Synchronization phenomena and statistical mechanics

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- What does an isolated unit? Noisy or deterministic dynamics?
- Units need not being identical (probably, should not)

A first example and the general structure

Single unit dynamics

$$\dot{X} = F_a(X), \qquad\qquad X \in \mathbb{R}^d$$

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- For $|a| < 1$ a stable limit cycle appears
- For $|a| > 1$ close to 1: moderate excitations produce large pulses

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N interaction units: $j = 1, \ldots, N$

$$\dot{X}_j = F_{a_j}(X_j) + \text{interaction term}_j(X_1, \ldots, X_N) + \text{noise}_j$$

The interaction works in favor of synchrony among the units.

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From a mathematical standpoint, very limited understanding: but [Scheutzow 85-86] has to be mentioned

Phase reduction and Kuramoto models

Isolated unit dynamics

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with $\varphi \in \mathbb{S} := \mathbb{R}/2\pi\mathbb{Z}$ and $U : \mathbb{S} \to \mathbb{R}$ smooth.

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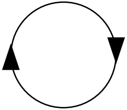
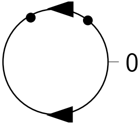
Case $|a| < 1$:  Case $|a| > 1$: 

Case $|a| > 1$: toy excitable unit.

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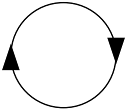
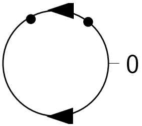
Case $|a| < 1$: 

Case $|a| > 1$: 

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- Stochastic isolated unit dynamics: $\mathrm{d}\varphi(t) = U(\varphi(t))\,\mathrm{d}t + \sigma\,\mathrm{d}w(t)$

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$$\mathrm{d}\varphi_j(t) \,=\, U_j(\varphi_j(t))\,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N}\sin\left(\varphi_j(t)-\varphi_i(t)\right)\,\mathrm{d}t + \sigma\,\mathrm{d}w_j(t)\,,$$

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Generalized prototypical examples:

- $U_j(\varphi) = \omega_j$ (Obs.: rotation invariance)
- $U_j(\varphi) = a_j \sin(\varphi) - 1$ (Obs.: no rotation invariance)

IID choice of $\{\omega_1, \omega_2, \ldots\}$ and $\{a_1, a_2, \ldots\}$

A statistical mechanics model?

If $\sigma > 0$, for every N

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Such a measure is reversible if (and only if)

$$U_j = V_j' \qquad \longleftarrow \quad \text{not an innocent condition!}$$

in particular for the case $U_j(\cdot) \equiv \omega_j$ the dynamics is *stochastically reversible* only if $\omega_j = 0$ for every j.

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In the reversible case the equilibrium measure is proportional to

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Of course there would be plenty to say about the case $\sigma = 0$ case!

Statistical mechanics: plane rotator model

Let us focus on the reversible and rotation invariant case

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- for $K > K_c = \sigma^2$: $\mu_{\infty,K}$ is a superposition of product measures

The empirical measure and the $N \to \infty$ limit

Useful tool: the empirical measure

$$\nu_{N,t}(\mathrm{d}\theta) = \frac{1}{N} \sum_{j=1}^{N} \delta_{\varphi_j(t)}(\mathrm{d}\theta)$$

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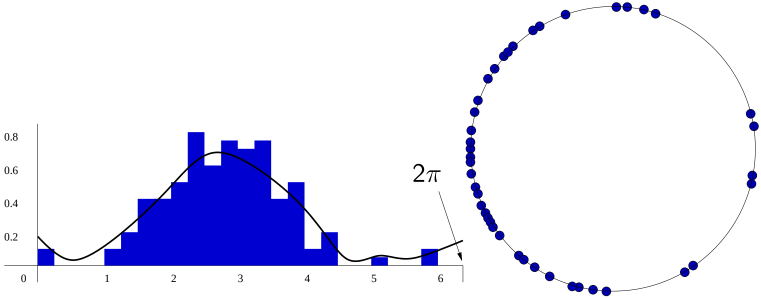

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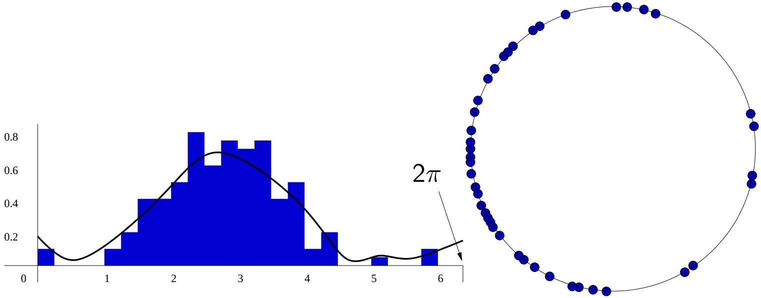

In fact, as $N \to \infty$, we have $\nu_{N,t}(\mathrm{d}\theta) \stackrel{N\to\infty}{\Longrightarrow} p_t(\theta)\,\mathrm{d}\theta$ where $p_t(\theta)$ solves a Fokker-Planck (McKean-Vlasov) PDE.

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- (FP) inherits the rotation symmetry: $p_t(\cdot + \psi)$ solves (FP) too

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From [Silver, Frankel, Ninham, Pearce, Kuramoto,. . .], [Bertini, G, Pakdaman 2010] and [G, Pakdaman, Pellegrin 2012]

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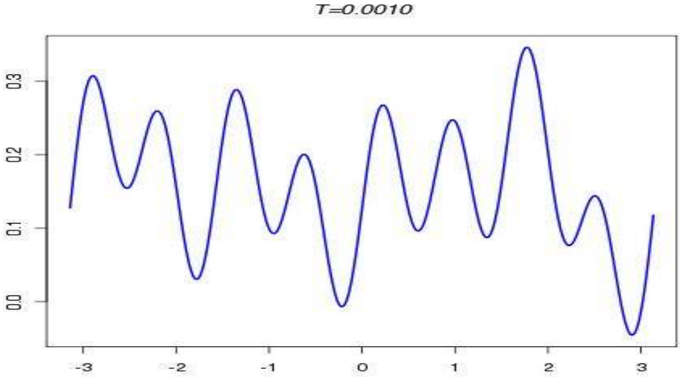
- We know all the stationary solutions and their stability properties (in a strong sense)
- The stationary solutions are ($\sigma = 1$)

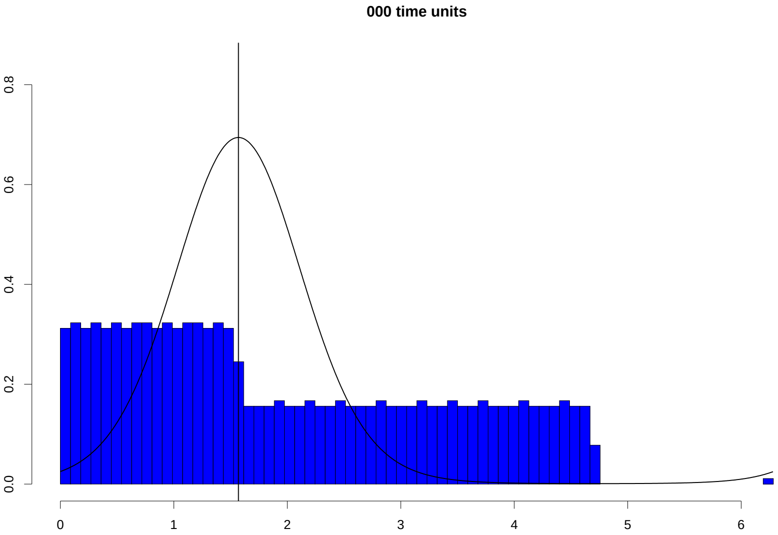
$$q_\psi(\theta) \propto \exp\left(c(K)\cos(\theta - \psi)\right)$$

with $c(K) \geq 0$ coming out of a fixed point problem

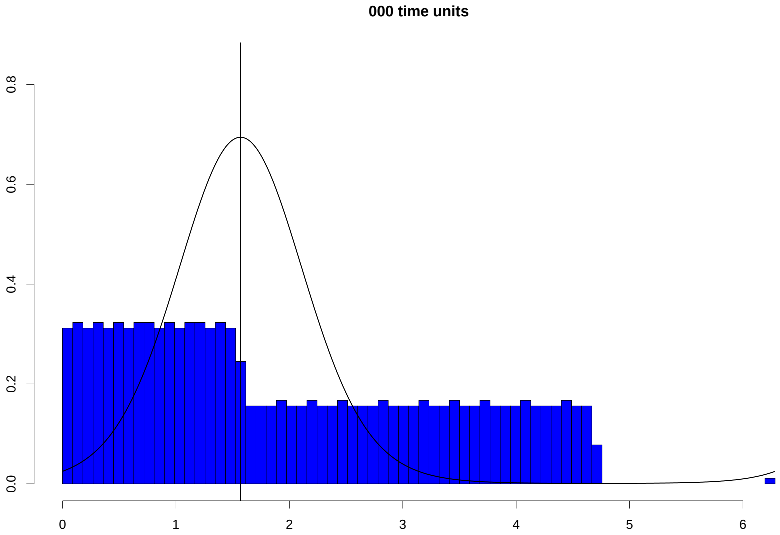
$N = \infty$ (Fokker-Planck), $K = 2$, $\sigma = 1$



$N = 1000$, $K = 2$, $\sigma = 1$



$N = 1000$, $K = 2$, $\sigma = 1$, but much faster



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- In the empirical measure at time 0 is close for N large to a density profile p_0 such that $\int_{\mathbb{S}} p_0(\theta) \exp(i\theta)\, \mathrm{d}\theta \neq 0$, in a finite (i.e. N independent) time the empirical measure reaches any given neighborhood of q_ψ, where ψ is determined by p_0

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- the empirical measure remains close to a q_ψ for much longer times (notably, N^c, any $c > 0$), though the center of synchronization ψ may change
- and in fact the center of synchronization, speeded up by a time factor N, converges as $N \to \infty$ to a Brownian motion on the circle (and we have a formula for the diffusion coefficient)

What about the non-equilibrium case?

$$\mathrm{d}\varphi_j(t) = U(\varphi_j(t))\,\mathrm{d}t - \frac{K}{N}\sum_{i=1}^{N} \sin\left(\varphi_j(t) - \varphi_i(t)\right)\,\mathrm{d}t + \sigma\,\mathrm{d}w_j(t)$$

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For $U \equiv 0$ we have observed:

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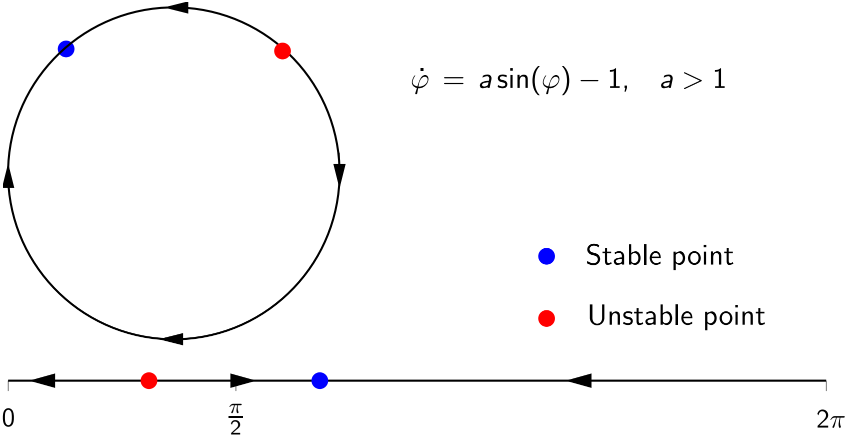
Let us now look at the non-equilibrium case $U(\varphi) = a\sin(\varphi) - 1$

$U(\theta) = (a\sin(\theta) - 1)/2$, $N = 4000$, $K = 2$, $a = 0.7$,$\sigma = 1$

Reminder for $U(\theta) = a\sin(\theta) - 1$

The single unit (no noise, no interaction) dynamics for $a > 1$ is:

$$\dot{\varphi} = a\sin(\varphi) - 1, \quad a > 1$$



$U(\theta) = (a\sin(\theta) - 1)/2$, $N = 4000$, $K = 2$, $a = 1.4$, $\sigma = 1$

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$U(\theta) = (a\sin(\theta) - 1)/2$, $K = 2$, $N = \infty$, $a = 1.1$, $\sigma = 1$

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We retain form the simulations:

1. emergence of synchronization (and of a natural synchronization center)
2. the synchronization center may have various dynamics, but the circle of stationary solutions of the $U \equiv 0$ case, M_0, has not completely disappeared

How can we get our hands on these phenomena?

The new Fokker-Planck PDE is now

$$\partial p_t(\theta) = \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta) - \partial_\theta \left[p_t(\theta)(J * p_t)(\theta)\right] - \delta\, \partial_\theta \left[p_t(\theta) U(\theta)\right]$$

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- It keeps the (local) stability property, but (in general) it is no longer a set of stationary solutions: the dynamics is locally attracted by M_δ and if we are on M_δ we stay on it (and we can still define a center of synchronization)
- We can get a quantitative characterization on M_δ and we can get sharp control on the dynamics on M_δ (notably: the dynamics of the center of synchronization) in the limit of small δ by perturbation technics [G., Pakdaman, Pellegrin, Poquet 2012], [G., Lucon, Poquet]

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- qualitatively this is not a serious restriction (if Kuramoto type models capture the essence of much more general cases)
- quantitatively this is a substantial restriction: *typical* biological models aren't so naturally linked to reversible cases