Synchronization in Ensembles of Oscillators: Theory of Collective Dynamics

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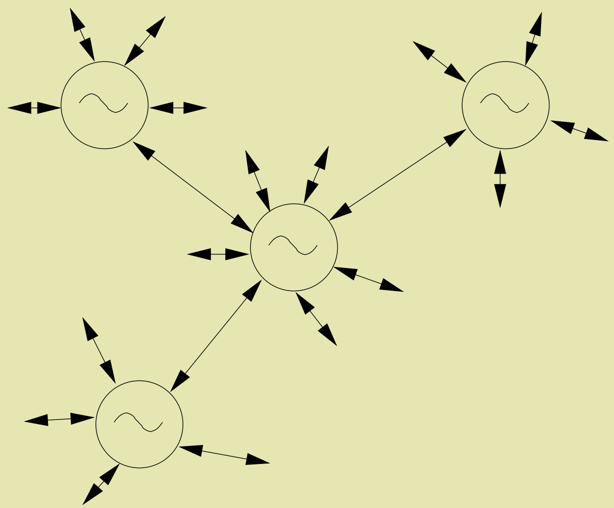
Content

- Synchronization in ensembles of coupled oscillators
- Watanabe-Strogatz theory, its relation to Ott-Antonsen equations and its generalization for hierarchical populations
- Partial synchronization due to nonlinear coupling
- Self-organizing chimera
- Populations with resonant and nonresonant coupling

Ensembles of globally (all-to-all) couples oscillators

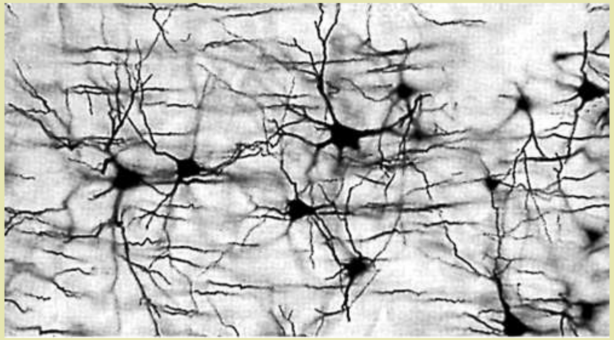
- Physics: arrays of Josephson junctions, multimode lasers,...
- Biology and neuroscience: cardiac pacemaker cells, population of fireflies, neuronal ensembles...
- Social behavior: applause in a large audience, pedestrians on a bridge,...



Main effect: Synchronization

Mutual coupling adjusts phases of indvidual systems, which start to keep pace with each other

Synchronization can be treated as a nonequilibrium phase transition!



Attempt of a general formulation

$$\dot{\vec{x}}_k = \vec{f}(\vec{x}_k, \vec{X}, \vec{Y})$$ individual oscillators (microscopic)

$$\vec{X} = \frac{1}{N}\sum_k \vec{g}(\vec{x}_k)$$ mean fields (generalizations possible)

$$\dot{\vec{Y}} = \vec{h}(\vec{X}, \vec{Y})$$ macroscopic global variables

Typical setup for a synchronization problem:

$\vec{x}_k(t)$ – periodic orchaotic oscillators

$\vec{X}(t), \vec{Y}(t)$ periodic or chaotic $\Rightarrow$ collective synchronous rhythm

$\vec{X}(t), \vec{Y}(t)$ stationary $\Rightarrow$ desynchronization

Description in terms of macroscopic variables

The goal is to describe the ensemble in terms of macroscopic variables $\vec{W}$, which characterize the distribution of $\vec{x}_k$,

$$\dot{\vec{W}} = \vec{q}(\vec{W}, \vec{Y}) \qquad \text{generalized mean fields}$$

$$\dot{\vec{Y}} = \vec{h}(\vec{X}(\vec{W}), \vec{Y}) \qquad \text{global variables}$$

as a possibly low-dimensional dynamical system

Below: how this program works for phase oscillators by virtue of Watanabe-Strogatz and Ott-Antonsen approaches

Kuramoto model: coupled phase oscillators

Phase oscillators ($\varphi_k \sim x_k$) with all-to-all pair-wise coupling

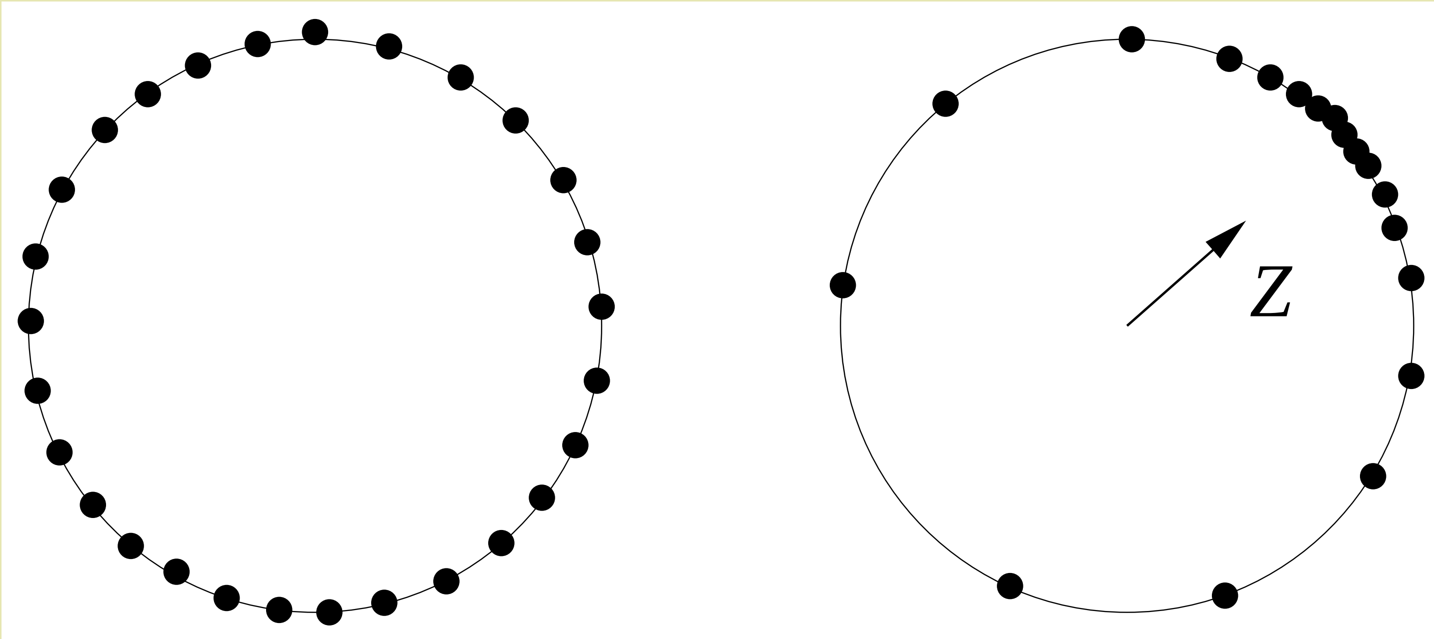
$$\begin{aligned}
\dot{\varphi}_k &= \omega_k + \varepsilon \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_k) \\
&= \varepsilon \left[\frac{1}{N} \sum_{j=1}^{N} \sin \varphi_j \right] \cos \varphi_k - \varepsilon \left[\frac{1}{N} \sum_{j=1}^{N} \cos \varphi_j \right] \sin \varphi_k \\
&= \omega_k + \varepsilon R(t) \sin(\Theta(t) - \varphi_k) = \omega_k + \varepsilon \mathrm{Im}(Z e^{-i\varphi_k})
\end{aligned}$$

System can be written as a mean-field coupling with the mean field (complex order parameter $Z \sim X$)

$$Z = R e^{i\Theta} = \frac{1}{N} \sum_k e^{i\varphi_k}$$

Synchronisation transition

$\varepsilon_c \sim$ width of distribution of frequecies $g(\omega) \sim$ “temperature”



small ε: no synchronization, phases are distributed uniformly, mean field vanishes $Z = 0$

large ε: synchronization, distribution of phases is non-uniform, finite mean field $Z \neq 0$

Watanabe-Strogatz (WS) ansatz

[S. Watanabe and S. H. Strogatz, PRL 70 (2391) 1993; Physica D 74 (197) 1994]

Ensemble of **identical** oscillators driven by the same complex field $H(t)$

$$\frac{d\varphi_k}{dt} = \omega(t) + \mathrm{Im}\left(He^{-i\varphi_k}\right) \qquad k = 1, \ldots, N$$

Möbius transformation to WS variables

$$\rho(t), \quad \Phi(t), \quad \Psi(t), \quad \psi_1 = \text{const}, \ \ldots \ \psi_N = \text{const}$$

$$e^{i\varphi_k} = e^{i\Phi(t)} \frac{\rho(t) + e^{i(\psi_k - \Psi(t))}}{\rho(t) e^{i(\psi_k - \Psi(t))} + 1}$$

yields WS equations

$$\frac{d\rho}{dt} = \frac{1-\rho^2}{2}\mathrm{Re}\left(He^{-i\Phi}\right) , \qquad \frac{d\Phi}{dt} = \omega + \frac{1+\rho^2}{2\rho}\mathrm{Im}\left(He^{-i\Phi}\right) ,$$

$$\frac{d\Psi}{dt} = \frac{1-\rho^2}{2\rho}\mathrm{Im}\left(He^{-i\Phi}\right) .$$

or in a complex form for $z = \rho e^{i\Phi}$, $\alpha = \Phi - \Psi$

$$\frac{dz}{dt} = i\omega z + \frac{1}{2}\left(H - z^2 H^*\right) \qquad \frac{d\alpha}{dt} = \omega + \mathrm{Im}\left(z^* H\right)$$

Why Möbius?

Phase equation $\dot{\varphi} = \omega(t) - i(He^{-i\varphi} - H^* e^{i\varphi})$

can be rewritten for $z = e^{i\varphi}$ as $\dot{z} = i\omega z + H - H^* z^2$

This Ricatti equations, for constant coefficients, has as solutions rational functions $z(t) = \frac{Az(0)+B}{Cz(0)+D}$

Combination of rational functions is rational

Even for non-constant coefficients the solution can be represented as a rational function with time-dependent parameters

cf.: Bicycle rear wheel governed by arbitrary trajectory of the front wheel, by M. Levi

Interpretation of WS variables

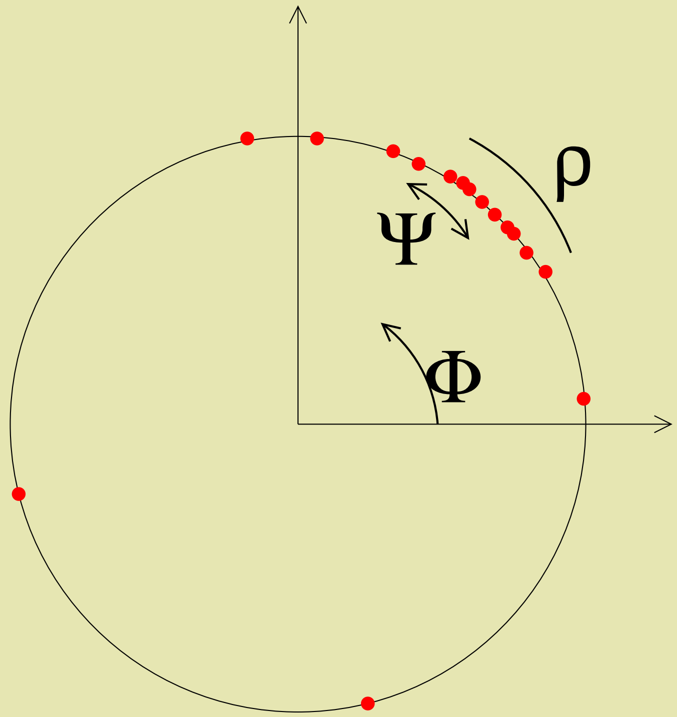
ρ measures the width of the bunch:

$\rho = 0$ if the mean field $Z = \sum_k e^{i\varphi_k}$ vanishes

$\rho = 1$ if the oscillators are fully synchronized and $|Z| = 1$

Φ is the phase of the bunch

Ψ measures positions of individual oscillators with respect to the bunch



Synchronization of uncoupled oscillators by external forces

Ensemble of **identical** oscillators driven by the same complex field $H(t)$

$$\frac{d\varphi_k}{dt} = \omega(t) + \text{Im}\left(H(t)e^{-i\varphi_k}\right) \qquad k = 1,\ldots,N$$

What happens to the WS variable ρ?

$\rho \to 1$: synchronization

$\rho \to 0$: desynchronization

Two basic examples oscillators and Jusephson junctions:

$$\dot{\varphi}_k = \omega - \sigma\xi(t)\sin\varphi_k \qquad \frac{\hbar}{2eR}\frac{d\varphi_k}{dt} + I_c\sin\varphi_k = I(t)$$

Hamiltonian reduction

$$\dot{\rho} = \frac{1-\rho^2}{2}\mathrm{Re}\left(H(t)e^{-i\Phi}\right) ,$$

$$\dot{\Phi} = \Omega(t) + \frac{1+\rho^2}{2\rho}\mathrm{Im}\left(H(t)e^{-i\Phi}\right) .$$

in variables

$$q = \frac{\rho\cos\Phi}{\sqrt{1-\rho^2}} , \qquad p = -\frac{\rho\sin\Phi}{\sqrt{1-\rho^2}} ,$$

reduces to a Hamiltonian system with Hamiltonian,

$$\mathcal{H}(J,\Phi,t) = \Omega(t)J - H(t)\frac{\sqrt{2J(2J+1)}}{2}\sin\Phi .$$

Action-angle variables

$$J = \frac{\rho^2}{2(1-\rho^2)}, \qquad \Phi$$

Hamiltonian reads

$$\mathcal{H}(J,\Phi,t) = \Omega(t)J - H(t)\frac{\sqrt{2J(2J+1)}}{2}\sin\Phi$$

Synchrony: $\mathcal{H}, J \to \infty$

Asynchrony: $\mathcal{H}, J \to 0$

For general noise: “Energy” grows $\Rightarrow$ synchronization by common noise

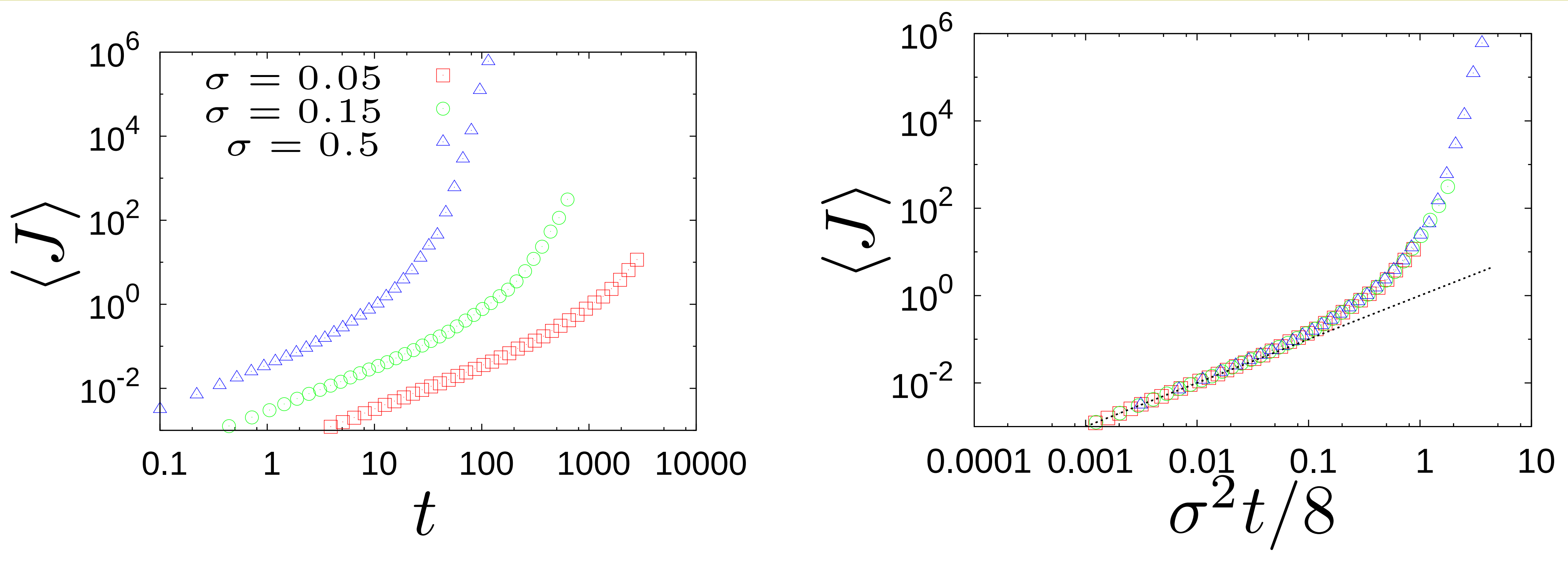
Analytic solution for the initial stage

Close to asynchrony: energy is small, equations can be linearized $\Rightarrow$ exact solution

$$\mathcal{H}, J \sim \sigma^2 t$$

Close to synchrony:

$$\mathcal{H}, J \sim \exp[\lambda t]$$


$\sigma = 0.05$
$\sigma = 0.15$
$\sigma = 0.5$
$\langle J \rangle$
t
0.1 1 10 100 1000 10000
10^6 10^4 10^2 10^0 10^{-2}
$\sigma^2 t/8$
0.0001 0.001 0.01 0.1 1 10

Globally coupled ensembles

Kuramoto model with equal frequencies

$$\dot{\varphi}_k = \omega + \varepsilon \mathrm{Im}(Z e^{-i\varphi_k})$$

belongs to the WS-class

$$\frac{d\varphi_k}{dt} = \omega(t) + \mathrm{Im}\left(H(t) e^{-i\varphi_k}\right) \qquad k = 1, \ldots, N$$

where H is the order parameter

$$Z = R e^{i\Theta} = \frac{1}{N} \sum_k e^{i\varphi_k}$$

Complex order parameters via WS variables

Complex order parameter can be represented via WS variables as

$$Z = \sum_k e^{i\varphi_k} = \rho e^{i\Phi}\gamma(\rho,\Psi) \qquad \gamma = 1 + (1-\rho^{-2}) \sum_{l=2}^{\infty} C_l (-\rho e^{-i\Psi})^l$$

where $C_l = N^{-1}\sum_k e^{il\psi_k}$ are Fourier harmonics of the distribution of constants ψ_k

Important simplifying case (adopted below):

Uniform distribution of constants ψ_k

$$C_l = 0 \quad \Rightarrow \quad \gamma = 1 \quad \Rightarrow \quad Z = \rho e^{i\Phi} = z$$

In this case WS variables yield the order parameter directly!

Closed equation for the order parameter for the Kuramoto-Sakaguchi model

Individual oscillators:

$$\dot{\varphi}_k = \omega + \varepsilon \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_k + \beta) = \omega + \varepsilon \mathrm{Im}(Z e^{i\beta} e^{-i\varphi_k})$$

Equation for the order parameter is just the WS equation:

$$\frac{dZ}{dt} = i\omega Z + \frac{\varepsilon}{2} e^{i\beta} Z - \frac{\varepsilon}{2} e^{-i\beta} |Z|^2 Z$$

Closed equation for the real order parameter $R = |Z|$:

$$\frac{dR}{dt} = \frac{\varepsilon}{2} R(1 - R^2) \cos\beta$$

Simple dynamics in the Kuramoto-Sakaguchi model

$$\frac{dR}{dt} = \frac{\varepsilon}{2} R(1 - R^2) \cos\beta$$

Attraction: $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ $\Longrightarrow$

Synchronization, all phases identical $\varphi_1 = \ldots = \varphi_N$, order parameter large $R = 1$

Repulsion: $-\pi < \beta < -\frac{\pi}{2}$ and $\frac{\pi}{2} < \beta < \pi$ $\Longrightarrow$

Asynchrony, phases distributed uniformely, order parameter vanishes $R = 0$

Linear vs nonlinear coupling I

- Synchronization of a periodic autonomous oscillator is a nonlinear phenomenon
- it occurs already for infinitely small forcing
- because the unperturbed system is singular (zero Lyapunov exponent)

In the Kuramoto model "linearity" with respect to forcing is assumed

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \varepsilon_1 \mathbf{f}_1(t) + \varepsilon_2 \mathbf{f}_2(t) + \cdots$$

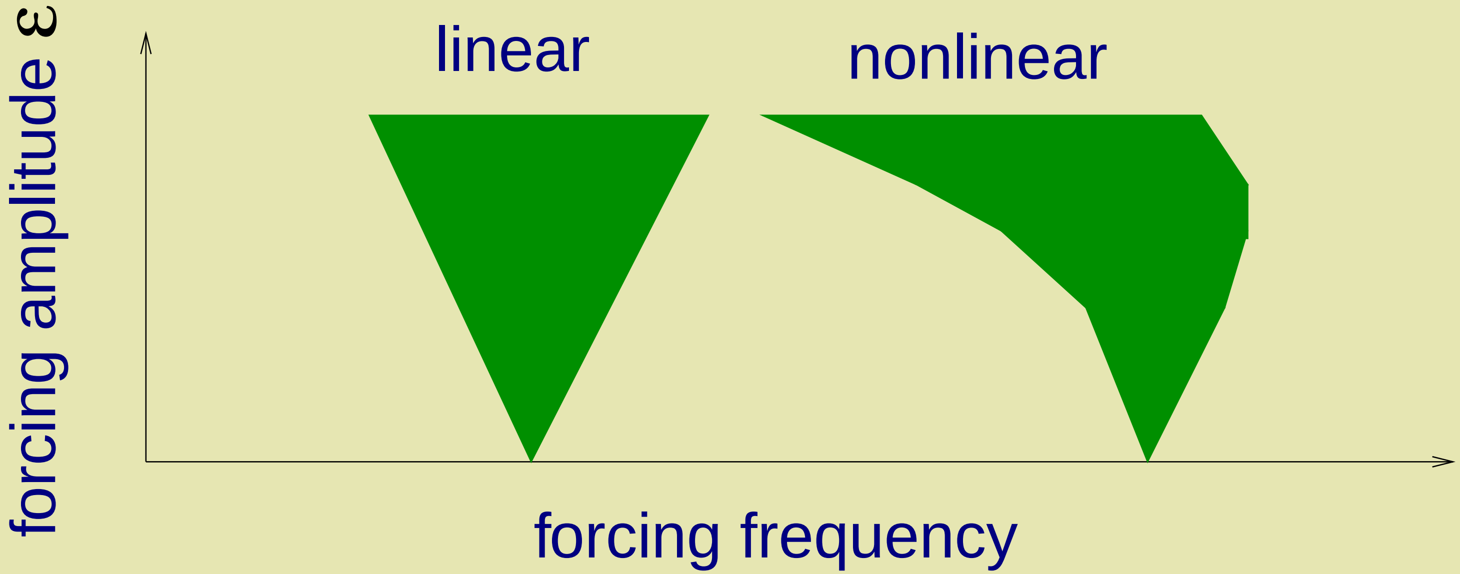
$$\dot{\varphi} = \omega + \varepsilon_1 q_1(\varphi, t) + \varepsilon_2 q_2(\varphi, t) + \cdots$$

Linear vs nonlinear coupling II

Strong forcing leads to “nonlinear” dependence on the forcing amplitude

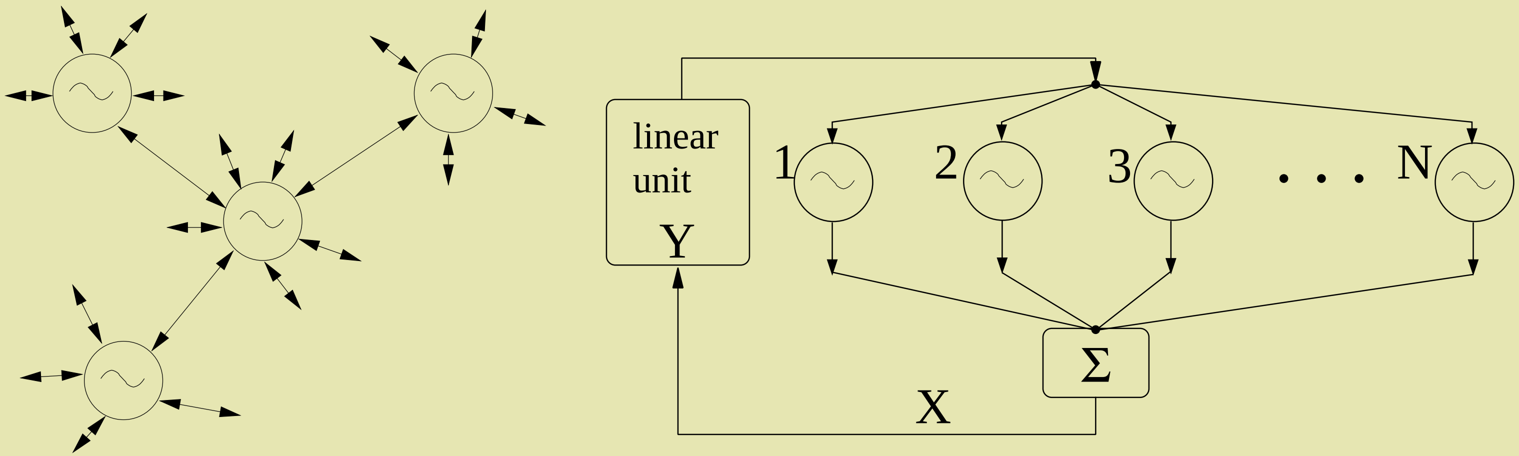
$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \varepsilon \mathbf{f}(t)$$
$$\dot{\varphi} = \omega + \varepsilon q^{(1)}(\varphi, t) + \varepsilon^2 q^{(2)}(\varphi, t) + \cdots$$

Nonlineraity of forcing manifests itself in the deformation/skeweness of the Arnold tongue and in the amplitude depnedence of the phase shift



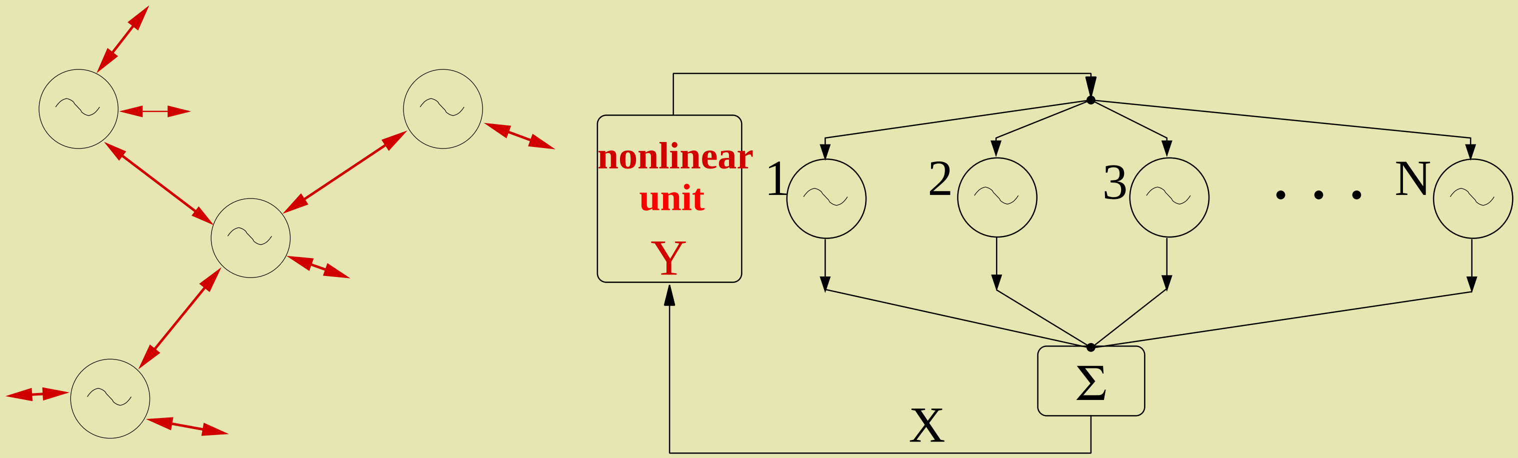
Linear vs nonlinear coupling III

Small each-to-each coupling $\Longleftrightarrow$ coupling via linear mean field



Strong each-to-each coupling $\Longleftrightarrow$ coupling via nonlinear mean field

[cf. Popovych, Hauptmann, Tass, Phys. Rev. Lett. 2005]



Nonlinear coupling: a minimal model

We take the standard Kuramoto-Sakaguchi model

$$\dot{\varphi}_k = \omega + \mathrm{Im}\left(He^{-i\varphi_k}\right) \qquad H \sim \varepsilon e^{-i\beta} Z \qquad Z = \frac{1}{N}\sum_j e^{i\varphi_j} = Re^{i\Theta}$$

and assume dependence of the acting force H on the “amplitude” of the mean field R:

$$\dot{\varphi}_k = \omega + A(\varepsilon R)\varepsilon R \sin(\Theta - \varphi_k + \beta(\varepsilon R))$$

E.g. attraction for small R vs repulsion for large R

WS equations for the nonlinearly coupled ensemble

$$\frac{dR}{dt} = \frac{1}{2}R(1-R^2)\varepsilon A(\varepsilon R)\cos\beta(\varepsilon R)$$

$$\frac{d\Phi}{dt} = \omega + \frac{1}{2}(1+R^2)\varepsilon A(\varepsilon R)\sin\beta(\varepsilon R)$$

$$\frac{d\Psi}{dt} = \frac{1}{2}(1-R^2)\varepsilon A(\varepsilon R)\sin\beta(\varepsilon R)$$

Full vs partial synchrony

All regimes follow from the equation for the order parameter

$$\frac{dR}{dt} = \frac{1}{2}R(1-R^2)\varepsilon A(\varepsilon R)\cos\beta(\varepsilon R)$$

Fully synchronous state: $R = 1$, $\dot{\Phi} = \omega + \varepsilon A(\varepsilon)\sin\beta(\varepsilon)$

Asynchronous state: $R = 0$

Partially synchronous bunch state

$0 < R < 1$ from the condition $A(\varepsilon R) = 0$:

No rotations, frequency of the mean field = frequency of the oscillations

Partially synchronized quasiperiodic state

$0 < R < 1$ from the condition $\cos\beta(\varepsilon R) = 0$:

Frequency of the mean field $\Omega = \dot{\varphi} = \omega \pm A(\varepsilon R)(1+R^2)/2$

Frequency of oscillators $\omega_{osc} = \omega \pm A(\varepsilon R)R^2$

Self-organized quasiperiodicity

- frequencies Ω and ω_{osc} depend on ε in a smooth way

  $\Longrightarrow$ generally we observe a quasiperiodicity

- attraction for small mean field vs repulsion for large mean field

  $\Longrightarrow$ ensemble is always at the stabilty border $\beta(\varepsilon R) = \pm\pi/2$, i.e. in a

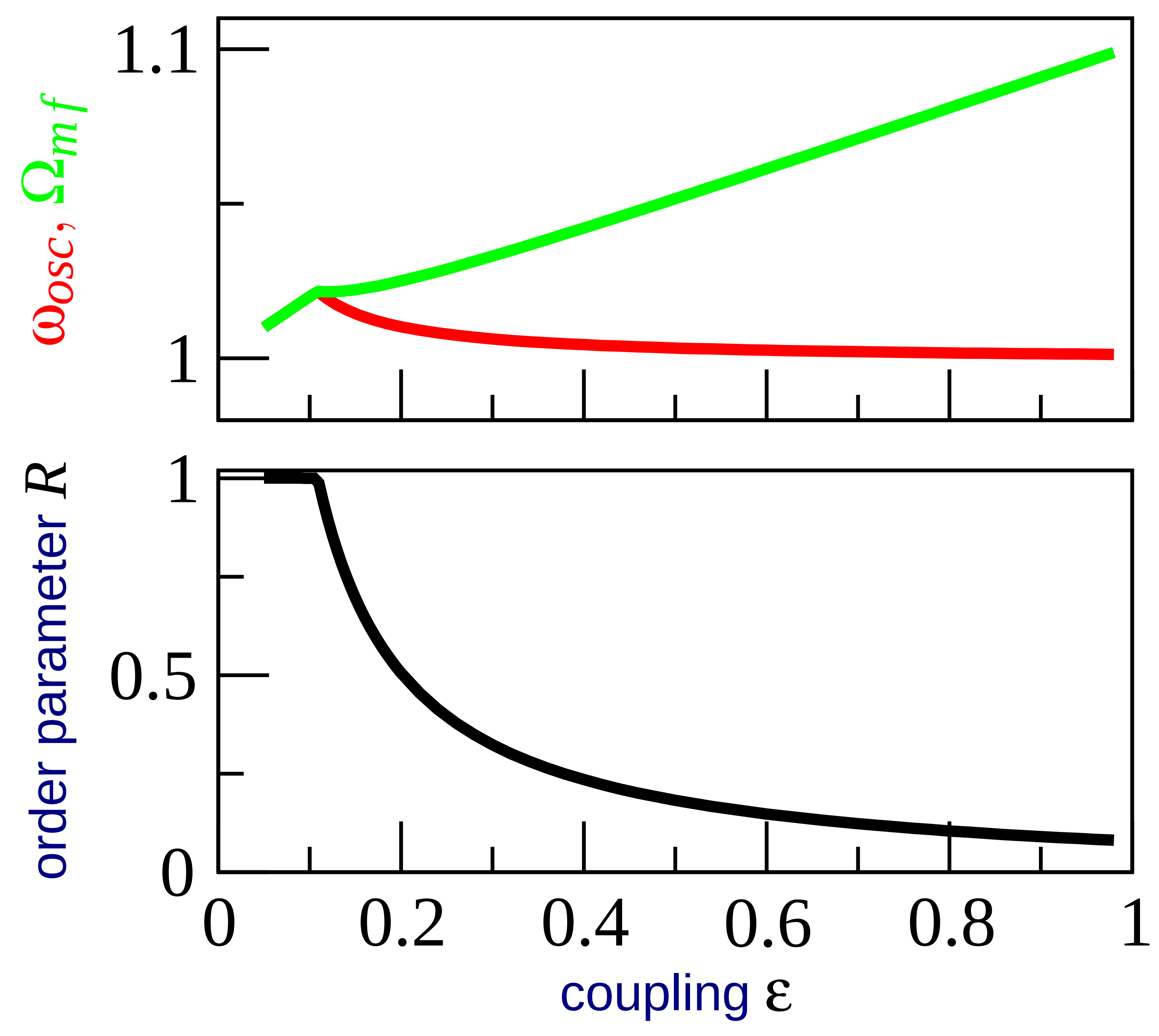
  self-organized critical state

- critical coupling for the transition from full to partial synchrony:

  $\beta(\varepsilon_q) = \pm\pi/2$

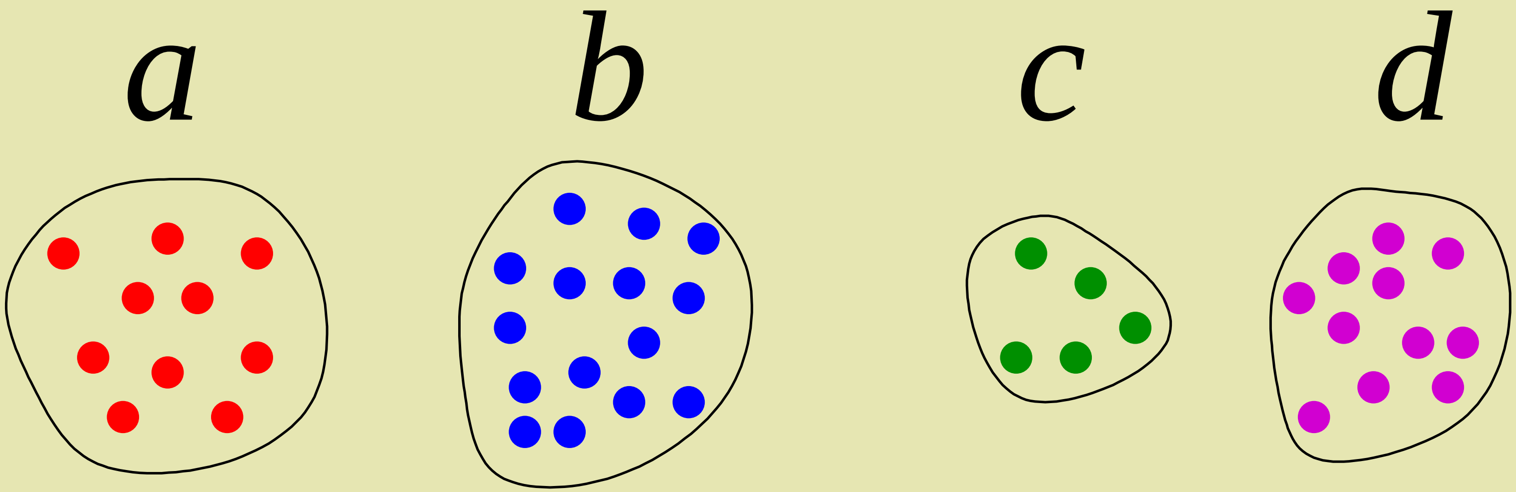
- transition at “zero temperature” like quantum phase transition

Simulation: loss of synchrony with increase of coupling



Hierarchically organized populations of oscillators

We consider populations consisting of M identical subgroups (of different sizes)



Each subgroup is described by WS equations

$\Rightarrow$ system of $3M$ equations completely describes the ensemble

$$\frac{d\rho_a}{dt} = \frac{1-\rho_a^2}{2}\mathrm{Re}\left(H_a e^{-i\Phi_a}\right) ,$$

$$\frac{d\Phi_a}{dt} = \omega_a + \frac{1+\rho_a^2}{2\rho_a}\mathrm{Im}\left(H_a e^{-i\Phi_a}\right) ,$$

$$\frac{d\Psi_a}{dt} = \frac{1-\rho_a^2}{2\rho_a}\mathrm{Im}\left(H_a e^{-i\Phi_a}\right) .$$

General force acting on subgroup a:

$$H_a = \sum_{b=1}^{M} n_b E_{a,b} Z_b + F_{ext,a}(t)$$

n_b: relative subgroup size

$E_{a,b}$: coupling between subgroups a and b

Thermodynamic limit

If the number of subgroups M is very large, one can consider a as a continuous parameter and get a system

$$\frac{\partial \rho(a,t)}{\partial t} = \frac{1-\rho^2}{2}\mathrm{Re}\left(H(a,t)e^{-i\Phi}\right)$$

$$\frac{\partial \Phi(a,t)}{\partial t} = \omega(a) + \frac{1+\rho^2}{2\rho}\mathrm{Im}\left(H(a,t)e^{-i\Phi}\right)$$

$$\frac{\partial \Psi(a,t)}{\partial t} = \frac{1-\rho^2}{2\rho}\mathrm{Im}\left(H(a,t)e^{-i\Phi}\right)$$

$$H(a,t) = F_{ext}(a,t) + \int db\, E(a,b)n(b)Z(b)$$

Relation to Ott-Antonsen equations

[E. Ott and T. M. Antonsen, CHAOS 18 (037113) 2008]

- in the case when subgroups differ only by frequency
- the coupling is global but nonlinear $E(\omega,\omega') = \varepsilon A(\varepsilon R)e^{i\beta(\varepsilon R)}$
- for a particular case when the complex order parameter for each subgroup is expressed via the WS variables as $Z(\omega) = \rho(\omega)e^{i\Phi(\omega)}$

we obtain Ott-Antonsen integral equations

$$\frac{\partial Z(\omega,t)}{\partial t} = i\omega Z + \frac{1}{2}H - \frac{Z^2}{2}H^*$$

$$H = \varepsilon A(\varepsilon R)e^{i\beta(\varepsilon R)}Y \qquad Y = Re^{i\Phi} = \int d\omega\, n(\omega)Z(\omega)$$

OA equations for Lorentzian distribution of frequencies

If

$$n(\omega) = \frac{\Delta}{\pi((\omega - \omega_0)^2 + \Delta^2)}$$

then the integral $Y = \int d\omega\, n(\omega) Z(\omega)$ can be calculated via residues as

$Y = Z(\omega_0 + i\Delta)$

This yields an ordinary differential equation for the order parameter Y

$$\frac{dY}{dt} = (i\omega_0 - \Delta)Y + \frac{1}{2}\varepsilon A(\varepsilon R)(e^{i\beta(\varepsilon R)} - e^{-i\beta(\varepsilon R)}|Y|^2)Y$$

Hopf normal form / Landau-Stuart equation/ Poincaré oscillator

$$\frac{dY}{dt} = (a + ib - (c + id)|Y|^2)Y$$

Nonidentical oscillators with nonlinear coupling

Lorentzian distribution of natural frequencies $n(\omega)$

$\Rightarrow$ standard “finite temperature” Kuramoto model of globally coupled oscillators with **nonlinear** coupling

(attraction for a small force, repulsion for a large force)

Novel effect: Multistability

Different partially synchronized states coexist for the same parameter range

Multistability of synchronous and asynchronous states in a Kuramoto model with nonlinear coupling

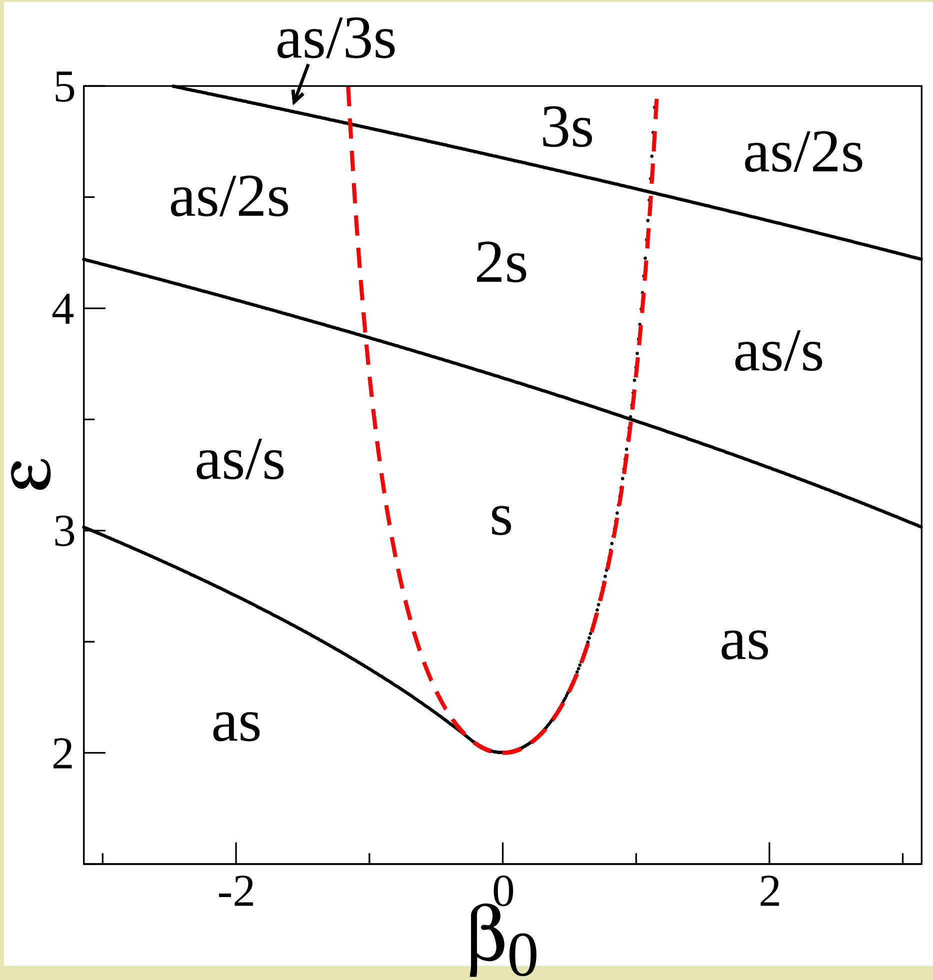
Nonlinear phase shift:

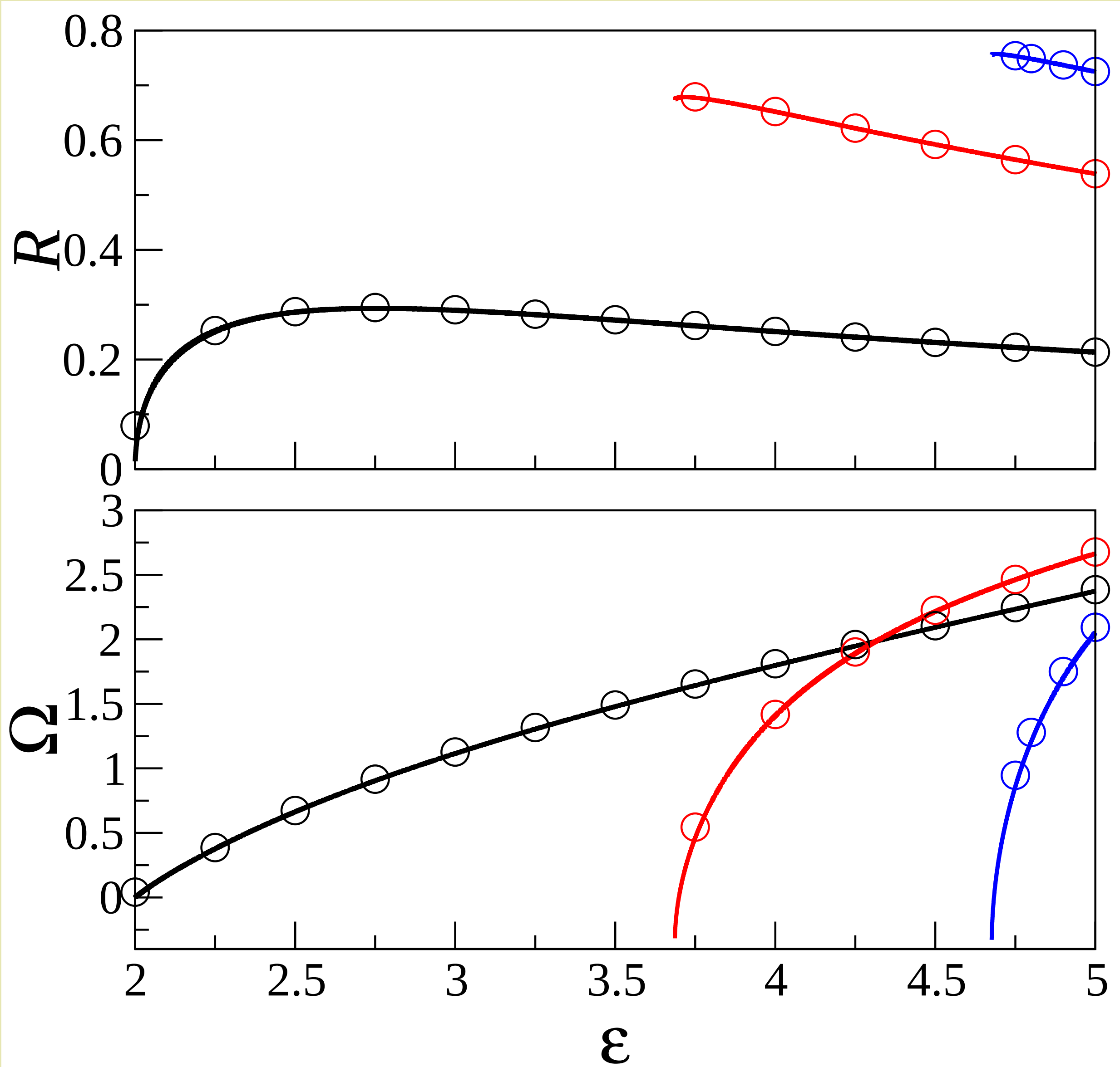
$$\beta = \beta_0 + \varepsilon^2 R^2$$

as: asynchronous

s: (partially) synchronous

ns: n coexisting synchronous states

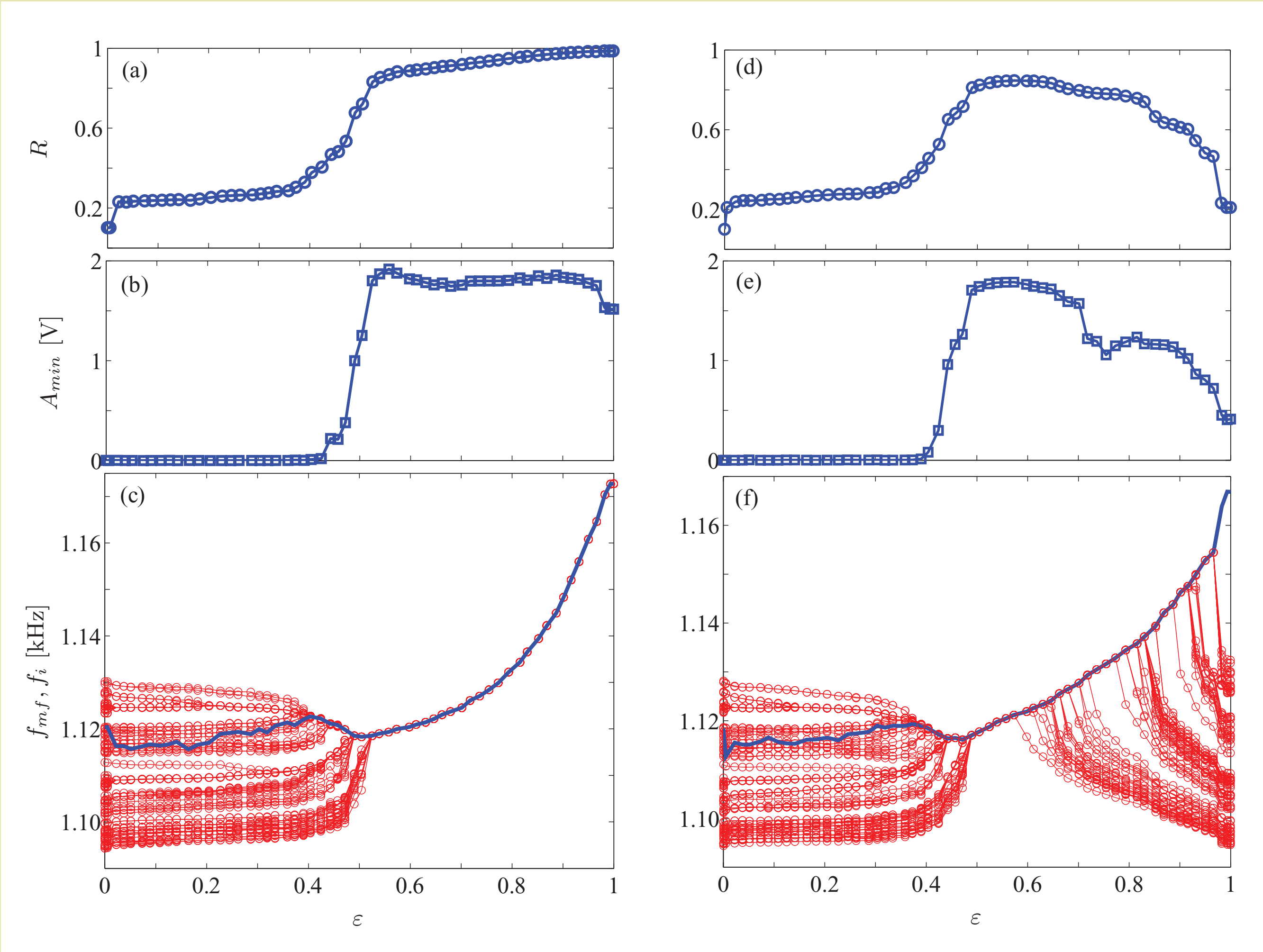



0.8
0.6
R
0.4
0.2
0
3
2.5
2
Ω
1.5
1
0.5
0
2
2.5
3
3.5
4
4.5
5
ε

Experiment

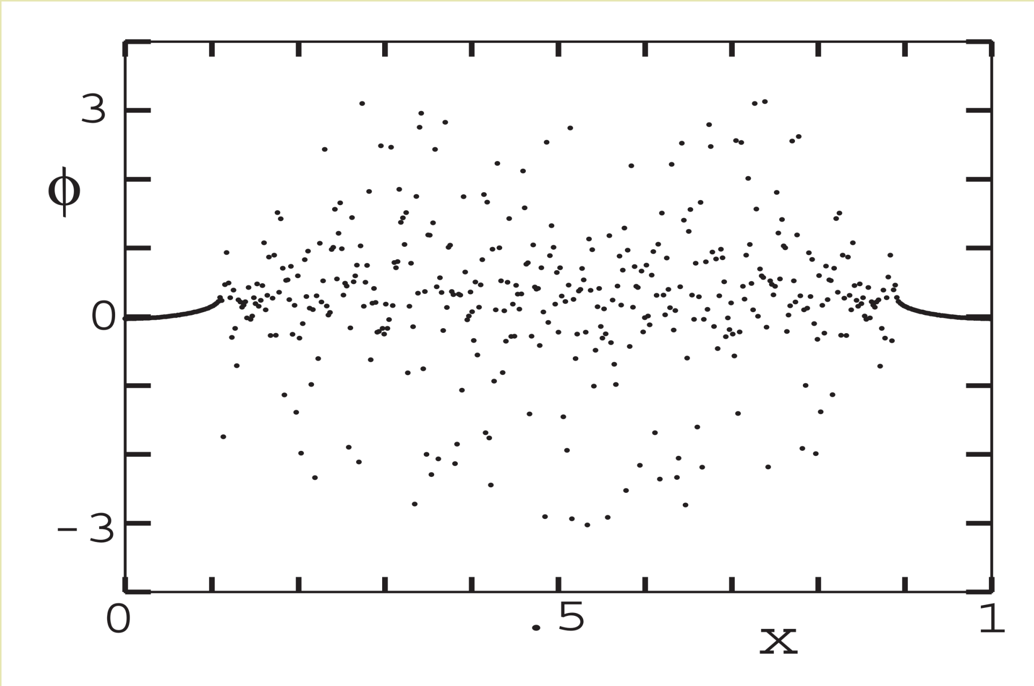
[Temirbayev et al, PRE, 2013]

Linear coupling　　Nonlinear coupling



Chimera states

Y. Kuramoto and D. Battogtokh observed in 2002 a symmetry breaking in non-locally coupled oscillators $H(x) = \int dx' \exp[x' - x] Z(x')$



This regime was called “chimera” by Abrams and Strogatz

Chimera in two subpopulations

Model by Abrams et al:

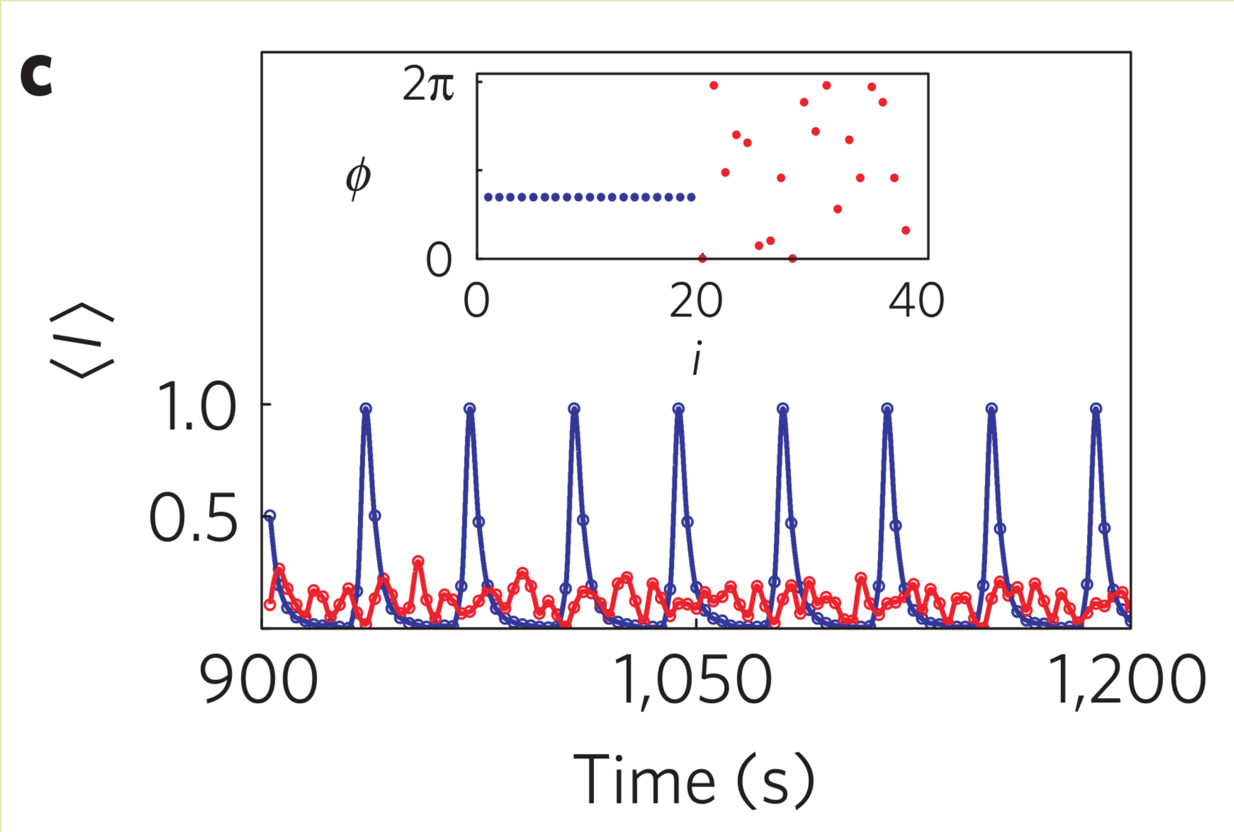
$$\dot{\varphi}_k^a = \omega + \mu \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j^a - \varphi_k^a + \alpha) + (1-\mu) \sum_{j=1}^{N} \sin(\varphi_j^b - \varphi_k^a + \alpha)$$

$$\dot{\varphi}_k^b = \omega + \mu \frac{1}{N} \sum_{j=1}^{N} \sin(\varphi_j^b - \varphi_k^b + \alpha) + (1-\mu) \sum_{j=1}^{N} \sin(\varphi_j^a - \varphi_k^b + \alpha)$$

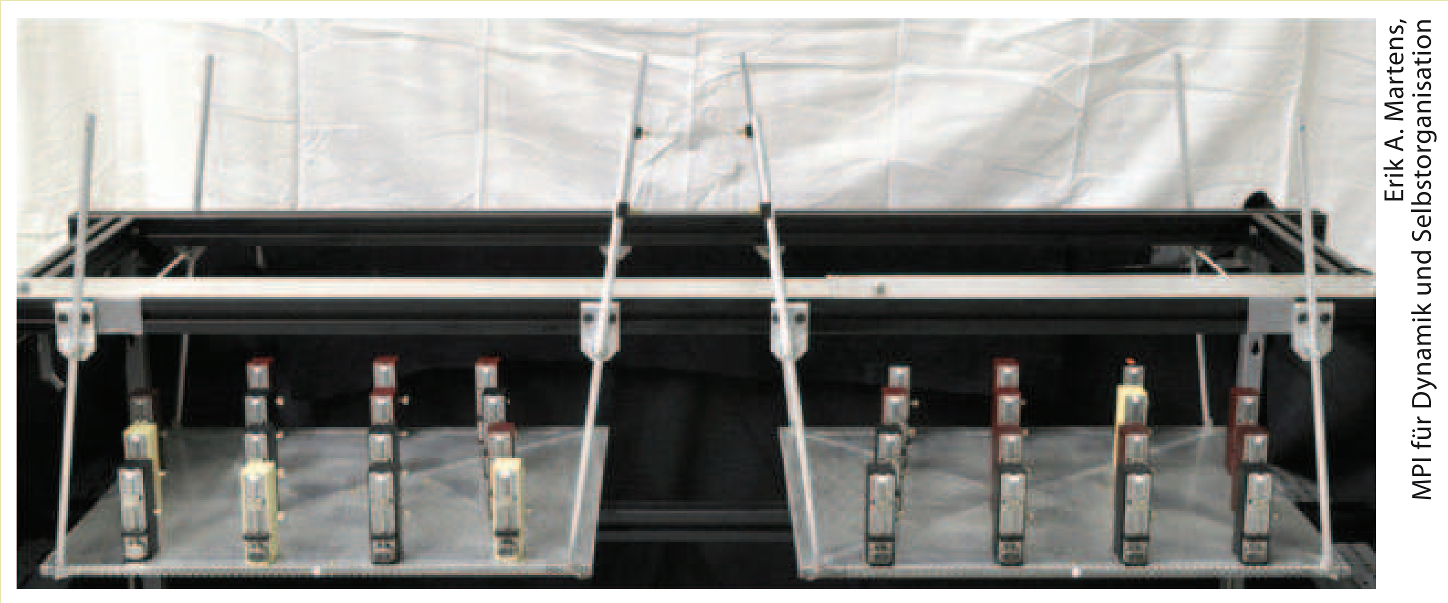
Two coupled sets of WS equations: $\rho^a = 1$ and $\rho^b(t)$ quasiperiodic are observed

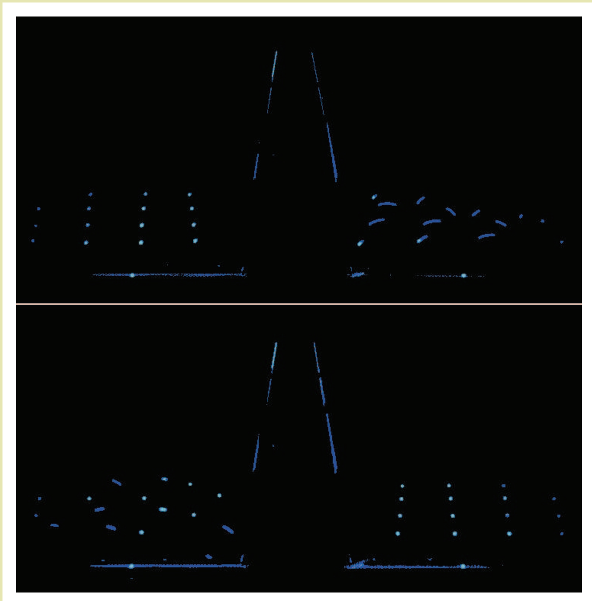
Chimera in experiments I

Tinsley et al: two populations of chemical oscillators



Chimera in experiments II





Self-emerging and turbulent chimera states at nonlinear coupling (with G. Bordyugov)

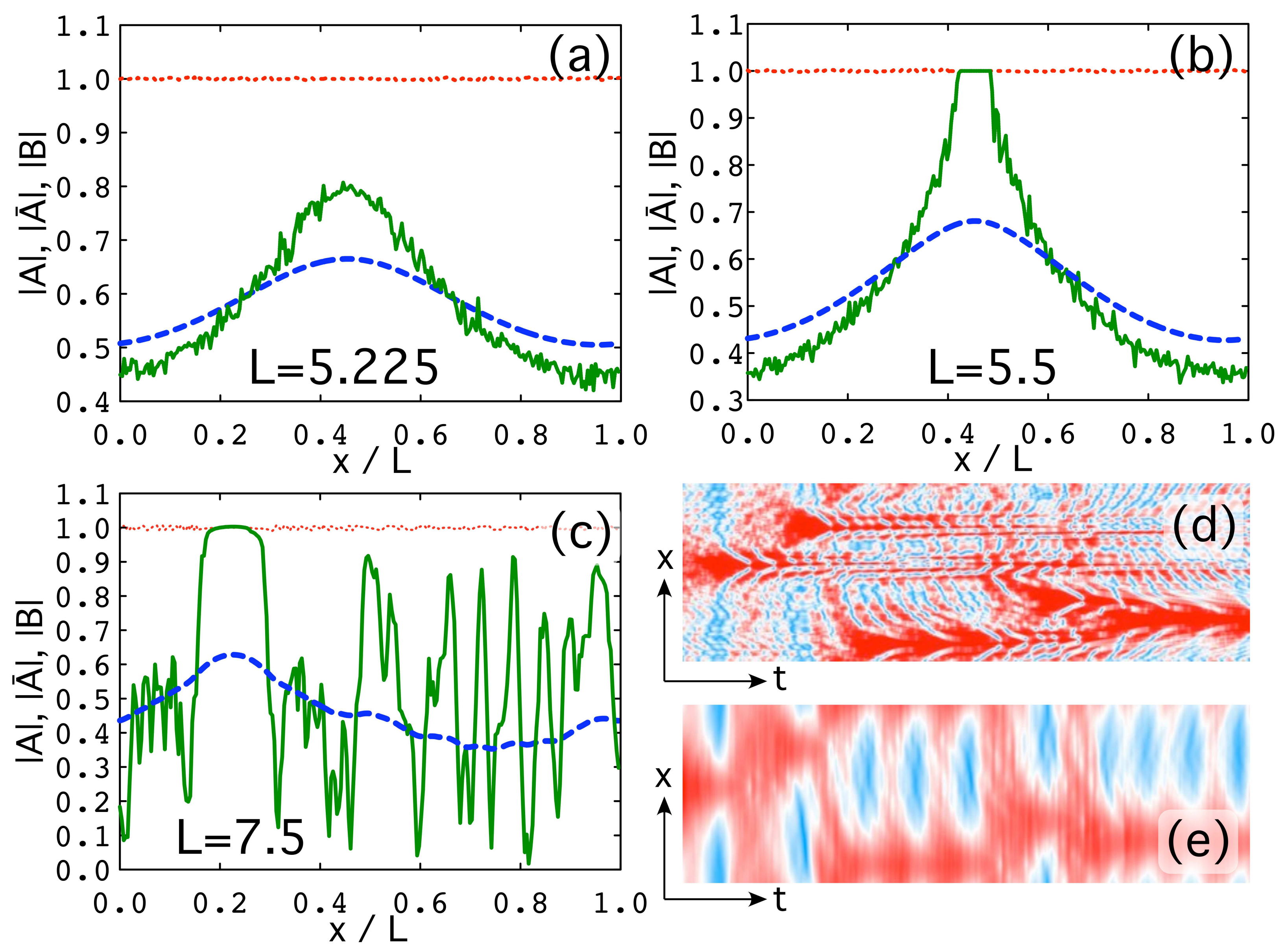
A one-dimensional array of oscillators with non-local coupling

$$\frac{\partial A}{\partial t} = A - |A|^2 + \varepsilon e^{i\beta_0 + i\beta_1 |B|^2} B$$

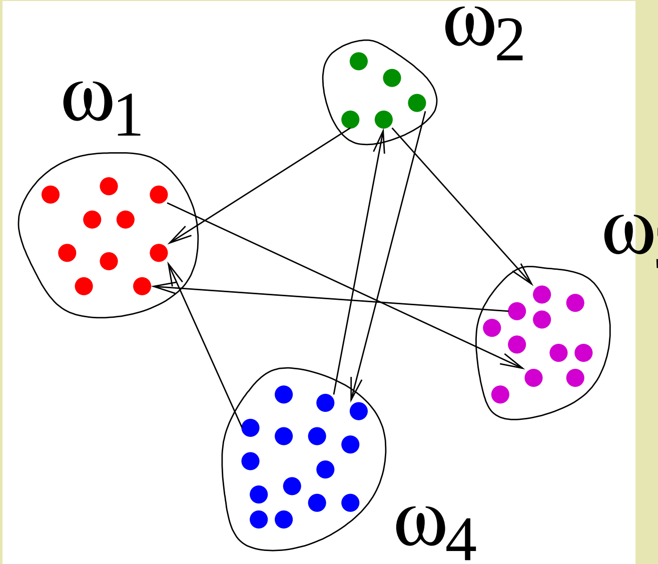
$$B(x,t) = \int_{-l}^{l} dx' \, G(x-x') A(x',t)$$

$\beta_1 = 0$ corresponds to linear coupling where chimera states (part of oscillator synchronous, part quasiperiodic) coexist with a stable synchronous state [Y. Kuramoto and D. Battogtokh, 2002].

Transition to static and turbulent chimera



Multifrequency populations



Each subpopulation is described by WS/OA equations for an effective ”oscillator”

Equivalent to many coupled oscillators

Amplitude = order parameter in the subpopulation

Phases = collective phases of mean fields

Multifrequency I: Non-resonantly interacting ensembles

[M. Komarov, A. P., Phys. Rev. E, v. 84, 016210 (2011)] Frequencies are different – all interactions are non-resonant

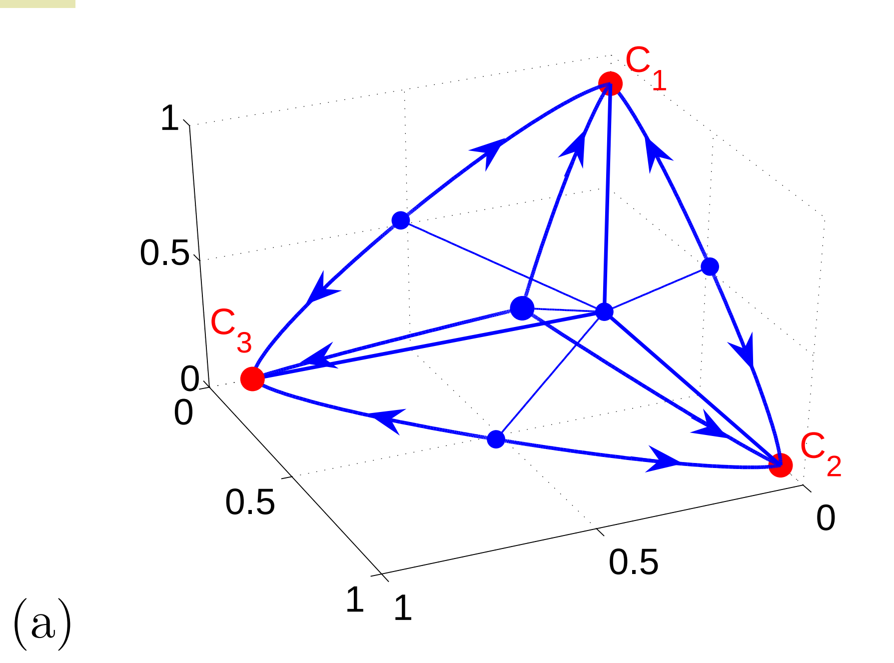
Only amplitudes of the order parameters can be involved in the coupling between subpopulations

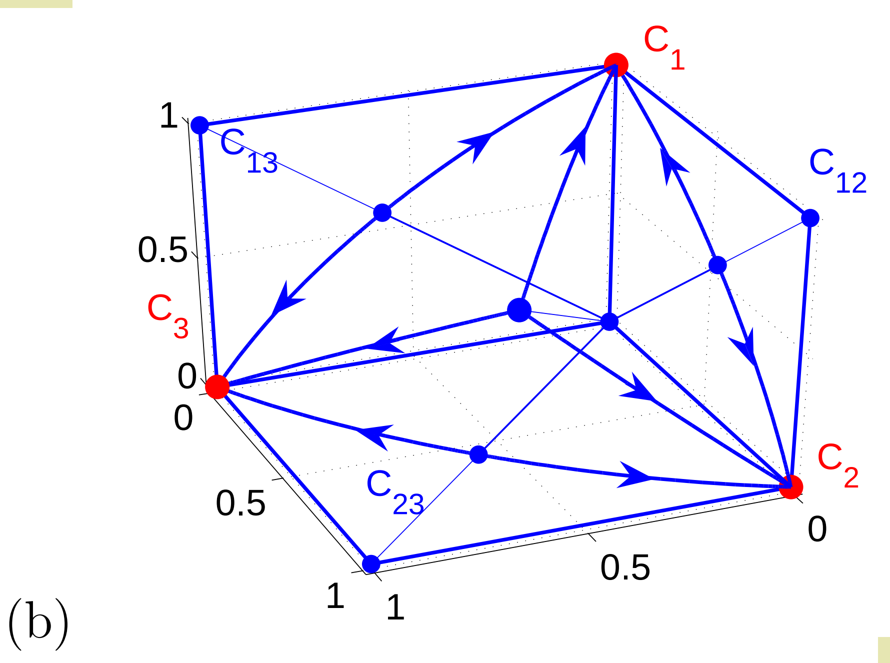
General equations are of type

$$\dot{\rho}_l = (-\Delta_l - \Gamma_{lm}\rho_m^2)\rho_l + (a_l + A_{lm}\rho_m^2)(1-\rho_l^2)\rho_l, \qquad l = 1,\ldots,L$$

where Γ_{lm} and A_{lm} decsribe the coupling

Competition for synchrony

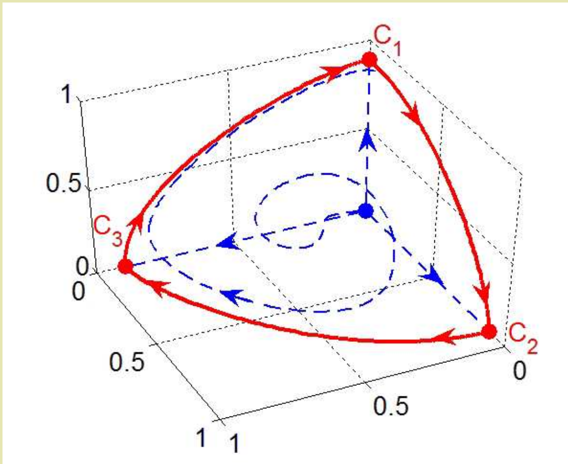


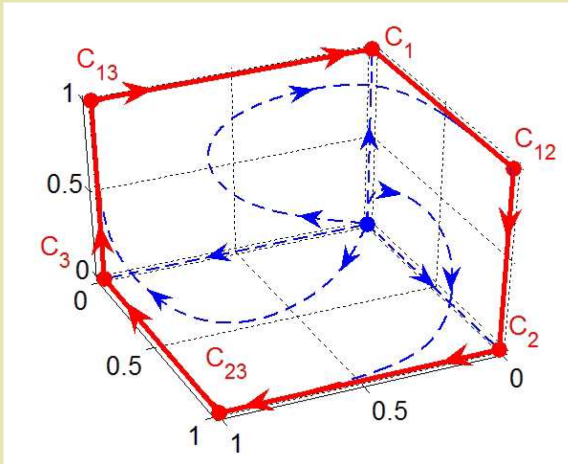


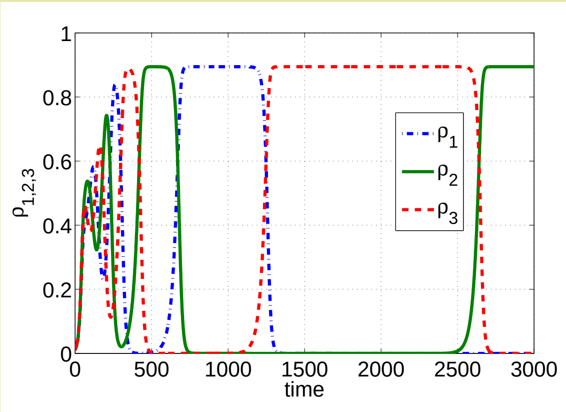
Only one ensemble is synchronous – depending on initial conditions

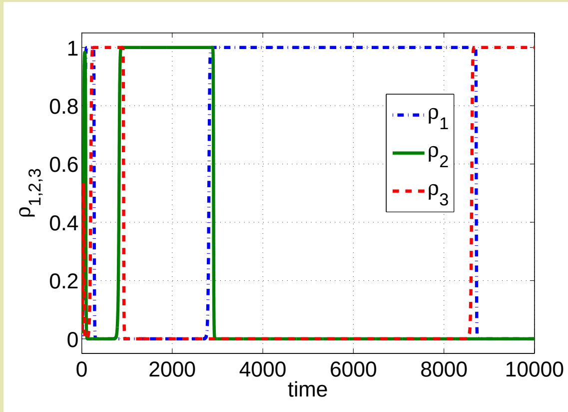
Heteroclinic synchrony cycles

Sequential synchrony (partial or full) in populations

(a)


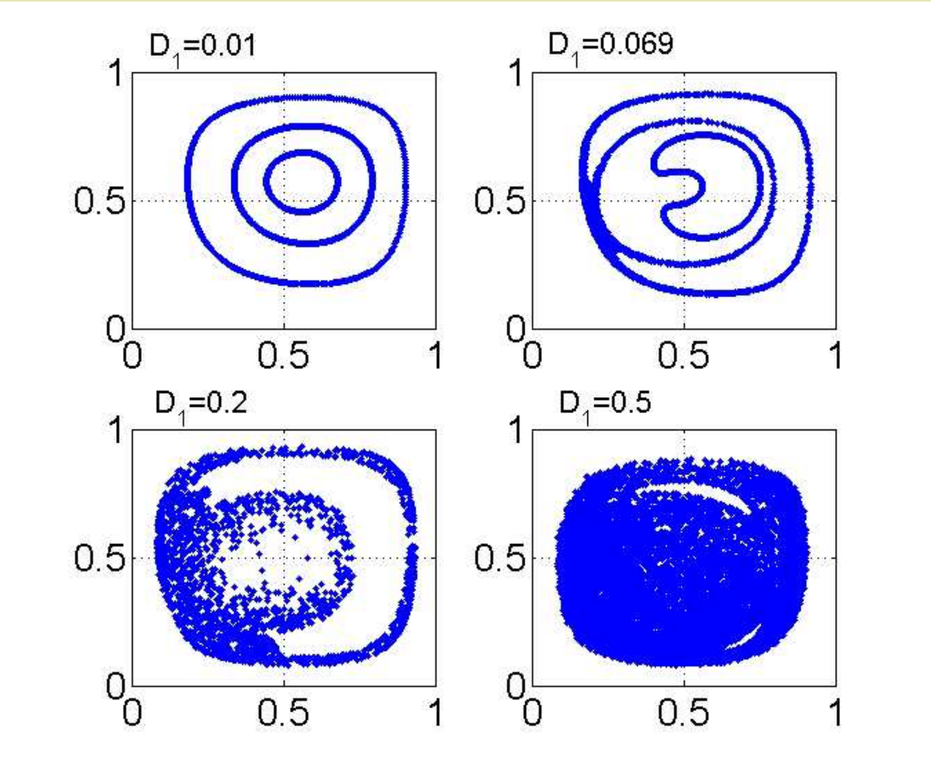
(b)


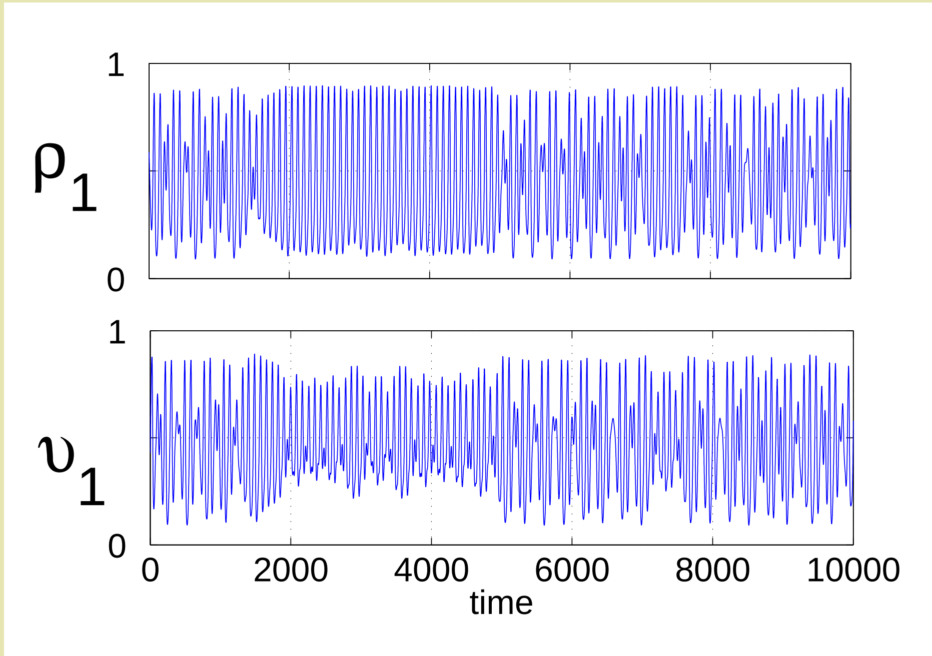
(c)


(d)


Chaotic synchrony cycles

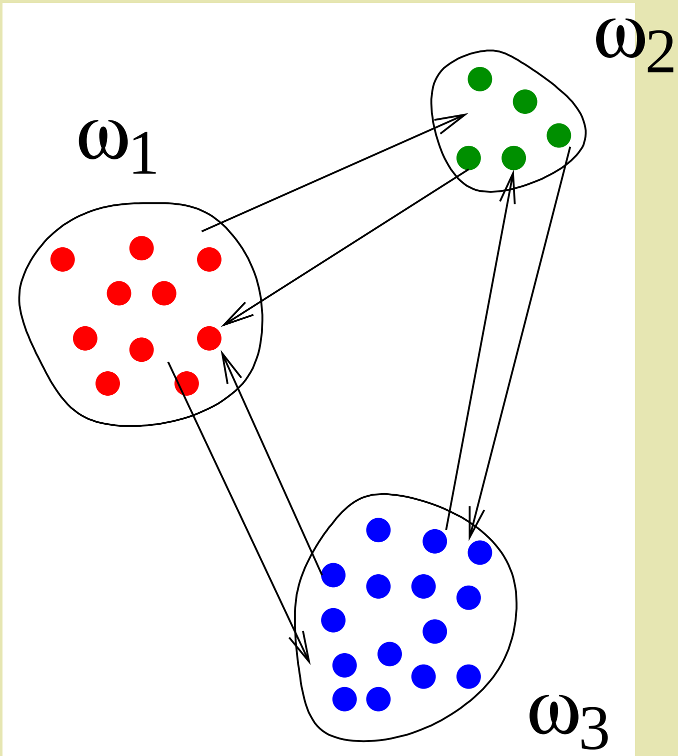
Order parameters demonstrate chaotic oscillations





Multifrequency II: Resonantly interacting ensembles

[M. Komarov and A. P., Phys. Rev. Lett. 110, 134101 (2013)]



Most elementary nontrivial resonance $\omega_1 + \omega_2 = \omega_3$

On the level of individual oscillators (phase ϕ from ω_1, phase ψ from ω_2, phase θ from $\omega_3 = \omega_2 + \omega_1$) one has to take into account **triple interactions**:

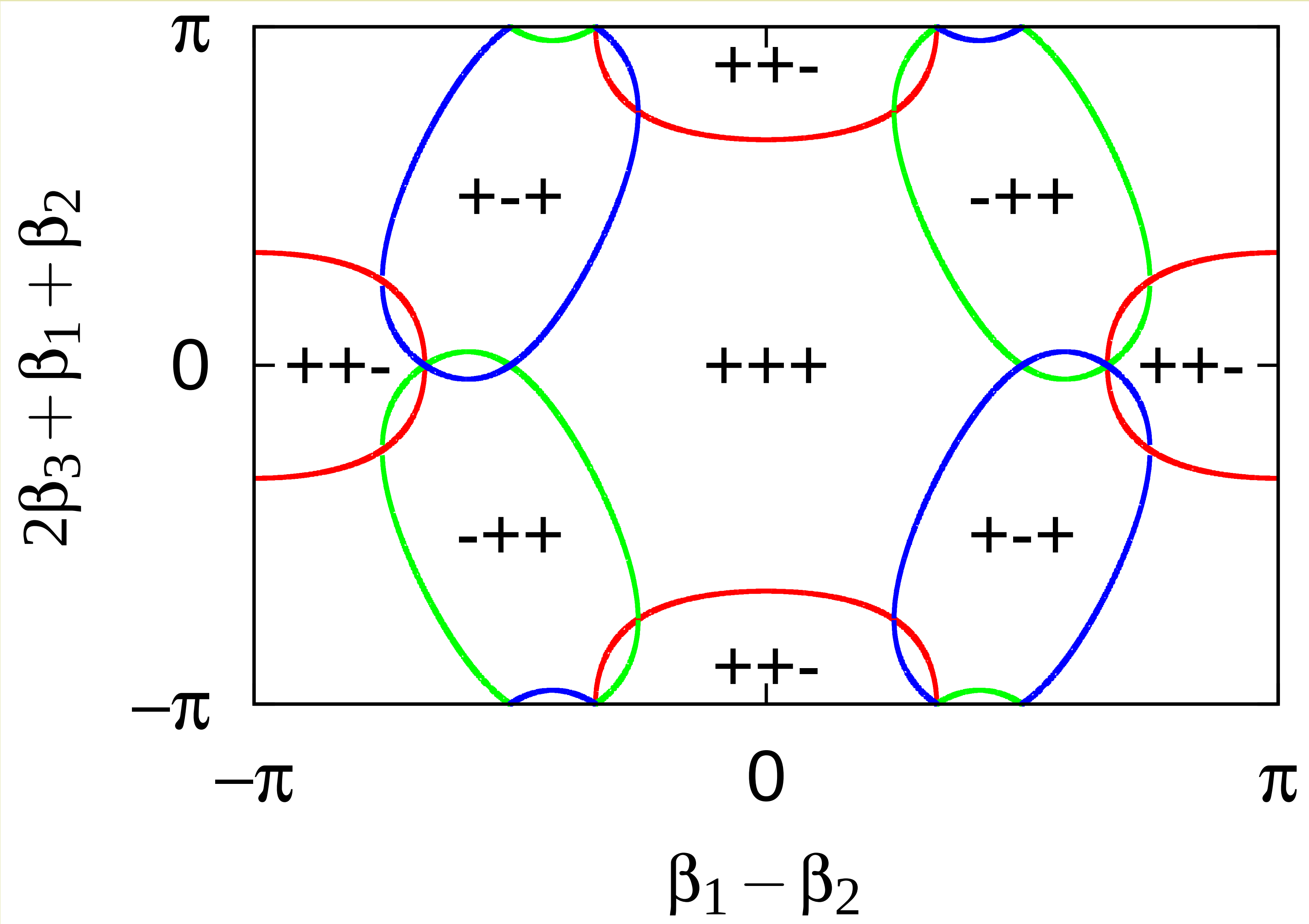
$$\dot{\phi}_k = \ldots + \Gamma_1 \sum_{m,l} \sin(\theta_m - \psi_l - \phi_k + \beta_1)$$
$$\dot{\psi}_k = \ldots + \Gamma_2 \sum_{m,l} \sin(\theta_m - \phi_l - \psi_k + \beta_2)$$
$$\dot{\theta}_k = \ldots + \Gamma_3 \sum_{m,l} \sin(\phi_m + \psi_l - \theta_k + \beta_3)$$

Set of three OA equations

On the level of effective oscillators describing order parameters, one has a triplet of Stuart-Landau equations with resonant coupling terms

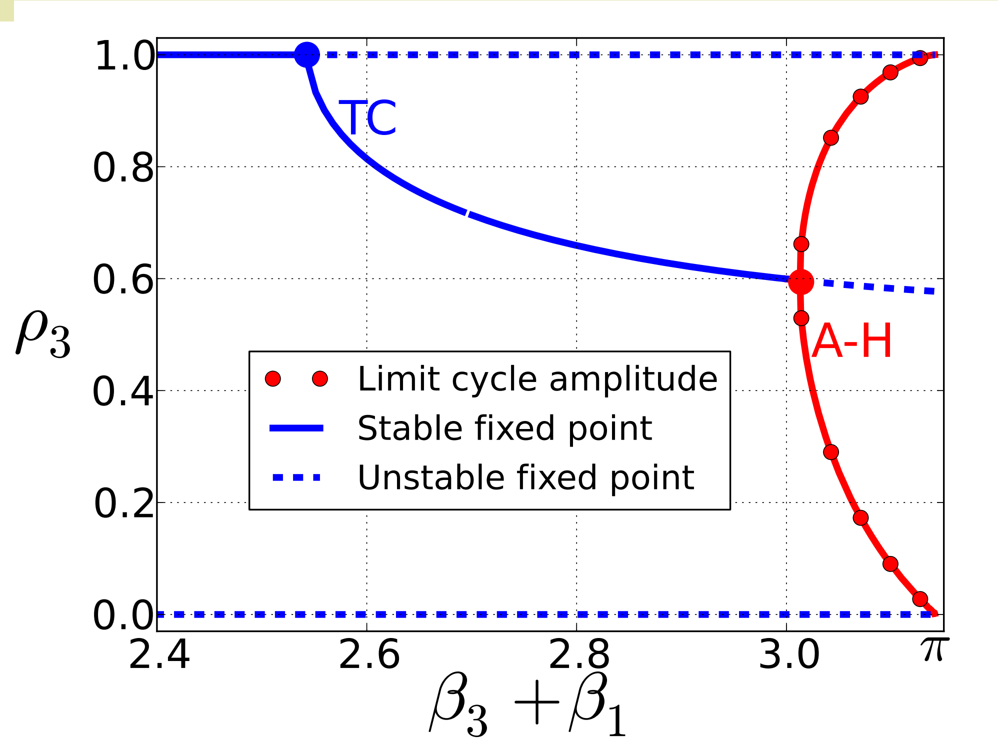
$$\dot{z}_1 = z_1(i\omega_1 - \delta_1) + (\varepsilon_1 z_1 + \gamma_1 z_2^* z_3 - z_1^2(\varepsilon_1^* z_1^* + \gamma_1^* z_2 z_3^*))$$
$$\dot{z}_2 = z_2(i\omega_2 - \delta_2) + (\varepsilon_2 z_2 + \gamma_2 z_1^* z_3 - z_2^2(\varepsilon_2^* z_2^* + \gamma_2^* z_1 z_3^*))$$
$$\dot{z}_3 = z_3(i\omega_3 - \delta_3) + (\varepsilon_3 z_3 + \gamma_3 z_1 z_2 - z_3^2(\varepsilon_3^* z_3^* + \gamma_3^* z_1^* z_2^*))$$

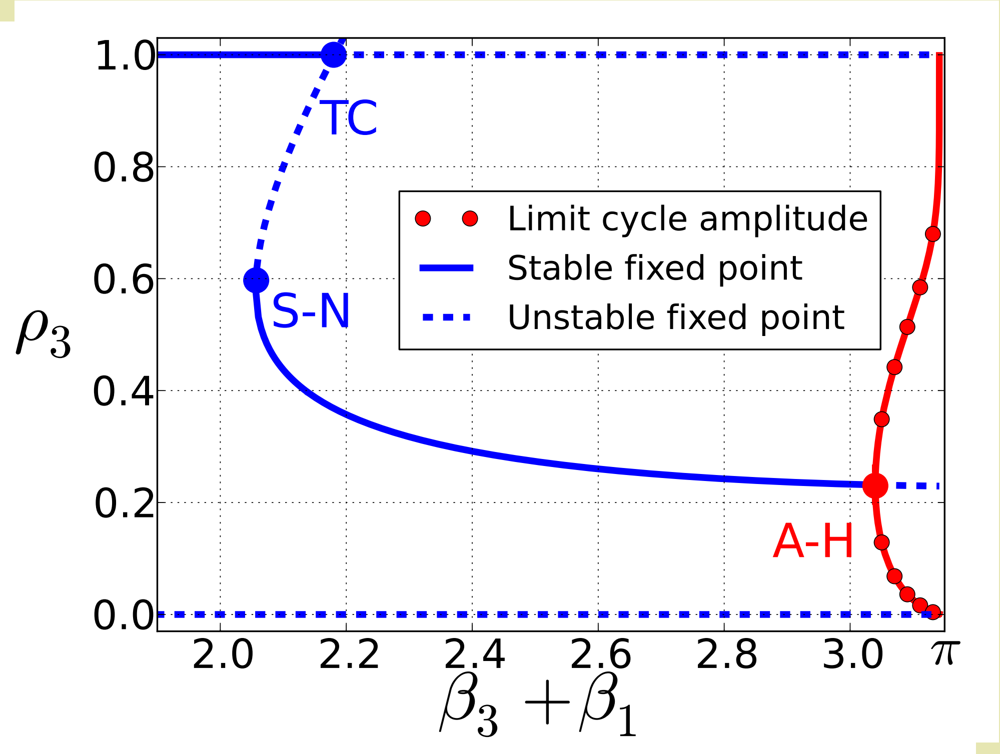
Regions of synchronizing and desynchronizing effect from triple coupling



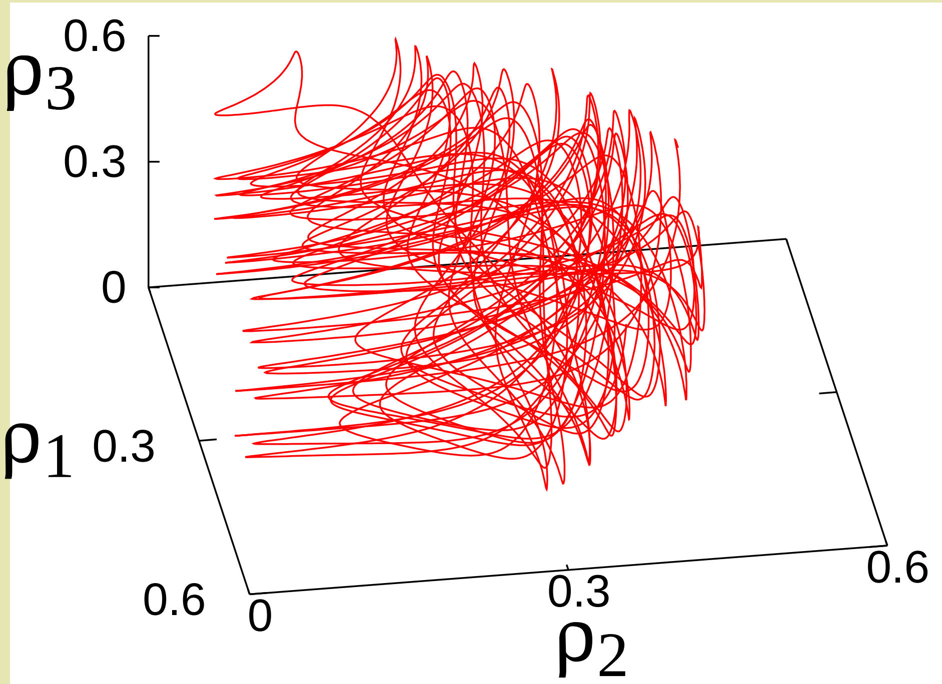
Bifurcations in dependence on phase constants

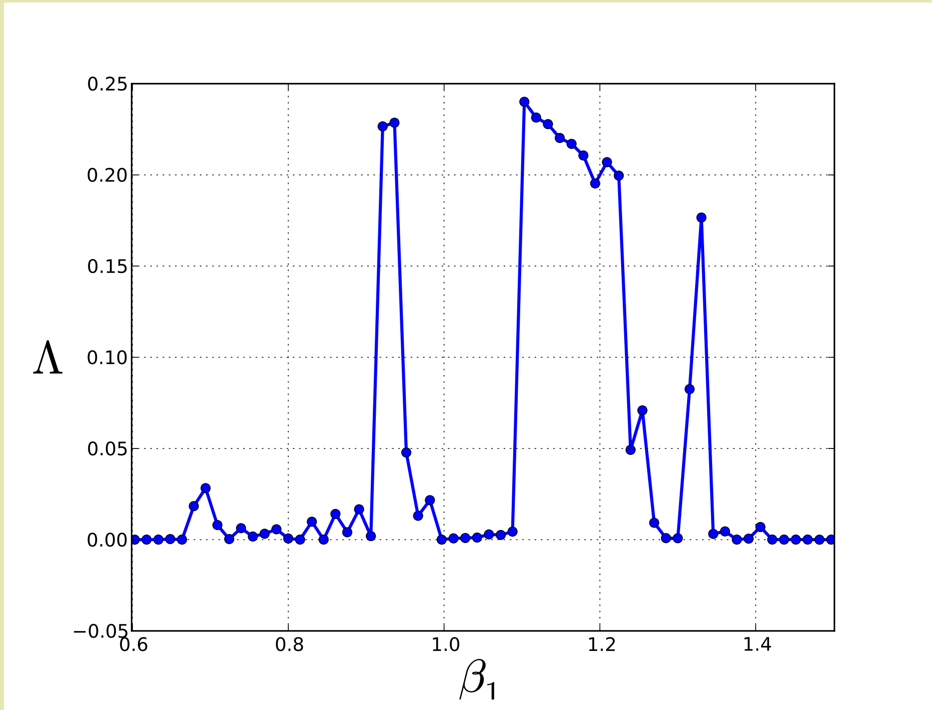
Different transitions from full to partial to oscillating synchrony





Chaos of order parameters





Conclusions

- Closed system for order parameters evolution (WS variables for identical, OA for non-identical)
- Nonlinear coupling I: “nonequilibrium quantum phase transition” from full to partial synchrony
- Nonlinear coupling II: self-organized chimera
- Multifrequency populations can be described in terms of order parameters as ”coupled oscillators”
- Resonantly and nonresonantly interacting populations – quasiperiodic partial synchrony, competition for synchrony and synchronization death, heteroclinic synchrony cycles, chaotic synchrony dinamics