Random tilings and surfaces

An elementary combinatorial introduction

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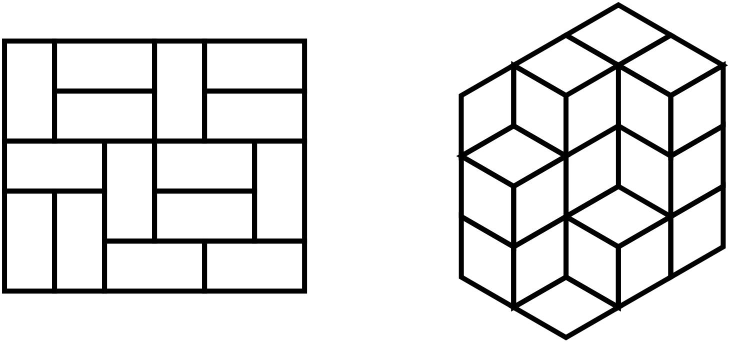
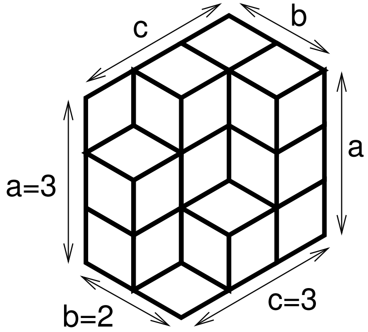
Introduction



Consider tilings of a fixed finite region made of dominos (2×1 rectangles) or lozenges (60° unit rhombi). How many such tilings exist?

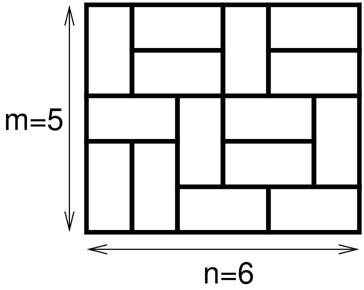
MacMahon formula (1915)



The number of lozenge tilings of a $a \times b \times c$ semiregular hexagon is

$$\mathcal{M}(a,b,c) = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}.$$

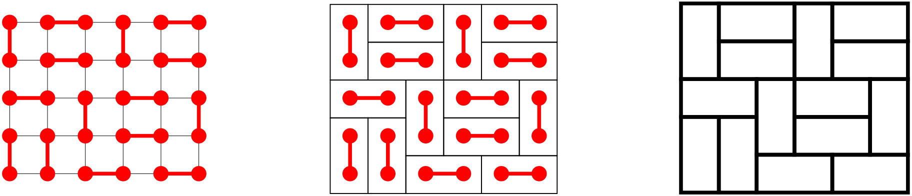
Fisher-Kasteleyn-Temperley formula (1961)



The number of domino tilings of a $m \times n$ rectangle (with a corner removed if mn odd) is

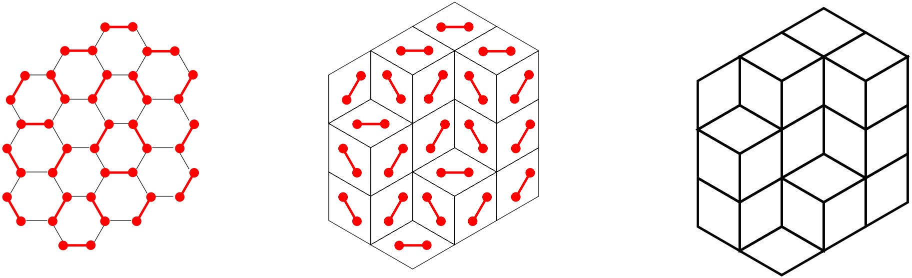
$$\mathcal{D}(m,n) = 4^{\lfloor m/2 \rfloor \lfloor n/2 \rfloor} \prod_{k=1}^{\lfloor m/2 \rfloor} \prod_{\ell=1}^{\lfloor n/2 \rfloor} \left(\cos^2 \frac{k\pi}{m+1} + \cos^2 \frac{\ell\pi}{n+1} \right).$$

Dimer model



By duality, domino tilings correspond to dimer configurations (perfect matchings) of a portion of the square lattice.

Dimer model



By duality, lozenge tilings correspond to dimer configurations of a portion of the honeycomb lattice.

Dimer model

Let $G = (V, E)$ be a finite graph. A dimer configuration (or perfect matching, 1-factor) is a subset E' of the set E of edges such that every vertex is incident to exactly one edge in E'.

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and the corresponding Boltzmann measure over dimer configurations.

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Kasteleyn gave a general method to compute Z for planar graphs:

$$Z = \begin{cases} |\,\mathrm{Pf}\, K\,| & \text{(general case)} \\ |\det K\,| & \text{(bipartite case)} \end{cases}$$

for a suitable "Kasteleyn" matrix K (which is a variant of the adjacency matrix of G). Entries of K^{-1} yield local dimer statistics.

Sketch of the proof of the FKT formula

Label the vertices of the rectangular grid by their complex coordinates:

$$V = \{0, \ldots, m-1\} + i\{0, \ldots, n-1\} = B \cup W$$

where B and W are respectively the even and odd subgrids.

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$$K(b, w) = \begin{cases} w - b & \text{if } |w - b| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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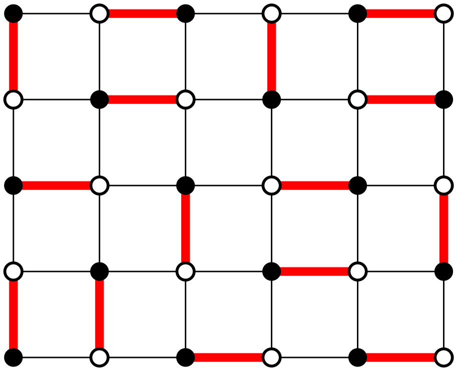

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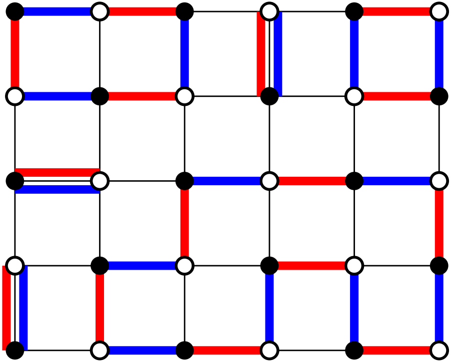

The superposition of two dimer configurations consists of doubly covered edges (OK) and cycles of even length $2k$, each of which yields:

- a factor $(-1)^{k-1}$ to the relative sign of the two permutations,
- a factor $(-1)^{k-1}$ to the ratio of edge factors.

Sketch of the proof of the FKT formula

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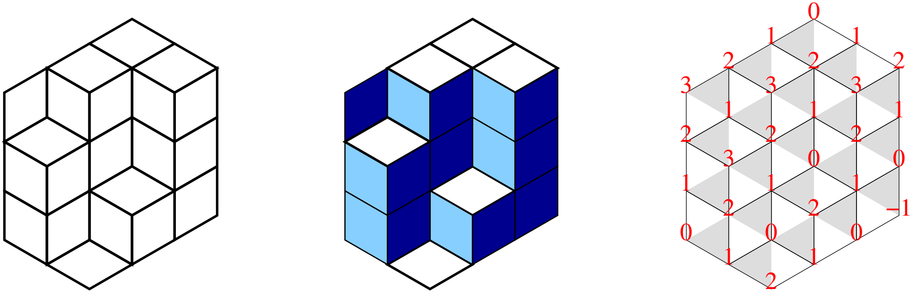
Finally $\det K$ may be evaluated in the case of the rectangular grid by an explicit diagonalization of K (via Fourier transform).

Limits of Kasteleyn's method and related techniques

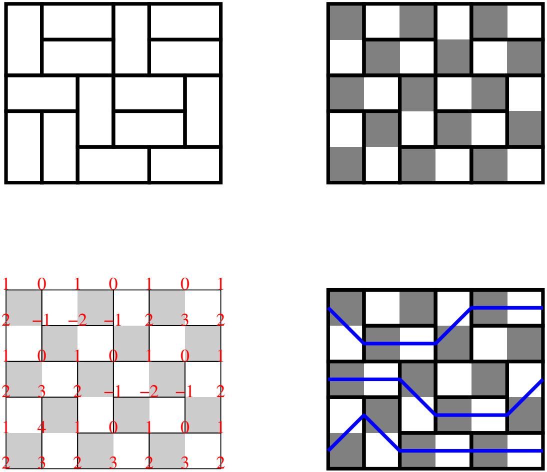
- Still works for graphs embedded in surfaces of higher genus, up to some adaptation (4^g Pfaffians/determinants for genus g).
- Essentially useless for higher-dimensional lattices.
- Does not work anymore when we add interactions between dimers (as in the talks of A. Giuliani and F. Colomo).
- Even for simple planar graphs, $\det K$ (for the partition function) or K^{-1} (for the dimer statistics) may not be easy to evaluate (but this might done using extra algebraic data in some cases, see the talks of S. Corteel and P. Ferrari).

Tilings and surfaces



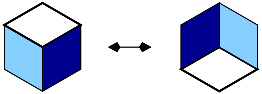
A lozenge tiling may be viewed as a “pile of cubes” or an interface in 3D. It is encoded by a height function.

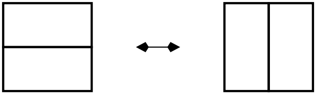
Tilings and surfaces



A similar construction exists for domino tilings (Levitov, Thurston).

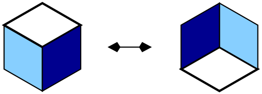
Local moves

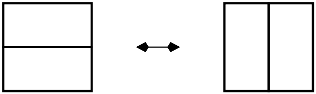




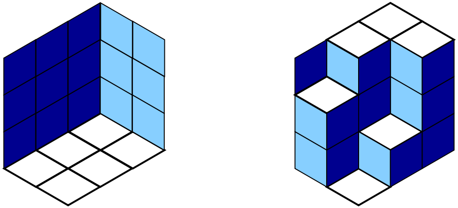
There are natural local moves ("flips") on lozenge/domino tilings. Such moves modify the height function by a unit.

Local moves

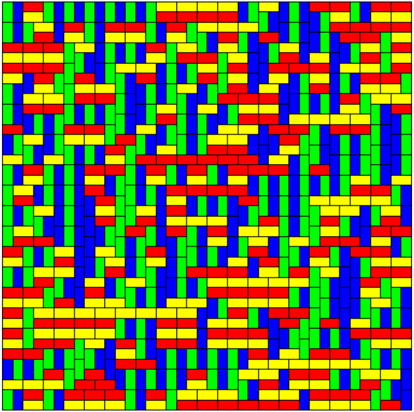


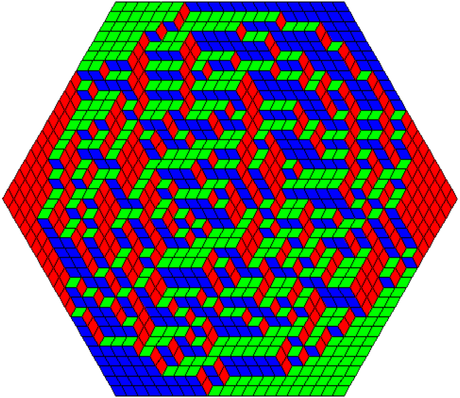


There are natural local moves ("flips") on lozenge/domino tilings. Such moves modify the height function by a unit. This allows to show that two tilings of a simply connected finite region are always related by a finite sequence of local moves, whose minimal length is equal to the volume between the two interfaces. See also the talk by T. Fernique.



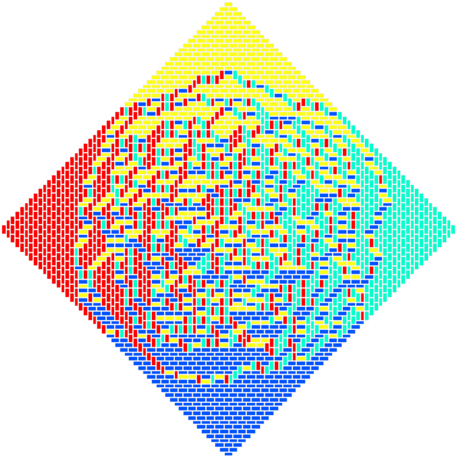
Limit shapes (images by J. Propp *et al.*)





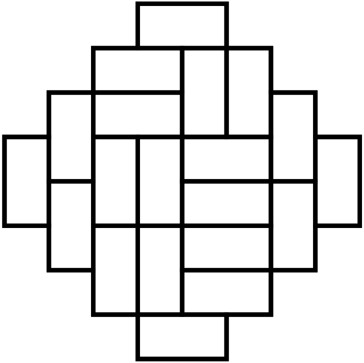
(Uniform) domino tilings of a rectangle have a "flat" limit shape (due to boundary conditions) while lozenge tilings of an hexagon display the famous "arctic circle" phenomenon (Cohn, Larsen, Propp).

Aztec diamond (image by J. Propp *et al.*)



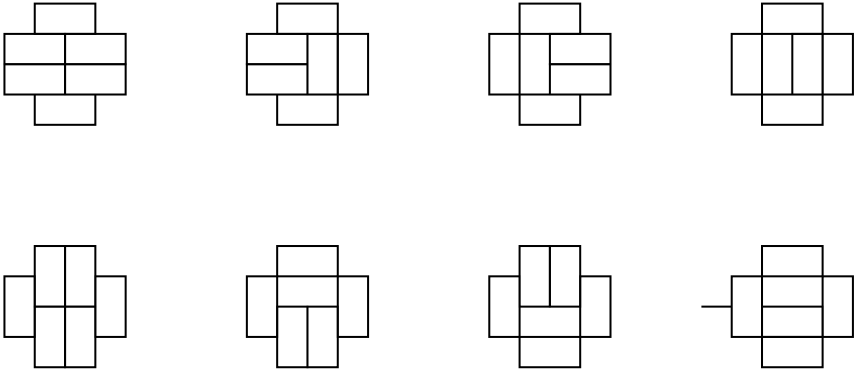
The arctic circle phenomenon also appears in domino tilings of the Aztec diamond (Jockusch, Propp, Shor).

Aztec diamond and ASMs



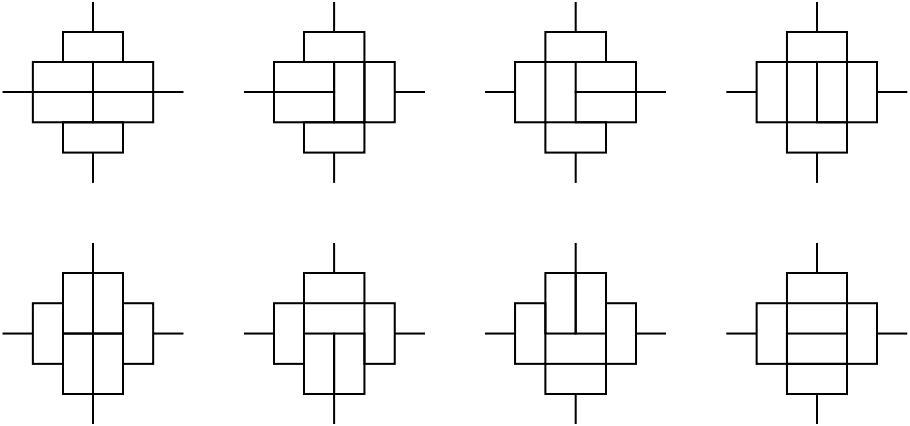
Domino tilings of the Aztec diamond were originally studied in connection with Alternating Sign Matrices or equivalently the six vertex model with domain wall boundary conditions (Elkies, Kuperberg, Larsen, Propp).

Aztec diamond and ASMs



8 domino tilings of the AD of size 2

Aztec diamond and ASMs



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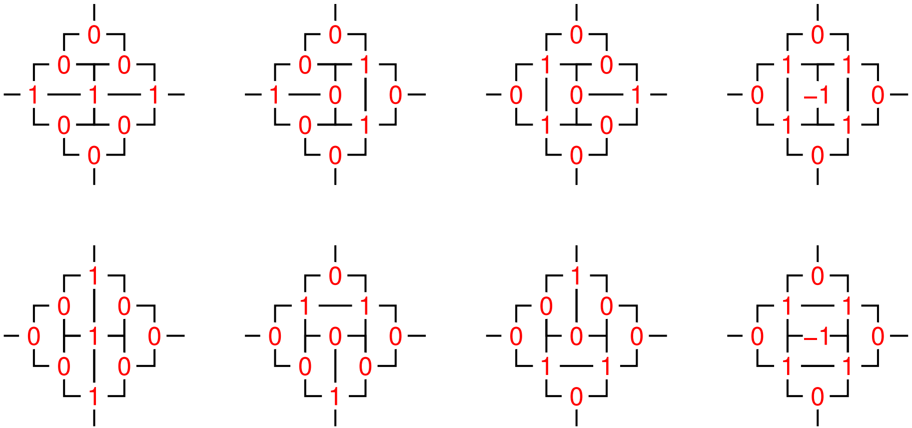


8 domino tilings of the AD of size 2

Aztec diamond and ASMs

```
   0              0              0              0
 0   0          0   1          1   0          1   1
1  1  1        1  0  0        0  0  1        0  −1  0
 0   0          0   1          1   0          1   1
   0              0              0              0

   1              0              1              0
 0   0          1   1          0   0          1   1
0  1  0        0  0  0        0  0  0        0  −1  0
 0   0          0   0          1   1          1   1
   1              1              0              0
```

8 domino tilings of the AD of size 2 $=$ 7 ASMs of size 3 !

Aztec diamond and ASMs

0
0 0
1 1 1
0 0
0

0
0 1
1 0 0
0 1
0

0
1 0
0 0 1
1 0
0

0
1 1
0 -1 0
1 1
0

1
0 0
0 1 0
0 0
1

0
1 1
0 0 0
0 0
1

1
0 0
0 0 0
1 1
0

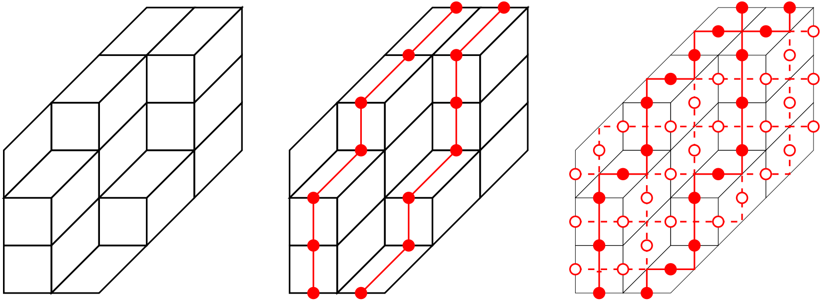
0
1 1
0 -1 0
1 1
0

8 domino tilings of the AD of size $2 = 7$ ASMs of size 3 !

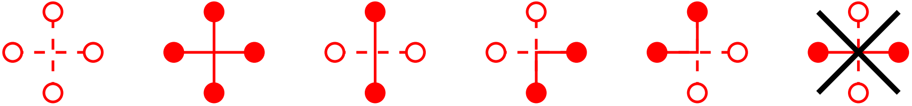
In this correspondence, -1's in ASMs must be counted with weight 2.

This maps to an instance of the six vertex model on the free fermion line.

Tilings and particles



Lozenge tilings may be mapped, via configurations of particles, to the five vertex model (another free fermion point).



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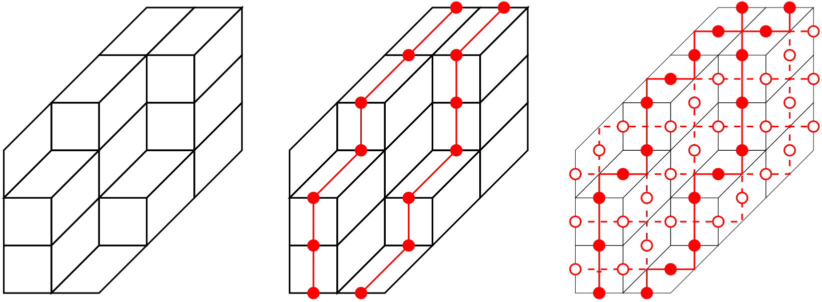

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We may define further discrete or continous Markov dynamics over such configurations of particles, which enjoy "integrability" properties: see the talks of P. Ferrari and L. Petrov.

Thanks for your attention and have a nice day!