



Properties of invariant graphs in discontinuous forced systems

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Synchronization in skew-product dynamical systems

$$\begin{cases} w^{t+1} = f(w^t) & w \in W, z \in Z \quad \text{Banach spaces} \quad 0 < \lambda < 1 \\ z^{t+1} = g(w^t, z^t) & f \text{ invertible}, \quad \|g(w, z) - g(w, z')\| < \lambda \|z - z'\| \end{cases}$$

Examples:

- $g(w, z) = (1 - \varepsilon)\tilde{f}(z) + \varepsilon f(w)$, with $\tilde{f} \approx f$ and $\varepsilon \approx 1$. Nearly identical coupled systems
- $g(w, z) = \gamma(w) + \lambda z$ Linear filters

Applications:

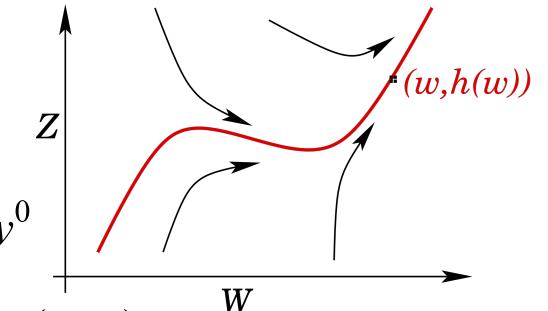
Signal analysis, Material science

(Generalized) Synchronization:

$\exists h: W \rightarrow Z$ such that $\lim_{t \rightarrow \infty} \|h(w^t) - g(w^t, z^t)\| = 0$ for all z^0 , unif.in w^0

Hint: $h(w) = \lim_{t \rightarrow \infty} g_{f^{-1}(w)} \circ g_{f^{-2}(w)} \circ \dots \circ g_{f^{-t}(w)}(z)$ where $g_w(z) := g(w, z)$

Rem: $g(w, h(w)) = h \circ f(w)$ Invariant graph h = Synchronization function



Question: Regularity (smoothness) properties of synchronization function?

Which features of the drive - e.g. dimensions, Lyapunov exponents, etc - carry over to the factor?

Motivation: Informations about attractor of f from the knowledge of the attractor of $h \circ f$

$$\begin{aligned} \text{Main results on sync functions: } & \begin{cases} w^{t+1} = f(w^t) & f \text{ invertible} \\ z^{t+1} = g(w^t, z^t) & \|g(w, z) - g(w, z')\| < \lambda \|z - z'\| \end{cases} \end{aligned}$$

- f homeomorphism & g continuous with $w \Rightarrow h$ continuous
- f diffeomorphism, $\|Df^{-1}\| = \mu (> 1)$ & g is C^1 & $\lambda\mu < 1 \Rightarrow h$ differentiable
[Campbell & Davies '96]
- non-uniformly hyperbolic case: $\lambda e^{\text{Lyap}_{f^{-1}}} < 1 \Rightarrow h$ Whitney differentiable [Stark, '97 & '99]
- f^{-1} Lipschitz with mod. μ & g Lipschitz $\Rightarrow h$ Hölder with exp. $\frac{-\log \lambda}{\log \mu}$ (< 1 if $\lambda\mu > 1$)
($+ \lambda\mu < 1 \Rightarrow h$ Lipschitz)

Consequence of theory of inertial manifolds [Hirsh, Pugh & Shub, '70 & '77]

- Nonlinear dynamics aspects (e.g. dimension estimates) [Pecora & Carroll '90, Hunt, Ott & Yorke '97]
- Extensions to non-invertible f [Rulkov & Afraimovich '03, Barreto et al. '03]

Sync functions in discontinuous forced systems?

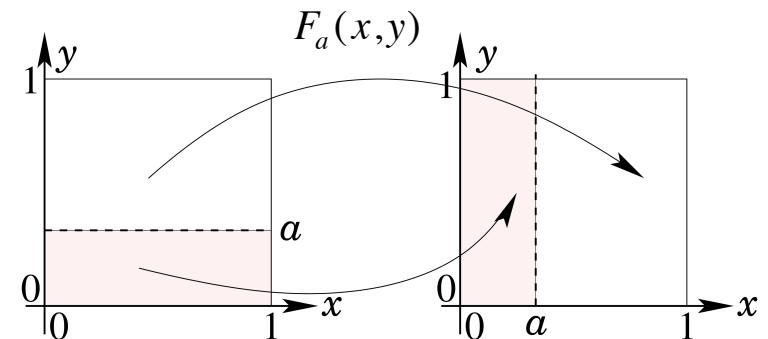
Motivation (applied): Low pass filtering of chaotic signals [Badii & Politi '87]

Motivation (theory): Regularity of sync function depends on topological features of forcing?

Discontinuous forced systems

Basic piecewise affine example:

$$\begin{cases} (x^{t+1}, y^{t+1}) = F_a(x^t, y^t) \quad (x, y) \in [0,1]^2 \\ z^{t+1} = (1 - \lambda)x^t + \lambda z^t \quad z \in R \end{cases}$$



Baker's map:

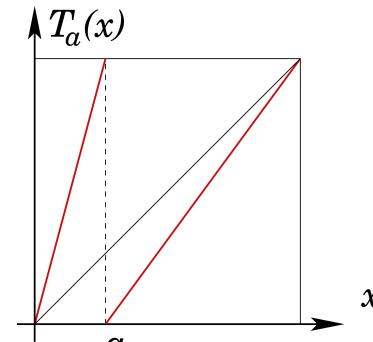
[Hunt, Ott & Yorke '97, Afraimovich, Chazottes & Cordonnet '01]

$$F_a(x, y) = (\Theta_a(x, y), T_a(y))$$

$$\Theta_a(x, y) = \begin{cases} a x & \text{if } 0 \leq y < a \\ a + (1 - a)x & \text{if } a \leq y \leq 1 \end{cases}$$

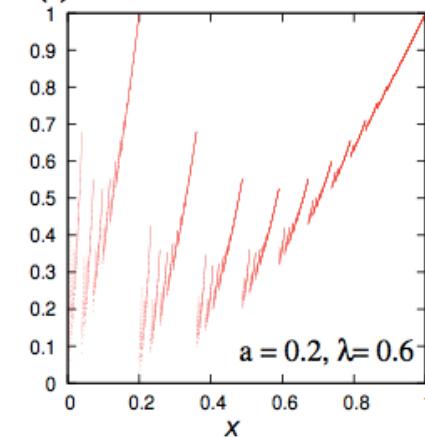
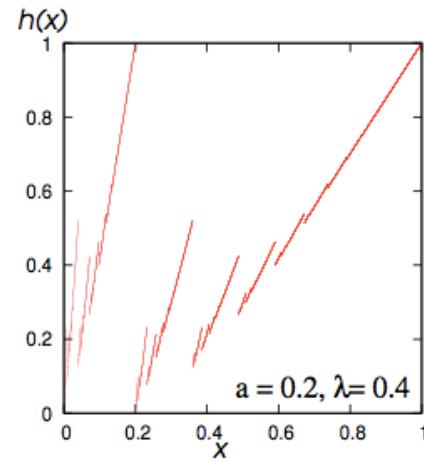
$$T_a(x) = \begin{cases} \frac{y}{a} & \text{if } 0 \leq y < a \\ \frac{y-a}{1-a} & \text{if } a \leq y \leq 1 \end{cases}$$

$$\text{Rem: } F_a^{-1}(x, y) = (T_a(x), \Theta_a(y, x))$$



Synchronization function:

$$h(x) = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t T_a^{t+1}(x)$$

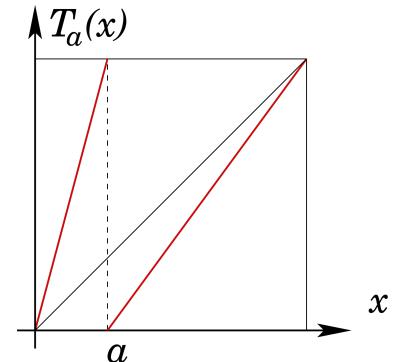


Ref. B.F., Discontinuous generalized synchronization of chaos, Dynamical Systems 27 (2012) 105-116

Properties of sync function (1)

$$h(x) = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t T_a^{t+1}(x)$$

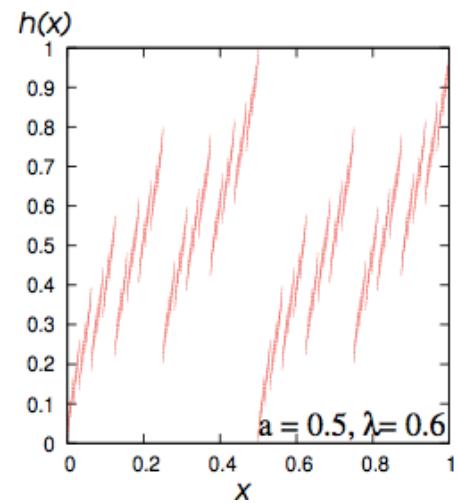
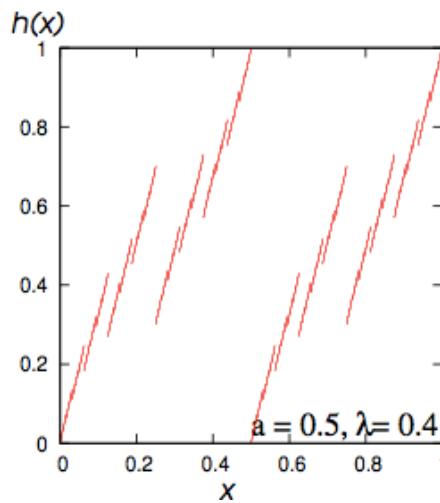
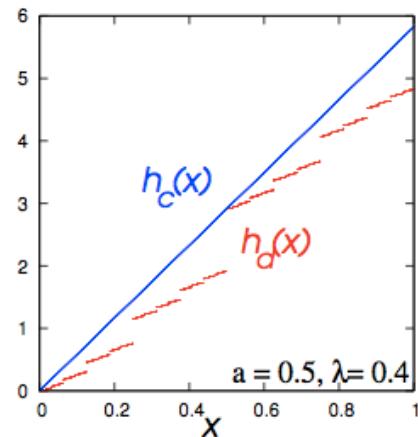
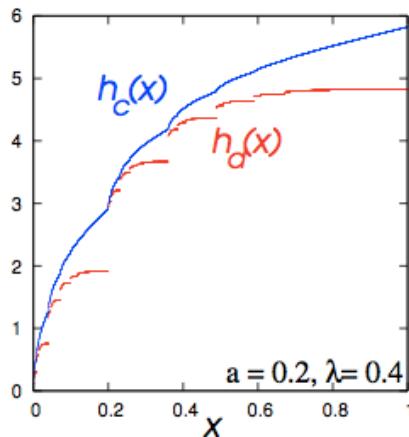
$$T_a(x) = \begin{cases} \frac{y}{a} & \text{if } 0 \leq y < a \\ \frac{y-a}{1-a} & \text{if } a \leq y \leq 1 \end{cases}$$



Proposition (basic properties):

- h is càdlàg
- h is continuous at $x \notin D_a = \bigcup_{t=0}^{\infty} T_a^{-t}(a)$
- h bounded variation $\Leftrightarrow 2\lambda < 1$
- If $2\lambda < 1$, $h_d(x) := \sum_{y \in D_a, y \leq x} h(y-0) - h(y)$ well-defined & $h_{ac} := h + h_d$ strictly incr. & abs. cont.
- $h(x-0) > h(x)$ if $x \in D_a$
- $h([0,1]) = [0,1]$
- D_a dense in $[0,1]$

NB: $e^{\text{Lyap}_{T_a}} = \frac{1}{a^a (1-a)^{1-a}} \leq 2 \leq \frac{1}{\min\{a, 1-a\}}$



Properties of sync function (2)

Proposition (fine properties 1): $h_{ac}'(x) = \sum_{t=0}^{\infty} \lambda^t (T_a^{t+1}(x))'$ exists $\forall x \notin D_a \Leftrightarrow \lambda < a \wedge (1-a)$

Rem: $h_{ac}'(x)$ exists for a.e. $x \in [0,1]$, for all $\lambda < 1/2$

Proposition (fine properties 2): Assume $a \neq 1/2$ and let $T'_a(a) = 1/(1-a)$. Then

- h_{ac}' bounded variation iff $\lambda < a(1-a)$
- For $\lambda < a(1-a)$, we have

* $(h_{ac}')_d(x) := \sum_{y \in D_a, y \leq x} h_{ac}'(y-0) - h_{ac}'(y)$ well-defined $\forall x$ & $(h_{ac}')_{ac} := h_{ac}' + (h_{ac}')_d$ abs. continuous

* $(h_{ac}')_d$ increasing & $(h_{ac}')_{ac}$ strictly incr. if $a < 1/2$ (decreasing if $a > 1/2$)

* $(h_{ac}')_{ac}'(x)$ exists and $= 0 \forall x \notin D_a$

Main ingredients of proofs (absolute continuity and differentiability):

- $h_n(x) = (1-\lambda) \sum_{t=0}^n \lambda^t T_a^{t+1}(x)$ uniformly converges to h
- Lebesgue convergence thm for integrals
- Fundamental thm of calculus
- h_{ac}' uniformly approx. by step functions and continuous for $x \notin D_a$

$\Rightarrow x \rightarrow \int_0^x h_{ac}'(y) dy$ differentiable at every $x \notin D_a$

Nonlinear extensions

Discontinuous forcing with N pieces:

$$\begin{cases} (x^{t+1}, y^{t+1}) = F(x^t, y^t) \\ z^{t+1} = g(x^t, z^t) \end{cases} \quad \begin{aligned} F(x, y) &= (\Theta(x, y), T(y)) \text{ where } \Theta(x, y) = S_i^{-1}(x) \text{ if } y \in J_i \\ F^{-1}(x, y) &= (S(x), \Sigma(y, x)) \text{ where } \Sigma(y, x) = T_i^{-1}(y) \text{ if } x \in I_i \end{aligned}$$

- $T|_{I_i} = T_i$ where $I_i = [x_i, x_{i+1})$, $0 < x_i < x_{i+1} < 1$, $i = 1, \dots, N$,

$T_i \in C^1(I_i, [0,1])$, increasing, 1-1 and onto

- $S|_{J_i} = S_i$ where $J_i = [x'_i, x'_{i+1})$, $0 < x'_i < x'_{i+1} < 1$, $i = 1, \dots, N$,

$S_i \in C^1(J_i, [0,1])$, increasing, 1-1 and onto

- $g(\cdot, z)$ strict. increasing and equi-continuous family, $|g(x, z) - g(x, z')| < \lambda |z - z'|$

Sync function: $h(x) = \lim_{t \rightarrow \infty} g_{f^{-1}(x)} \circ g_{f^{-2}(x)} \circ \dots \circ g_{f^{-t}(x)}(z)$ where $g_x(z) := g(x, z)$

Proposition (standard properties):

- h is càdlàg
- h bounded variation and $h = h_{ac} - h_d$ if $N\lambda < 1$
- h continuous at $x \notin \bigcup_{t=0}^{\infty} T^{-t}(\{x_i\})$
- h infinite variation if $N \left(\inf_{x, z \neq z'} \frac{|g(x, z) - g(x, z')|}{|z - z'|} \right) > 1$
- $h(x-0) > h(x)$
- h nowhere monotonous

Absolute cont. of induced measure (pw affine case - 1)

$$\begin{cases} (x^{t+1}, y^{t+1}) = F_a(x^t, y^t) \\ z^{t+1} = x^t + \lambda z^t \end{cases}$$

$$h(x) = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t T_a^{t+1}(x)$$

[with A. Quas]

- Rem: Leb² = SRB measure for F_a
- Question: Leb o h^{-1} abs. continuous?

Theorem (BV case): $\forall a \in (0,1)$ and $2\lambda < 1$, Leb o h^{-1} abs. cont. with density $\sum_{x \in h^{-1}(z)} \frac{1}{h'(x)}$

Main ideas of proof: Use $h = h_{ac} - h_d$ to show

(1) $\psi(z) = \sum_{x \in h^{-1}(z)} \frac{1}{h'(x)}$ is well-defined a.e. (In particular, prove that $\# h^{-1}(z) < \infty$ a.e.)

(2) ψ integrable and $|h^{-1}J| \geq \int_J \psi \ \forall$ sub-interval J

(3) $\int_{h([0,1])} \psi = 1$, then (2) + (3) $\Rightarrow |h^{-1}J| = \int_J \psi$

Ref. B.F. and A. Quas, *Statistical properties of invariant graphs in piecewise affine discontinuous forced systems*, Nonlinearity 24 (2011) 2477-2488

Absolute cont. of induced measure (pw affine case - 2)

$$\begin{cases} (x^{t+1}, y^{t+1}) = F_a(x^t, y^t) \\ z^{t+1} = x^t + \lambda z^t \end{cases}$$

$$h(x) = (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t T_a^{t+1}(x)$$

[with A. Quas]

- Rem: Leb² = SRB measure for F_a
- Question: Leb $o h^{-1}$ abs. continuous?

Theorem (Case of infinite variation): If $a = 1/2$, then Leb $o h^{-1}$ abs. cont. for a.e. $\lambda > 1/2$

Strategy of proof: Let $\pi(\omega) := \sum_{n \geq 0} 2^{-n} (\omega_n + 1)$ where $\omega_n = \pm 1$

Show that $h o \pi \propto \sum_{n \geq 1} \omega_n (\lambda^n - 2^{-n})$, where $\omega_n = \pm 1$ with proba 1/2, has abs. cont. distribution

Inspired by proof that $\sum_{n \geq 1} \omega_n \lambda^n$ has abs. cont. distribution for a.e. $\lambda > 1/2$ [Peres & Solomyak '96]

(Proof based on differentiation of measures and 'transversality condition' for related series)

Rem: $\sum_{n \geq 1} \omega_n \lambda^n$ = Bernoulli convolution

Question about abs. cont. 1st appeared in '35, solved by Solomyak in '95

Ref. B.F. and A. Quas, *Statistical properties of invariant graphs in piecewise affine discontinuous forced systems*, Nonlinearity **24** (2011) 2477-2488