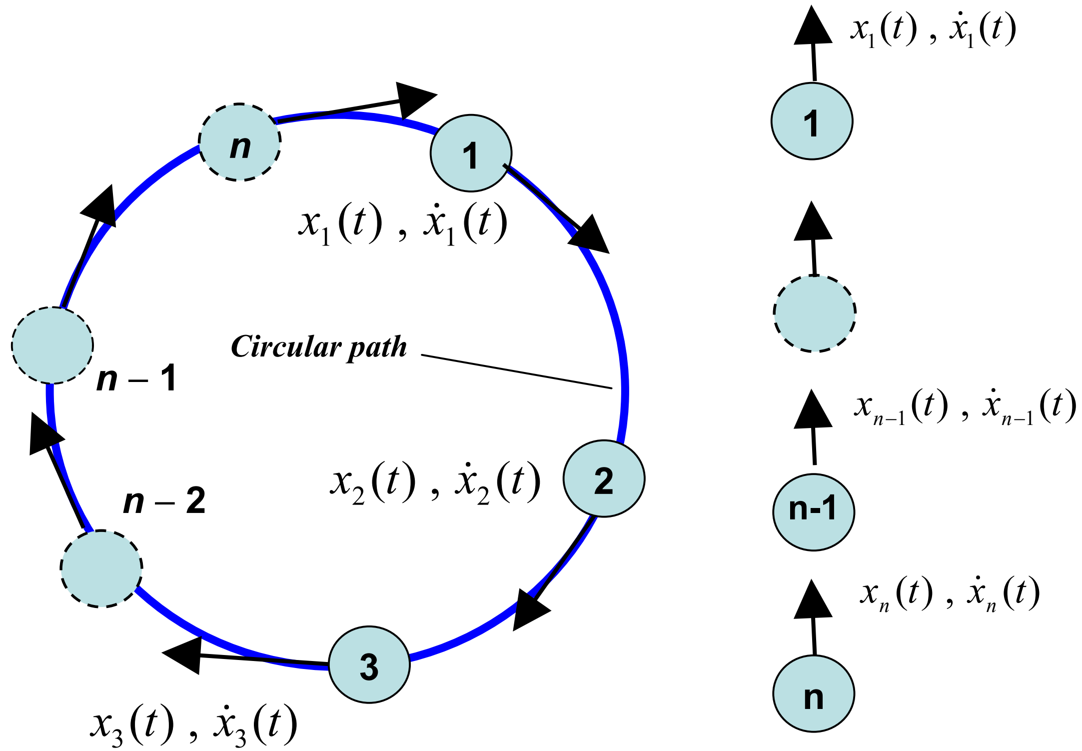
- Information processing delays

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} \Big(x_j(t-\tau) - x_i(t-\tau)\Big)$$

- Information propagation delays

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} \Big(x_j(t-\tau) - x_i(t)\Big)$$

Processing delays: Car-following problem



$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij} h(x_j(t) - x_i(t)), \qquad x_i = \text{speed of vehicle } i.$$

Desired: All vehicles travel at a common speed to avoid collisions.

Modeling driver reaction time

- τ = reaction delay

$$\dot{x}_i(t) = \sum_{j=1}^{N} a_{ij}(x_j(t-\tau) - x_i(t-\tau))$$

$$\dot{x}(t) = -Lx(t-\tau)$$

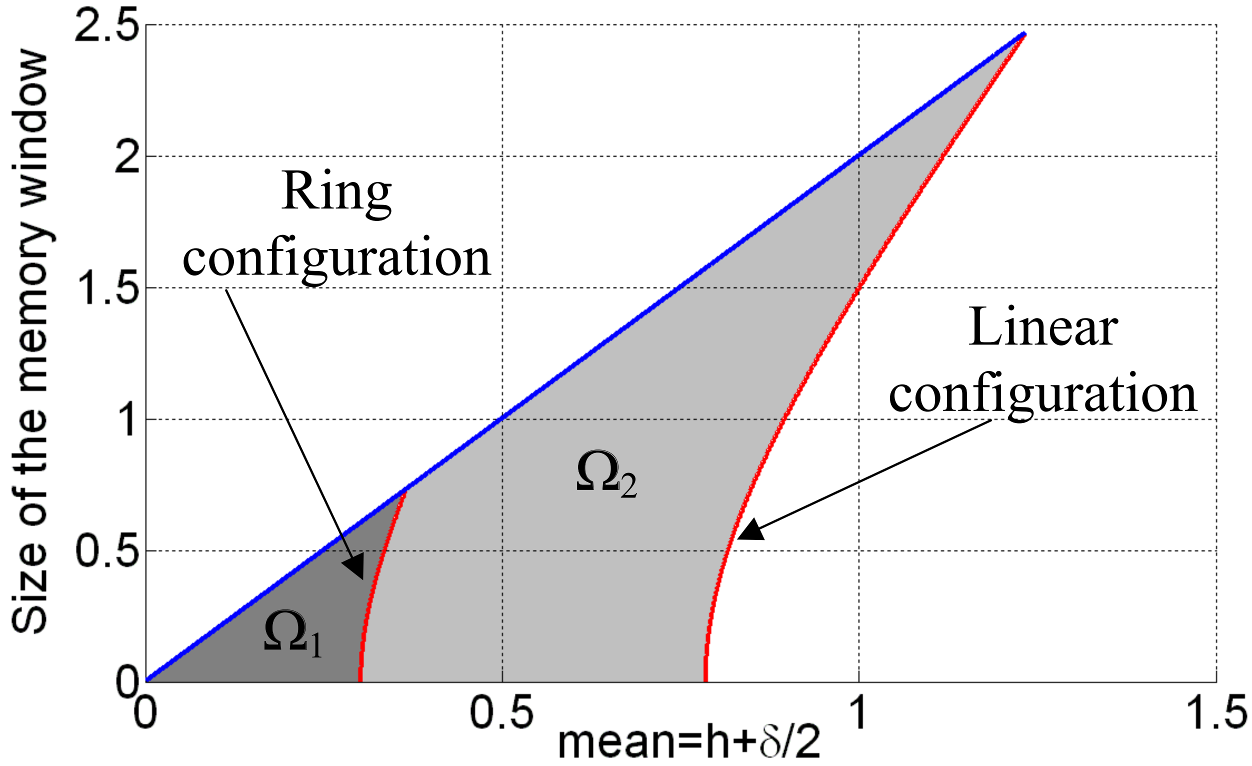
- More general delay model

$$\dot{x}(t) = -L \int_0^{\tau} x(t-s) f(s)\, ds$$

which includes single or multiple delays as well as distributed delays. Distributed delays model short-term driver memory.

Sipahi, FA, Niculescu, SIAM J. App. Math. 2007

Stability regions



Sipahi, FA, Niculescu, SIAM J. App. Math. 2007

Comparison: Discrete vs distributed delays

- For processing delays, increasing the variance of the delay distribution about a fixed mean value can stabilize an otherwise unstable system.

- Generally discrete delays represent an extremum among delay distributions. If the delays act towards destabilizing (resp., stabilizing) the system, then the discrete delay is locally the most destabilizing (resp., most stabilizing) one among delay distributions having the same mean.

FA, Discr. Cont. Dyn. Syst. S 2008

Propagation delays: Consensus problem

$$\dot{x}_i(t) = \frac{1}{d_i} \sum_{j=1}^{n} a_{ij} \left(\int_0^{\tau} f(s) x_j(t-s)\, ds - x_i(t) \right)$$

$f(s) \geq 0$ for $s \in [0, \tau]$, and $\int_0^{\tau} f(s)\, ds = 1$.

Theorem. The system reaches consensus if and only if zero is a simple eigenvalue of the Laplacian $\bar{L}$, i. e. the graph has a spanning tree. Furthermore, the consensus value is

$$c = \frac{1}{1+\bar{\tau}} \left\langle u^1, x(0) + \int_0^{\tau} \int_{-\theta}^{0} f(\theta) x(\xi)\, d\xi\, d\theta \right\rangle$$

where u^1 is the left eigenvector of $\bar{L}$ corresponding to the zero eigenvalue and $\bar{\tau} = \int_0^{\tau} s f(s)\, ds$ is the mean of the distribution f.

FA, Phil. Trans. Roy. Soc. A 2013

FA, Lect. Notes Contr. Inform. Sci. 2012

Comparison: Transmission vs processing delays

- Transmission delays: Stability independent of the delays.
  Processing delays: Stability for small delays, instability for large.
  The critical delay depends on network topology.

- Transmission delays: Consensus value depends on the past history.
  Processing delays: Consensus value depends only on the initial value at a single time instant.

Time-varying graphs

In many real networks, connections or the strength of connections change in time.

- Communication networks

- Networks of moving agents; obstacles between agents

- Failure and recovery of nodes in networks

- Social and biological networks with changing connectivities and/or connection weights.

In time-varying networks, the adjacency matrix $A = A(t)$ changes with time, randomly or deterministically with its own dynamics.

$\Rightarrow$ A changing graph $G(t)$ at each instant of time.

Synchronization problems on time-varying graphs

Model system

$$x_i(t+1) = f_i^t(x_1(t), \ldots, x_n(t)), \quad x_i \in \mathbb{R},\, i = 1, \ldots, n,$$

The system is said to synchronize if $\lim_{t\to\infty} |x_i(t) - x_j(t)| = 0$ for all i, j, though x_i not necessarily converging to a constant value.

Assumption There exists a C^1 function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f_i^t(s, \ldots, s) = f(s), \quad \forall s, t, i.$$

- Then the synchronization manifold

$$\mathcal{S} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 = \cdots = x_n\}$$

is invariant under the evolution of the system.

- If $x_1(t) = \cdots = x_n(t) = s(t)$ denotes the synchronized state, then
$$s(t+1) = f(s(t)).$$
Assume that there exists a compact asymptotically stable attractor $\mathcal{A}$ for this equation.

- (Local) synchronization refers to the local attractivity of $\mathcal{S} \cap \mathcal{A}^n$, where $\mathcal{A}^n$ denotes the n-fold Cartesian product $\mathcal{A} \times \cdots \times \mathcal{A}$.

- Note that by assumption,
$$\sum_{i=1}^{n} \frac{\partial f_i^t}{\partial x_j}(s, \ldots, s) = f'(s)$$
for all t, i.

- Hence, when the system is linearized around the synchronous solution,

$$u(t+1) = G(t)u(t),$$

  the matrix $G(t)$ has constant row sums (equal to $f'(s(t))$), $\forall t$.

- So, the subspace $E_0 \subset \mathbb{R}^n$ spanned by $(1, 1, \ldots, 1)$ is an invariant subspace of $G(t)$ for all t.

- Local conditions for synchronization are expressed as asymptotical decay of small perturbations in directions transverse to E_0.

- The latter can be characterized by the *projection joint spectral radius* and the *projection Lyapunov exponent*.

Further characterization of synchronization: Hajnal diameter

For a matrix L with row vectors $g_1, \cdots, g_n$ and a vector norm $\| \cdot \|$ in $\mathbb{R}^n$, the Hajnal diameter of L is given by *(Hajnal, 1958):*

$$\text{diam}(L, \| \cdot \|) = \max_{i,j} \|g_i - g_j\|.$$

We introduce the Hajnal diameter for a matrix sequence $\mathcal{L} = \{L_1, L_2, \ldots\}$:

Definition. The Hajnal diameter of the matrix sequence $\mathcal{L}$ is defined by

$$\text{diam}(\mathcal{L}) = \overline{\lim_{t\to\infty}} \sup_{t_0 \geq 0} \left\{ \text{diam} \left[\prod_{k=t_0}^{t_0+t-1} L_k \right] \right\}^{\frac{1}{t}}$$

where $\prod$ denotes the left matrix product:

$$\prod_{k=1}^{n} A_k = A_n \times A_{n-1} \times \cdots \times A_1.$$

- The definition does not depend on the choice of the norm.

Theorem. (Lu, FA & Jost, SIAM J. Math. Anal. 2007) The logarithm of the diameter equals the largest transverse Lyapunov exponent.

Synchronization theorem for time-varying networks

$$x_i(t+1) = f_i^t(x_1(t), \ldots, x_n(t)), \quad i = 1, \ldots, n$$

with $f_i^t(s, \ldots, s) = f(s)$ for all s, t, i.

Small perturbations u around the synchronized state evolve by

$$u(t+t_0) = \prod_{k=t_0}^{t+t_0-1} D_k(f^{(k-t_0)}(s_0))\, u(t_0)$$

where D_k are the appropriate Jacobians at time k.

Let $\mathcal{D} = \{D_1, D_2, \ldots\}$ and define

$$\mathrm{diam}(\mathcal{D}, s_0) = \overline{\lim_{t\to\infty}} \sup_{t_0 \geq 0} \left\{ \mathrm{diam} \left[\prod_{k=t_0}^{t+t_0-1} D_k(f^{(k-t_0)}(s_0)) \right] \right\}^{\frac{1}{t}}.$$

Theorem. (Lu, FA & Jost, SIAM J. Math. Anal. 2007) If

$$\sup_{s_0 \in \mathcal{A}} \mathrm{diam}(\mathcal{D}, s_0) < 1$$

then the compact set $\mathcal{A}^n \bigcap \mathcal{S}$ is a uniformly asymptotically stable attractor of the coupled system in $\mathbb{R}^n$; that is, the coupled system locally uniformly synchronizes.

Application to delayed networks

$$x_i(t+1) = x_i(t) + \varepsilon \sum_{j=1}^{n} a_{ij}(t)(x_j(t) - x(t - \tau_{ij}(t)))$$

with bounded delays $\tau_{ij}(t) \in \mathbb{Z}^+$ from vertex j to i. The time dependence may be driven by, e.g. some stochastic process.

Can be written as

$$x_i(t+1) = \sum_{j=1}^{n} G_{ij}(t) x_j(t - \tau_{ij}(t))$$

- Let $\mathcal{G} = \{G(t) = I - \varepsilon L(t)\}_{t \in \mathbb{Z}^+}$ be the corresponding graph sequence.
- We say that the self-links are undelayed if $\tau_{ii}(t) = 0 \ \forall i, t$.

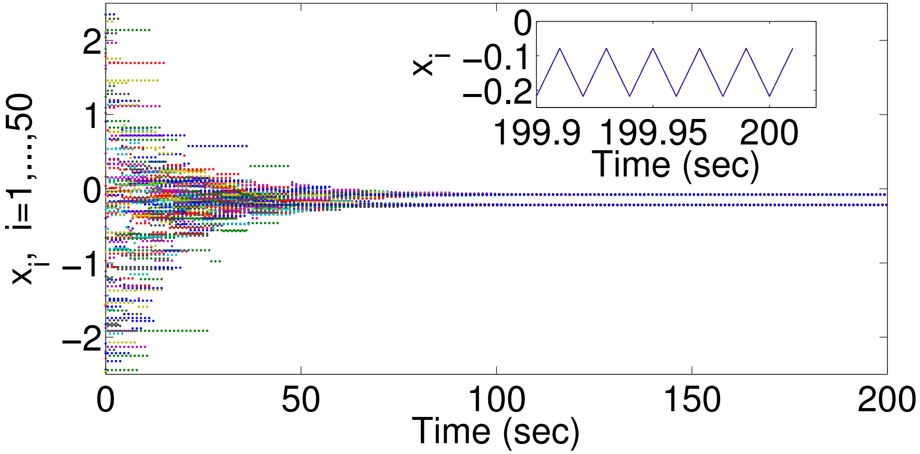
Theorem (Lu, FA, & Jost, *Networks and Heterogeneous Media*, 2011) Suppose the self-links are undelayed. Then the network reaches consensus if there exists $T \in \mathbb{N}$ such that the union graph across any time interval of length T has a spanning tree.

Theorem (Lu, FA & Jost, *SIAM J. Math. Anal.* 2007) The latter condition is satisfied if and only if diam($\mathcal{G}$) < 1.

Theorem (Lu, FA, & Jost, *Networks and Heterogeneous Media*, 2011) If the self-links are also delayed, and the (constant) delays satisfy certain integer patterns, the system does not reach consensus but instead synchronizes to a periodic trajectory whose period depends on the delay pattern.

Simulation examples of nonlinear consensus

Self-delay = 1:



Self-delay = 2: