Inhomogeneous Random Systems
Institut Henri Poincaré
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Arctic Curves of the six-vertex model

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Based on joint works with:

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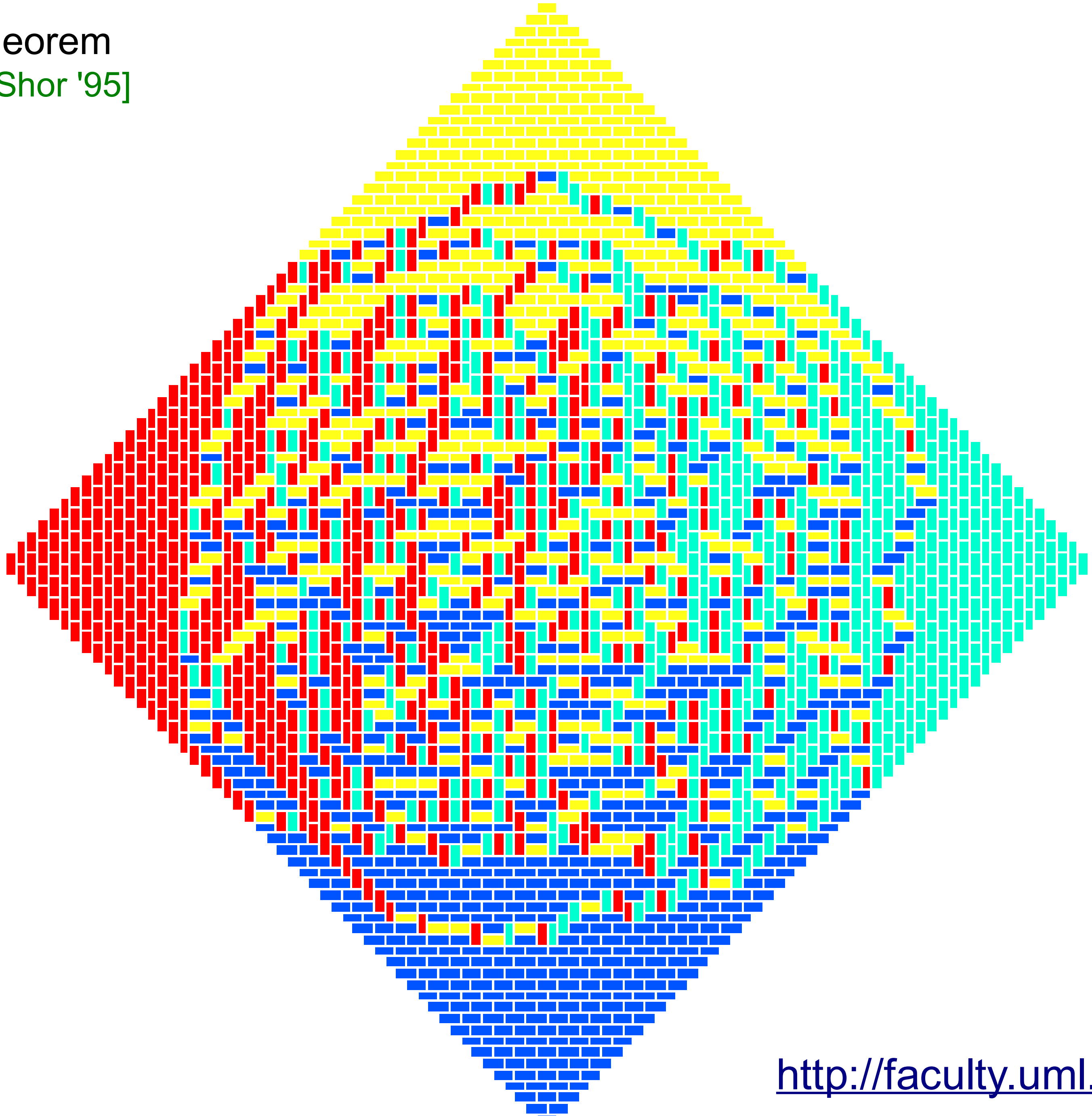
Paul Zinn-Justin (CNRS-UPMC, Paris 6)

Domino tilings of the Aztec Diamond

Arctic Circle theorem

[Jockush-Propp-Shor '95]

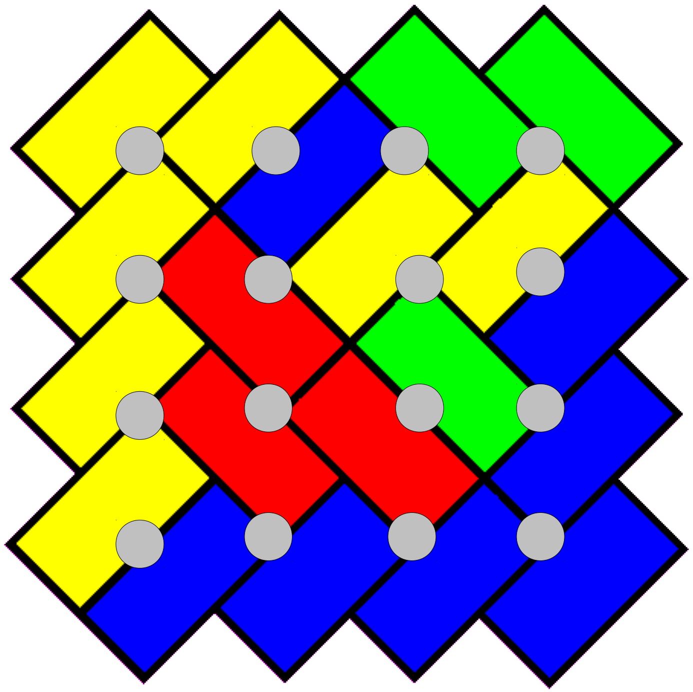


$N = 64$

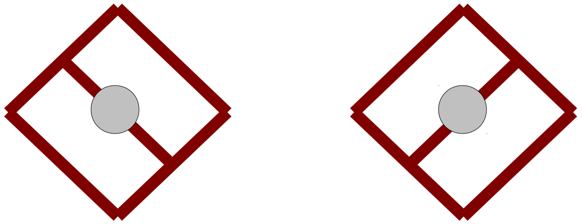
http://faculty.uml.edu/jpropp

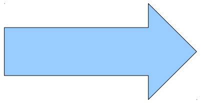
Introducing an interaction between dimers

[Elkies-Kuperberg-Larsen-Propp'92]



Assign a nontrivial weight to:

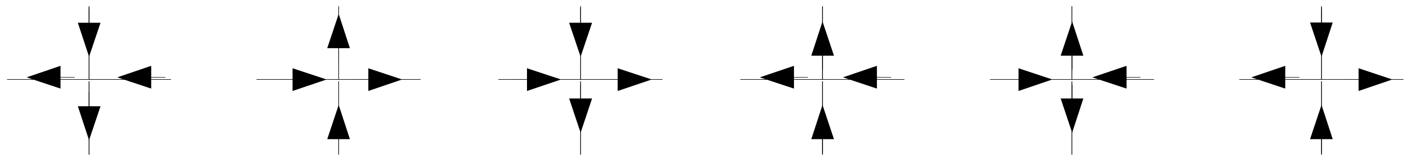


 Six-vertex model with Domain Wall b.c.

An exactly solvable model of statistical mechanics

The six-vertex model

[Lieb '67] [Sutherland'67]



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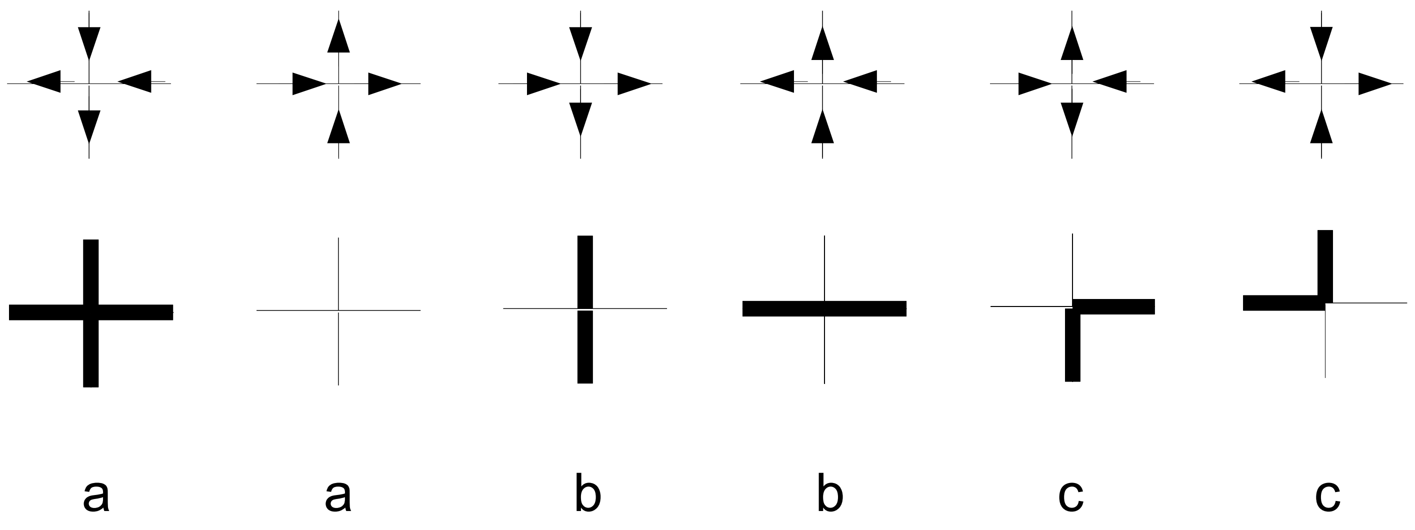

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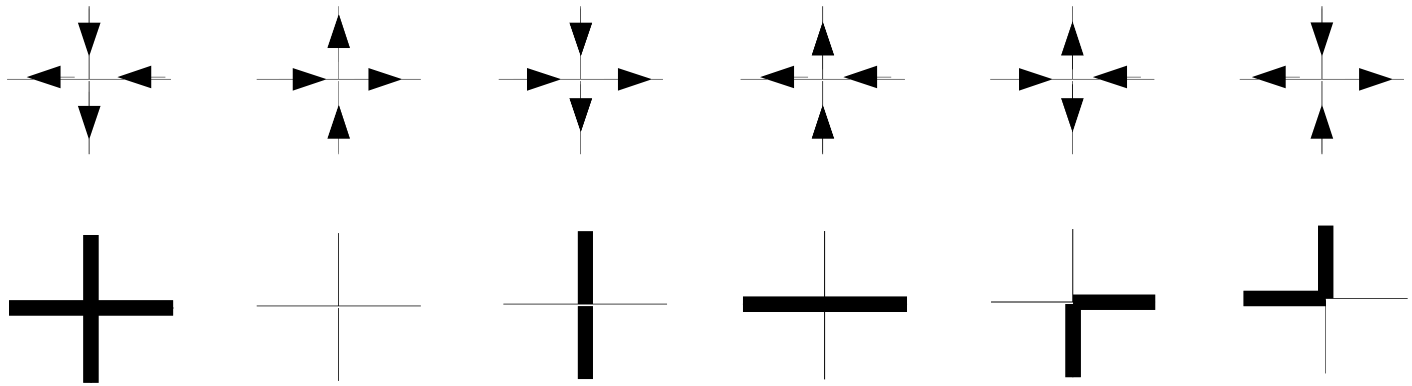
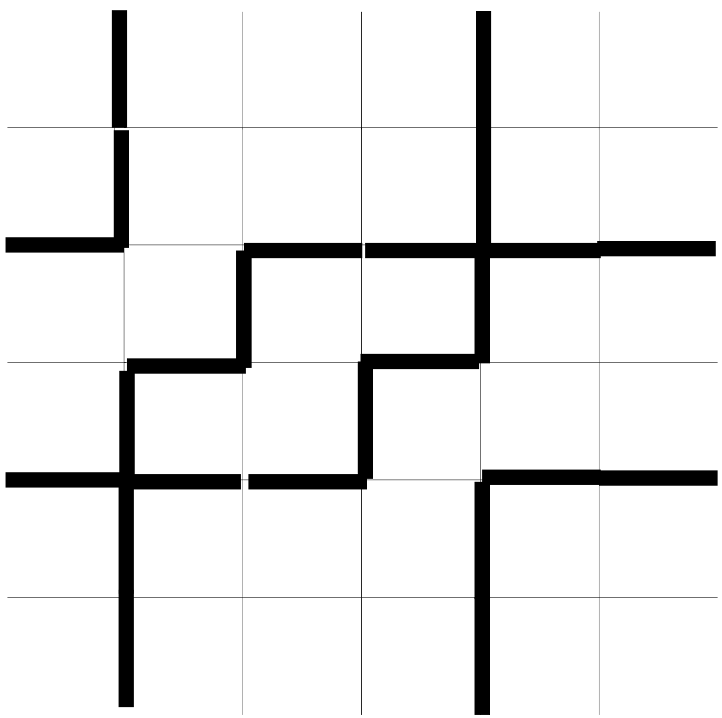


a a b b c c



$$w(\mathcal{C}) = a^{n_a(\mathcal{C})} b^{n_b(\mathcal{C})} c^{n_c(\mathcal{C})}$$

$$Z = \sum_{\mathcal{C}} w(\mathcal{C})$$

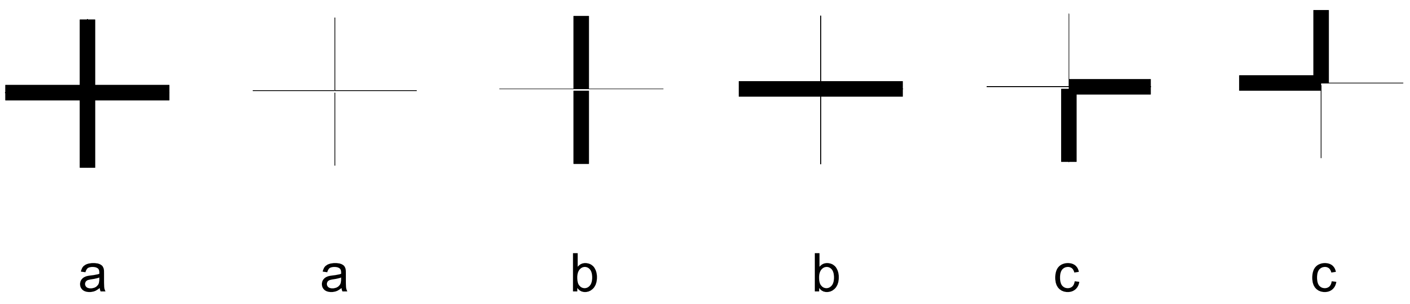
$$P(\mathcal{C}) = \frac{w(\mathcal{C})}{Z}$$

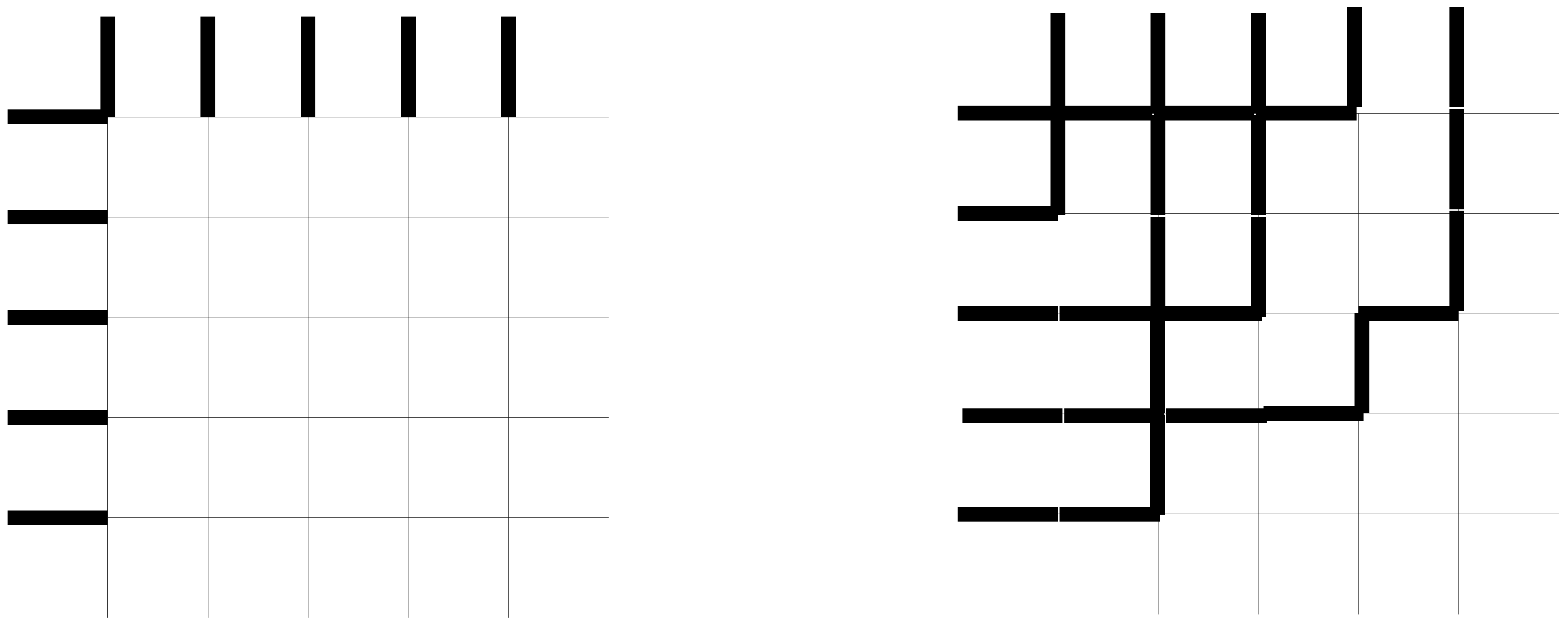
$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

$$t = \frac{b}{a}$$

The Domain Wall boundary conditions

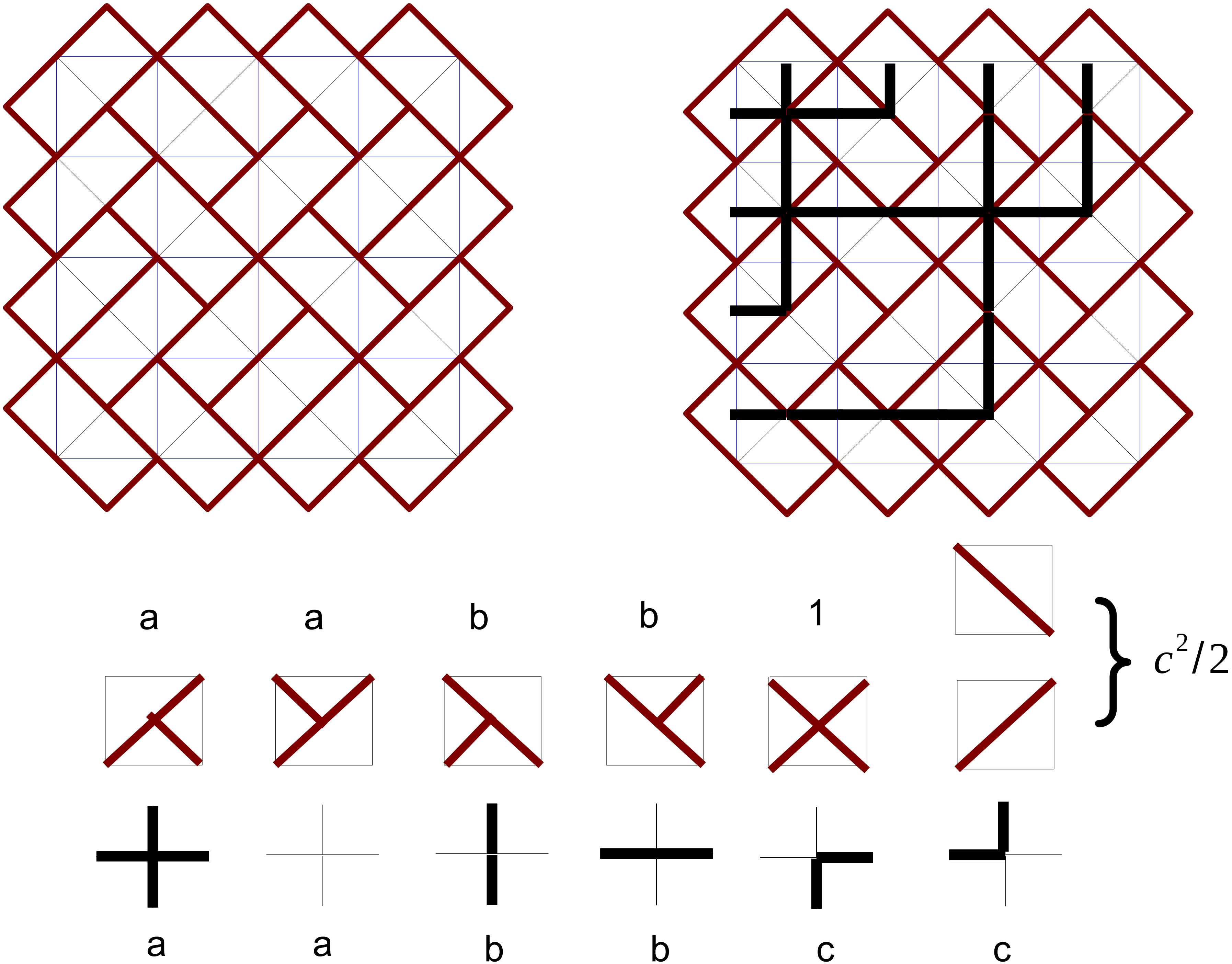
[Korepin'82, Izergin'87]

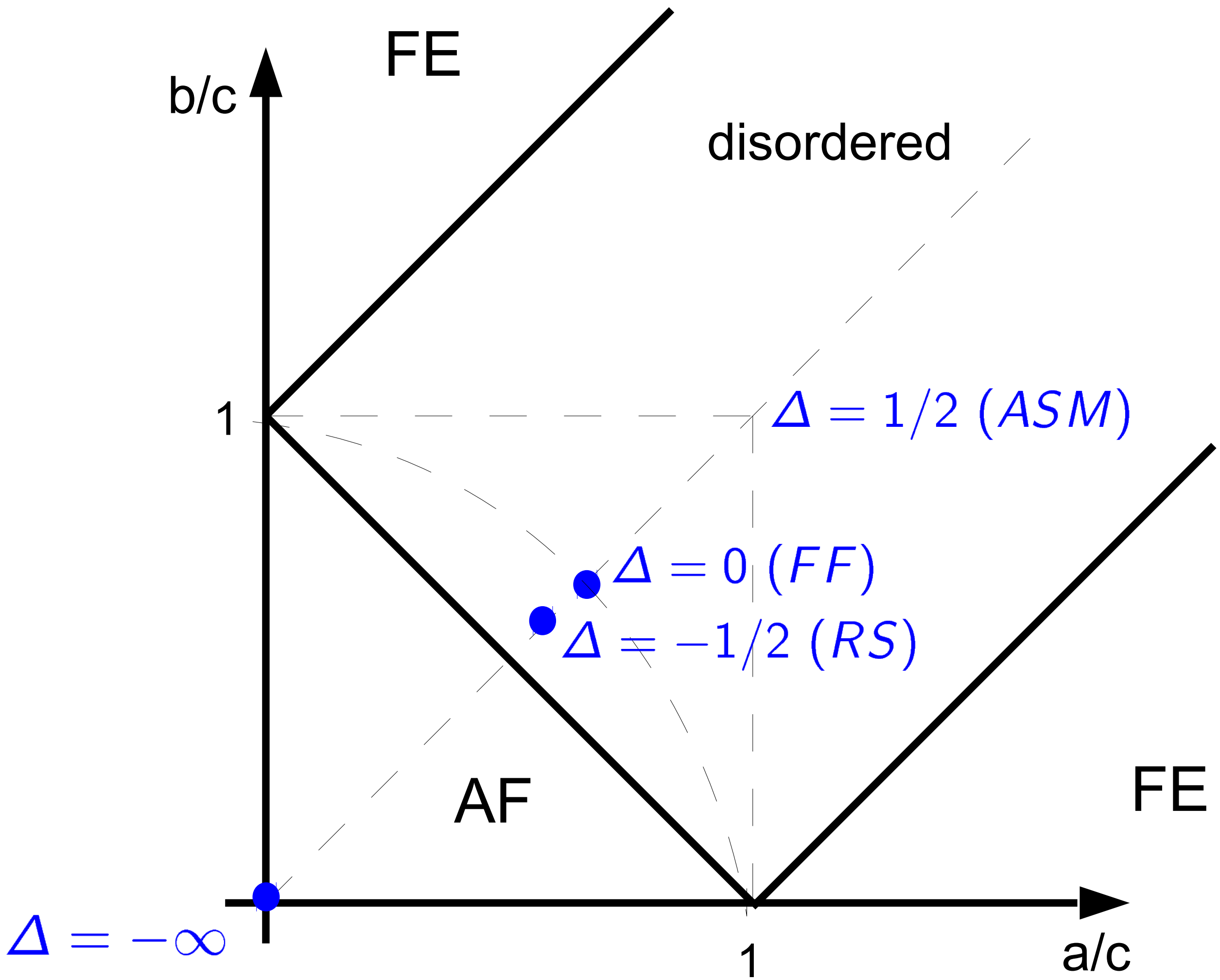




The domino-tilings/six-vertex-model correspondence

[Elkies-Kuperberg-Larsen-Propp '92]





FE
b/c
disordered
1
$\Delta = 1/2\ (ASM)$
$\Delta = 0\ (FF)$
$\Delta = -1/2\ (RS)$
AF
FE
$\Delta = -\infty$
1
a/c

Six-vertex model with Domain Wall boundary condition: main results

- Partition function of the model on the $N \times N$ lattice as an $N \times N$ determinant

[Korepin'82] [Izergin'87]

- Thermodynamics of the model

[Korepin,Zinn-Justin'00] [Bleher, Liechty...'06-'13]

- Numerous result related to combinatorics (i.e., at $\Delta = \pm 1/2$)

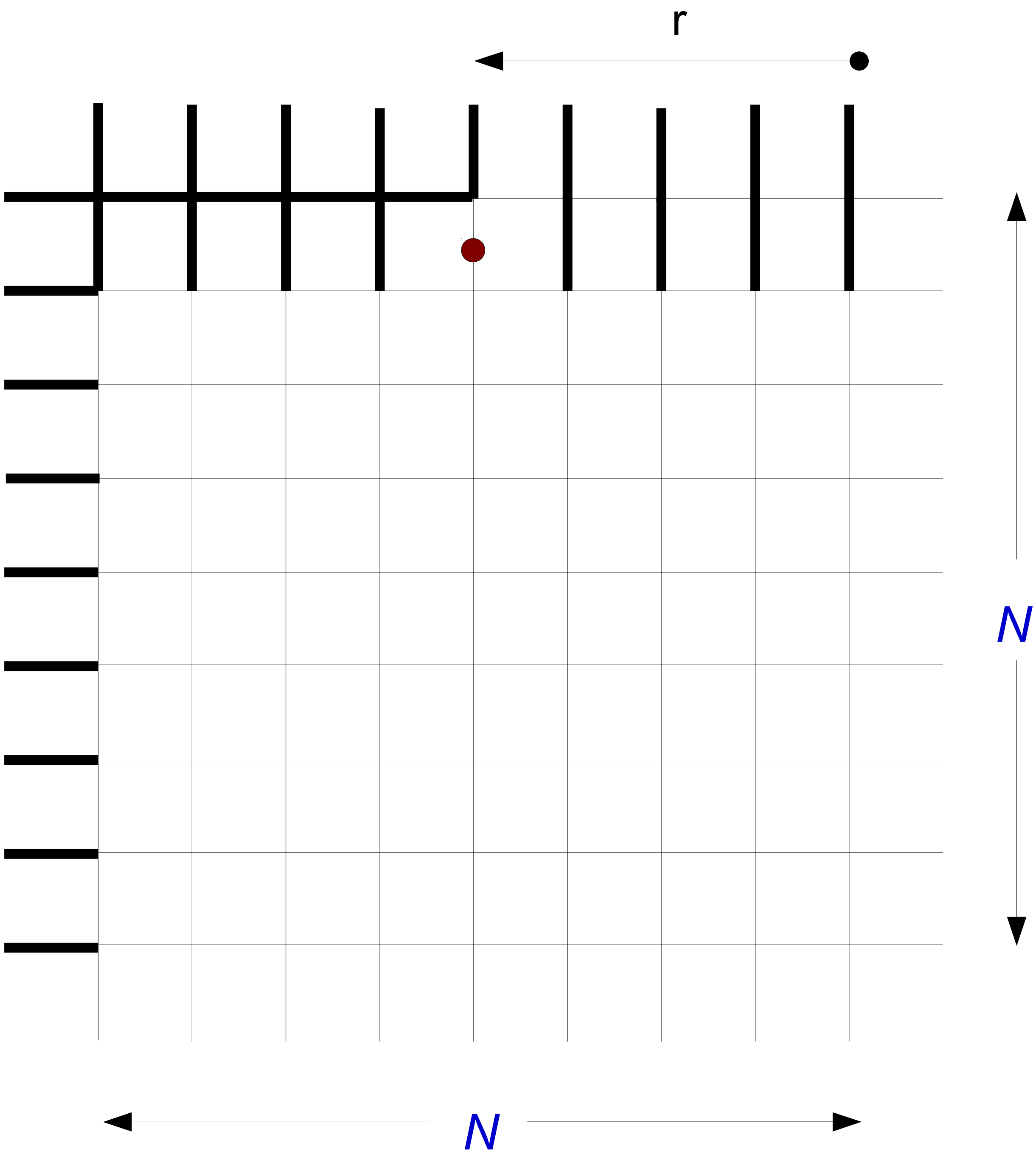
[Kuperberg '95] [Zeilberger '95]
[Razumov-Stroganov'00-'05]
[De Gier-Niehuis'00-'05]
[Di Francesco-ZinnJustin '03-'12]
[FC-Pronko'04-'06]
[Cantini-Sportiello'11]
[...and many others...]

- Boundary correlation functions

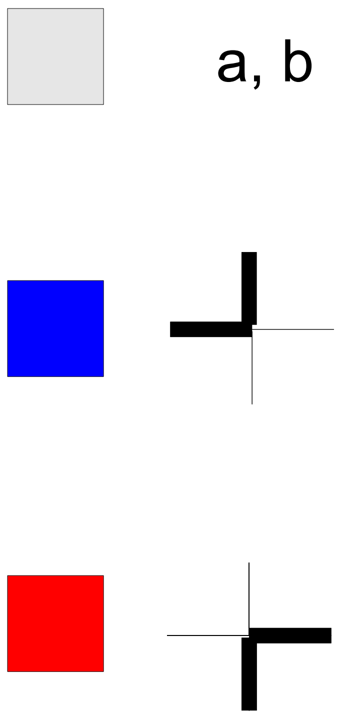
[Bogoliubov-Pronko-Zvonarev'01]
[Foda-Preston'04]
[FC-Pronko'05]

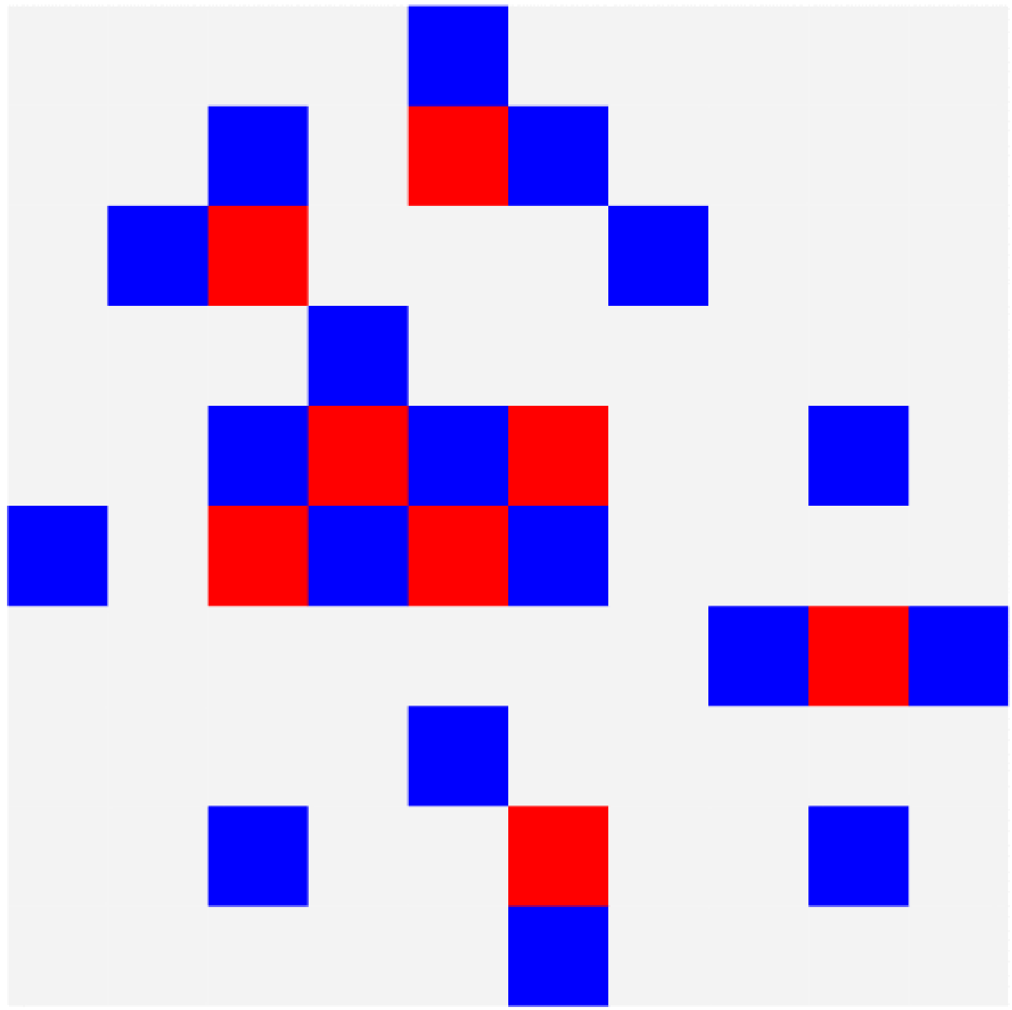
$H_N(r)$ One-point boundary correlation function

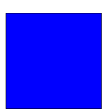
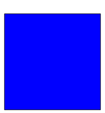
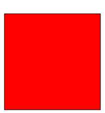
[Bogoliubov-Pronko-Zvonarev '01]



A typical configuration of the 6VM with DWBC, on the 10x10 lattice $(\Delta = 1/2)$

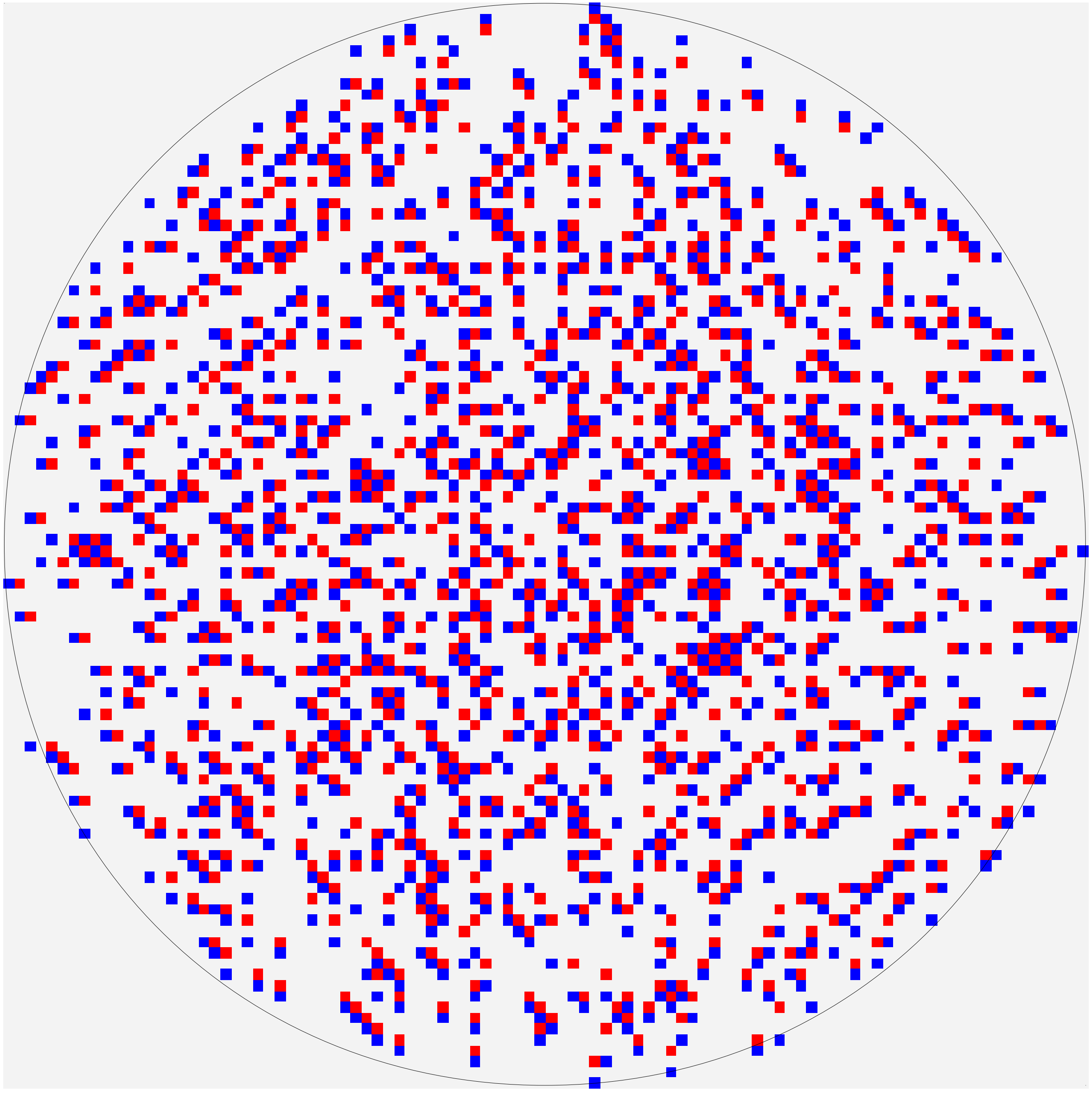




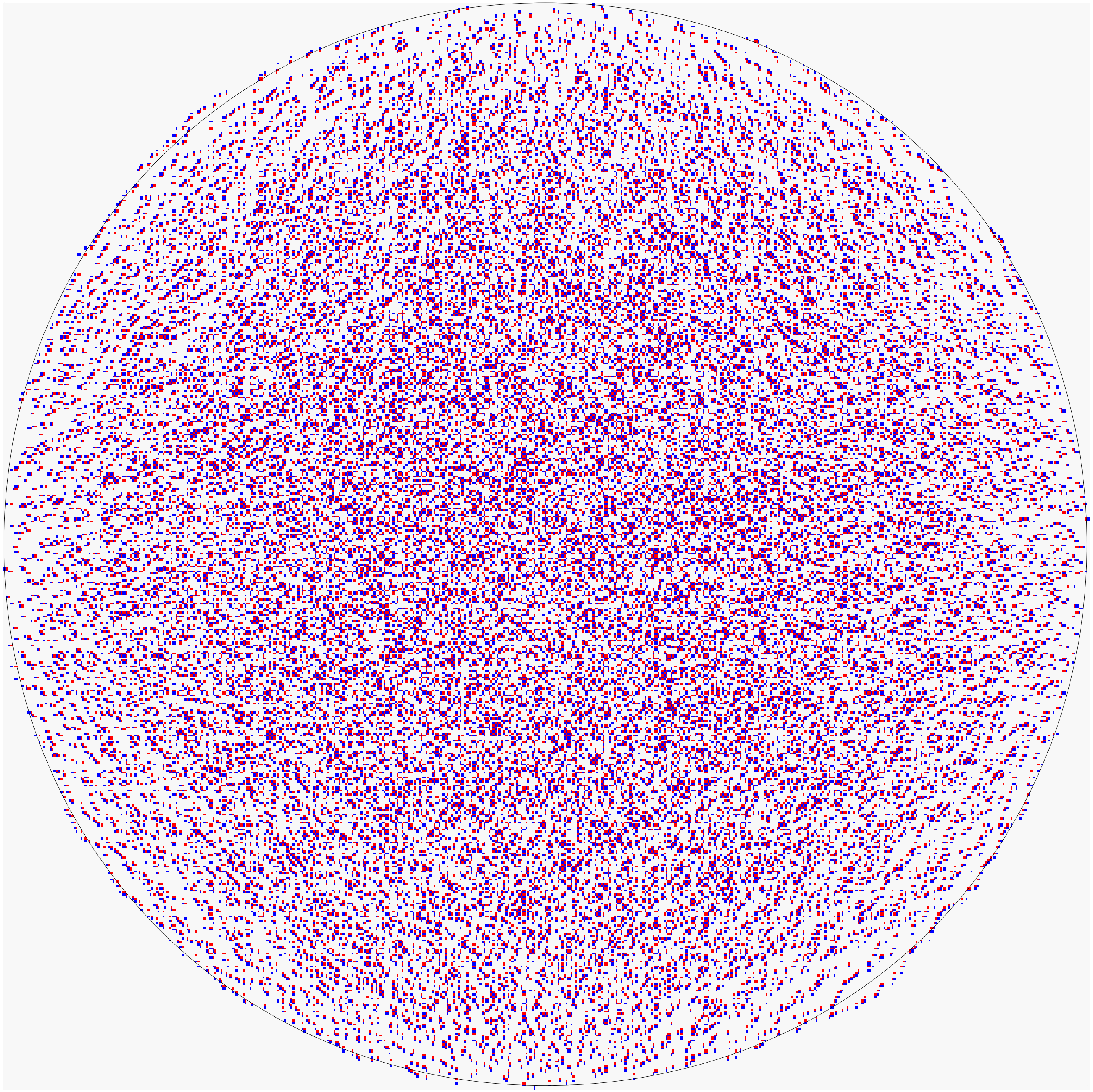
- Just a single  in first/last rows and columns
-  ,  stay away from the corners

From now on, all pictures produced with an improved version of a C code, kindly provided by Ben Wieland, and based on the `Coupling From The Past algorithm [Propp-Wilson'96]

A typical configuration of the 6VM with DWBC, on the 100x100 lattice $(\Delta = 1/2)$

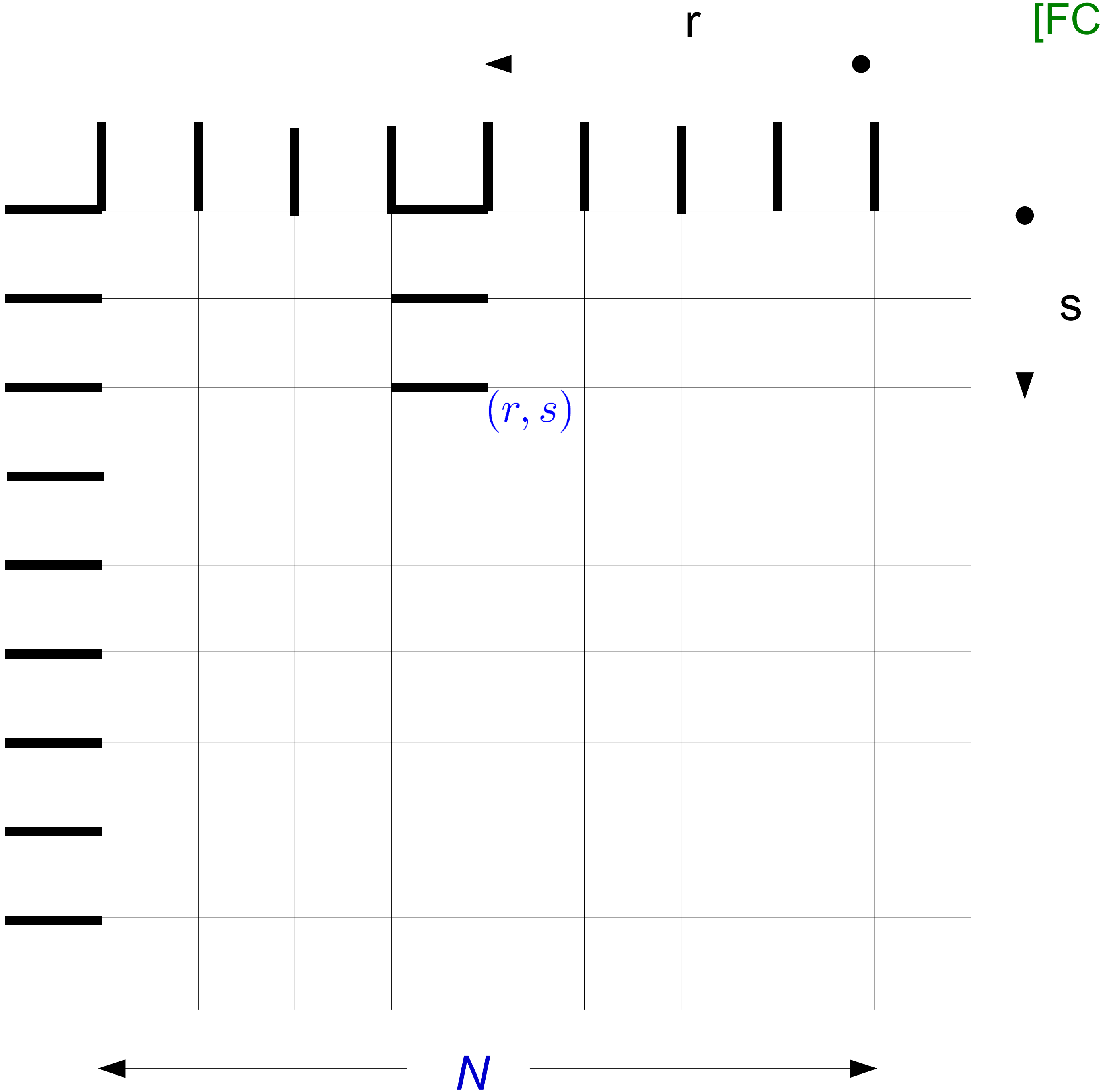


A typical configuration of the 6VM with DWBC, on the 500x500 lattice $(\Delta = 1/2)$



$F_N(r,s)$ Emptiness Formation Probability (EFP)

[FC-Pronko'08]



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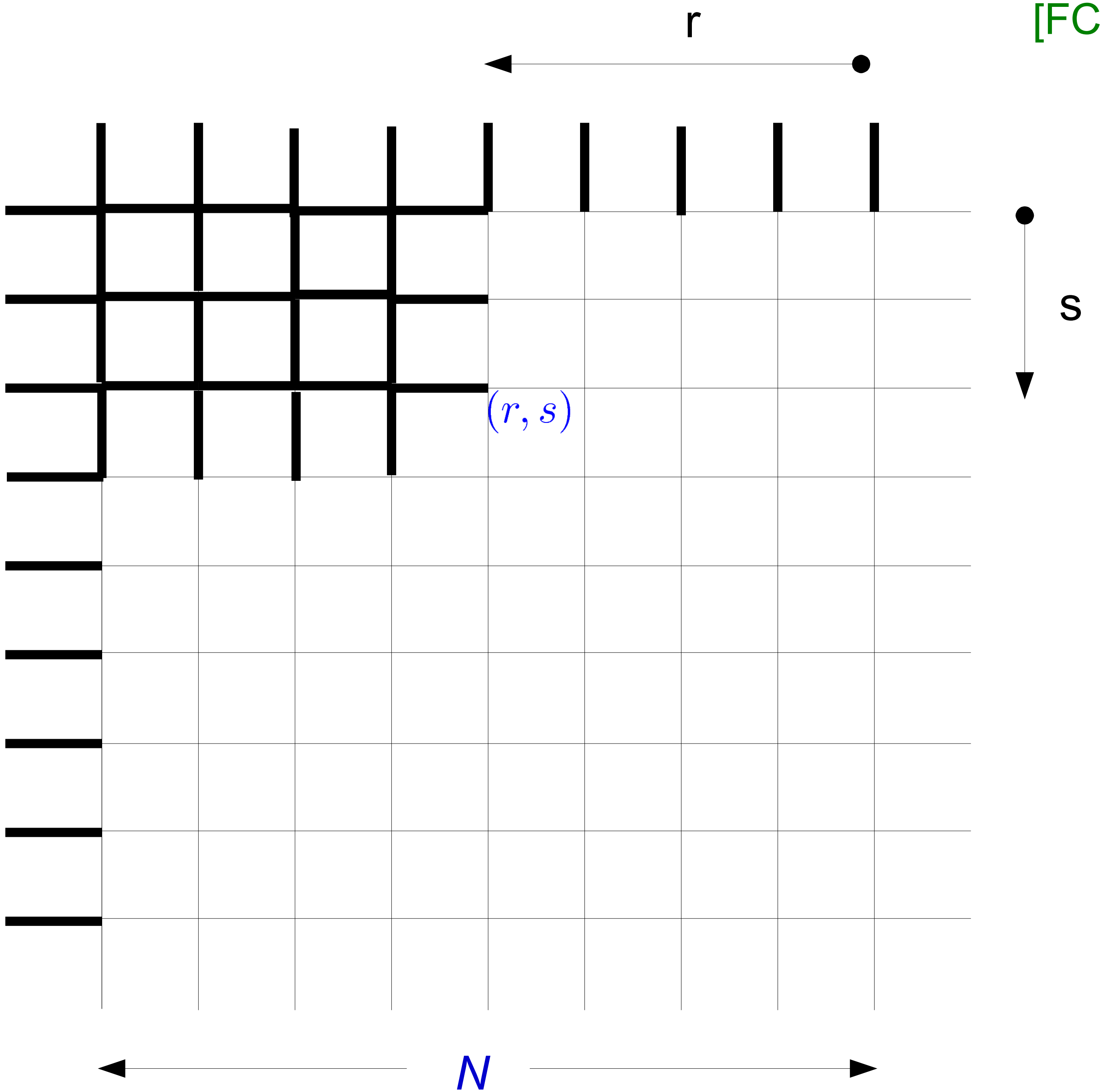

$F_N(r,s)$ Emptiness Formation Probability (EFP)

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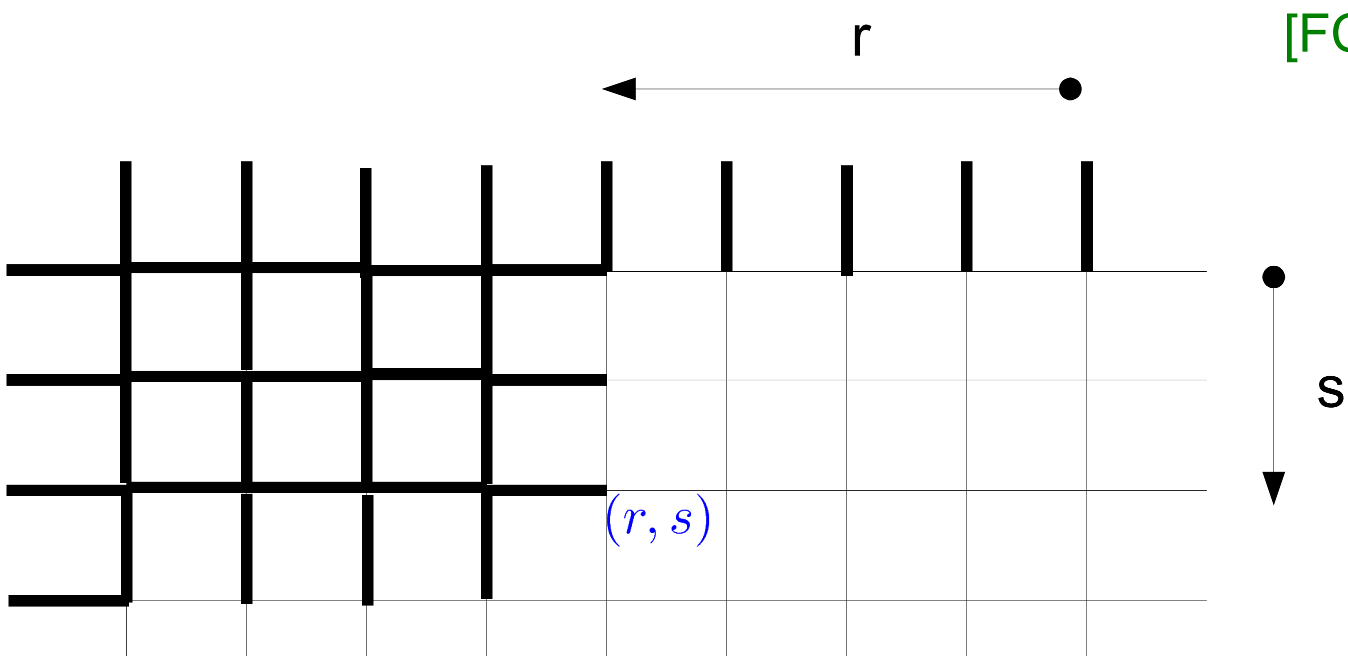

Clearly: $F_N^{(r,s)} \sim 1$ near the top-left corner

$F_N^{(r,s)} \sim 0$ deep inside the lattice

If the Arctic curve exist, in the scaling limit:

$$N, r, s \to \infty \qquad \frac{N-r}{N} = x \qquad \frac{s}{n} = y \qquad x, y \in [0,1]$$

$F_N^{(r,s)}$ should have a stepwise behaviour, from 1 outside the arctic curve, to 0 inside it.

NB: of course only the top-left `quarter' of the Arctic curve can be detected

Multiple Integral Representation for EFP

[FC-Pronko'08]

Define the generating function for the 1-point boundary correlator:

$$h_N(z) := \sum_{r=1}^{N} H_N(r) z^{r-1}\,, \qquad h_N(1) = 1\,.$$

Now define, for $s = 1, \dots, N$:

$$h_N^{(s)}(z_1, \dots, z_s) := \frac{1}{\Delta_s(z_1, \dots, z_s)} \det_{1 \le j,k \le s} \left[h_{N-s+k}(z_j)(z_j - 1)^{k-1} z_j^{s-k} \right]$$

- The functions $h_N^{(s)}(z_1, \dots, z_s)$ are totally symmetric polynomials of order $N-1$ in $z_1, \dots, z_s$.
- They define a new, alternative representation (with respect to Izergin-Korepin determinant) for the partition function Z_N.

Two important properties of $h_N^{(s)}(z_1, \dots, z_s)$:

$$h_N^{(s)}(z_1, \dots, z_{s-1}, 0) = h_N(0) h_{N-1}^{(s-1)}(z_1, \dots, z_{s-1})\,,$$

$$h_N^{(s)}(z_1, \dots, z_{s-1}, 1) = h_N^{(s-1)}(z_1, \dots, z_{s-1})\,.$$

Multiple Integral Representation for EFP

Theorem: [FC-Pronko'08]

The following Multiple Integral Representation is valid ($r, s = 1, 2, \ldots, N$):

$$F_N^{(r,s)} = \frac{(-1)^s Z_s}{s!(2\pi i)^s a^{s(s-1)} c^s} \oint_{C_0} \cdots \oint_{C_0} d^s z \prod_{j=1}^{s} \frac{[(t^2 - 2t\Delta) z_j + 1]^{s-1}}{z_j^r (z_j - 1)^s}$$

$$\times \prod_{\substack{j,k=1 \\ j \neq k}}^{s} \frac{z_k - z_j}{t^2 z_j z_k - 2t\Delta z_j + 1} h_{N,s}(z_1, \ldots, z_s) h_{s,s}(u(z_1), \ldots, u(z_s))$$

where $u(z) := -\frac{z-1}{(t^2 - 2t\Delta)z + 1}$.

Ingredients:

- Quantum Inverse Scattering Method to obtain a recurrence relation, which is solved in terms of a determinant representation on the lines of Izergin-Korepin formula;
- Orthogonal Polynomial and Random Matrices technologies to rewrite it as a multiple integral.

Remark:

Similar expressions occurs for correlation function in ASEP [Tracy-Widom'08-'11].

Scaling limit of EFP

[FC-Pronko'10]

Scaling limit of EFP

[FC-Pronko'10]

Evaluate: $F(x,y) := \lim_{N\to\infty} F_N(xN, yN)$ $\quad x, y \in [0,1]$

in the limit: $N, r, s \to \infty$ $\quad \frac{r}{N} = x \quad \frac{s}{N} = y$

using Saddle-Point method.

Saddle-point equations:

$$
-\frac{s}{z_j - 1} - \frac{r}{z_j} - \sum_{\substack{k=1\\k\neq j}}^{s} \left(\frac{z_k - 1}{z_j z_k - z_j + 1} + \frac{z_k}{z_j z_k - z_k + 1} + \frac{2}{z_k - z_j} \right)
$$

$$
+ \frac{\partial \ln h_{N,s}(z_1, \dots, z_s)}{\partial z_j} + \frac{\partial \ln h_{s,s}(1 - z_1, \dots, 1 - z_s)}{\partial z_j} = 0,
$$

$$(j = 1, \dots, s)$$

Scaling limit of EFP

[FC-Pronko'10]

NB1:

- $s \times s$ Vandermonde determinant
- s-order pole at $z = 1$

$\Rightarrow$ Penner Random Matrix model [Penner'88]

NB2:

- By construction, in the scaling limit, EFP is 1 in the frozen region, and 0 in the disordered one, with a stepwise behaviour in correspondence of the Arctic curve.
- From the structure of the Multiple Integral Representation, such stepwise behaviour can be ascribed to the position of the SPE roots with respect to the pole at $z = 1$.
- The considered generalized Penner model allows for condensation of `almost all' SPE roots at $z = 1$. [Tan'92] [Ambjorn-Kristjansen-Makeenko'94]

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Condensation of `almost all' SPE roots at $z = 1$ $\Leftrightarrow$ Arctic Curves

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Condensation of \`almost all' SPE roots at $z=1$ $\Leftrightarrow$ Arctic Curves

Mathematically, the condition of total condensation (i.e. the Arctic curve) is given by:

$$\frac{y}{z-1}-\frac{1-x+y}{z}+\lim_{N\to\infty}\frac{1}{N}\partial_z \ln h_N(z)=0$$

must have two coinciding real roots in the interval: $z\in[1,\infty)$.

Conjecture: the arctic curve is given by the condition that

$$\frac{y}{z-1} - \frac{1-x+y}{z} + \lim_{N\to\infty} \frac{1}{N} \partial_z \ln h_N(z) = 0$$

must have two coinciding real roots in the interval: $z \in [1, \infty)$.

If we are able to evaluate the large N behaviour of $h_N(z)$ then:

$$\begin{cases} x = X(z) \\ \\ y = Y(z) \end{cases} \qquad z \in [1, \infty).$$

Arbitrary ω ($\Delta = 1 - \omega/2$):

$$
\begin{aligned}
x &= \frac{1}{\Phi(\zeta+\lambda-\eta,2\eta)\Psi(\zeta,2\eta)-\Psi(\zeta+\lambda-\eta,2\eta)\Phi(\zeta,2\eta)} \\
&\qquad \times \left\{ \left[\Psi(\zeta,\lambda-\eta)-\gamma^2\Psi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Phi(\zeta,2\eta) \right. \\
&\qquad\qquad \left. -\left[\Phi(\zeta,\lambda-\eta)-\gamma\Phi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Psi(\zeta,2\eta)\right\} , \\
y &= \frac{1}{\Phi(\zeta+\lambda-\eta,2\eta)\Psi(\zeta,2\eta)-\Psi(\zeta+\lambda-\eta,2\eta)\Phi(\zeta,2\eta)} \\
&\qquad \times \left\{ \left[\Psi(\zeta,\lambda-\eta)-\gamma^2\Psi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Phi(\zeta+\lambda-\eta,2\eta) \right. \\
&\qquad\qquad \left. -\left[\Phi(\zeta,\lambda-\eta)-\gamma\Phi(\gamma\zeta,\gamma(\lambda-\eta))\right]\Psi(\zeta+\lambda-\eta,2\eta)\right\} .
\end{aligned}
$$

where

$$
\Phi(\mu) := \frac{\sin(2\eta)}{\sin(\mu+\eta)\sin(\mu-\eta)} , \qquad \gamma := \frac{\pi}{\pi - \arccos\Delta}
$$

$$
\Psi(\zeta) := \cot\zeta - \cot(\zeta+\lambda-\eta) - \gamma\cot\gamma\zeta + \gamma\cot\gamma(\zeta+\lambda-\eta) ,
$$

(Disordered regime, $-1 < \Delta < 1$)

or

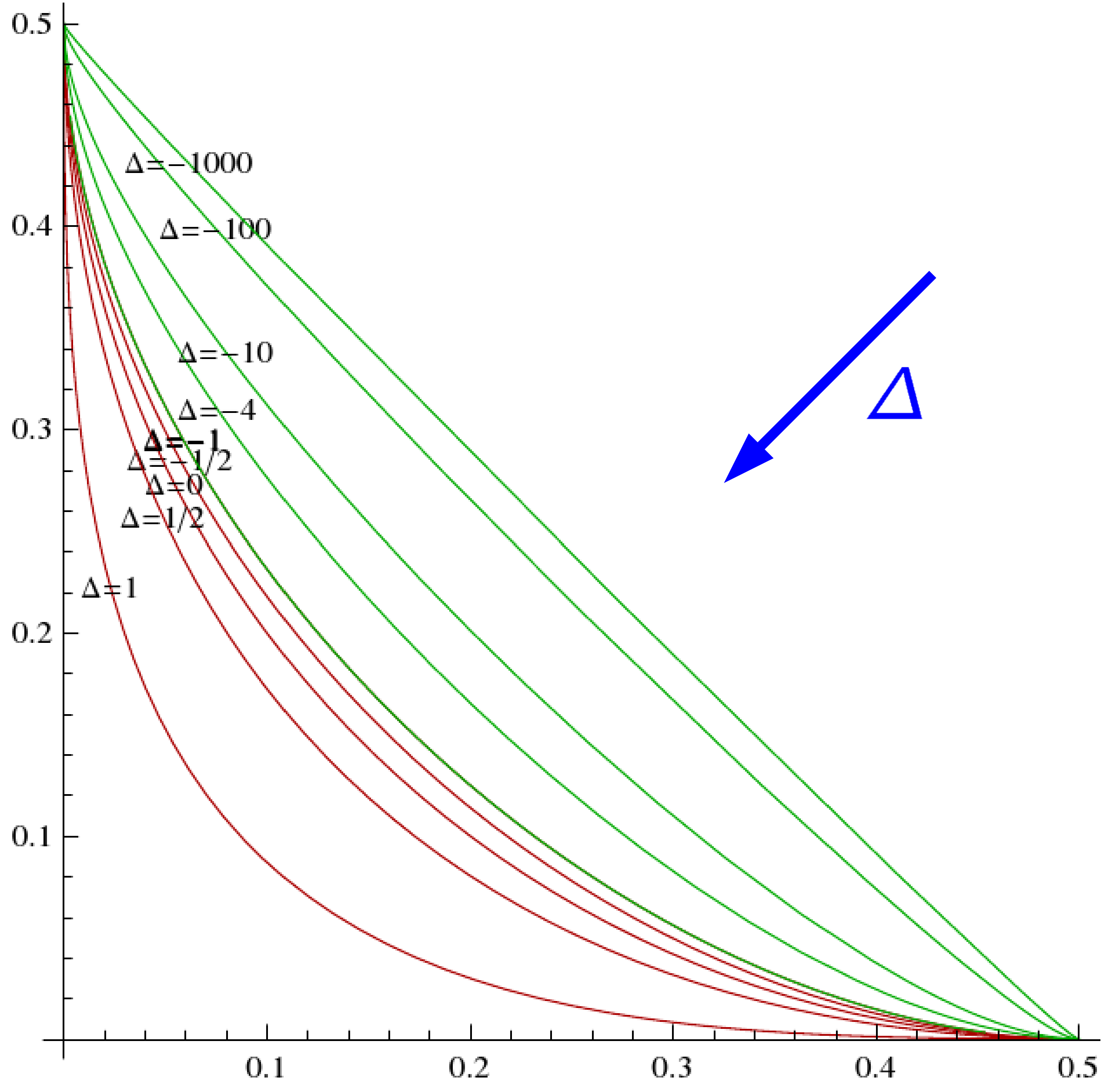
$$
\Phi(\mu) := \frac{\sinh(2\eta)}{\sinh(\eta-\mu)\sinh(\eta+\mu)} , \qquad \gamma := \frac{\pi}{\mathrm{arccosh}(-\Delta))}
$$

$$
\Psi(\zeta) := \cot\zeta - \coth(\eta-\lambda-\zeta) - \gamma\frac{\vartheta_1'(\gamma\zeta)}{\vartheta_1(\gamma\zeta)} + \gamma\frac{\vartheta_1'(\gamma(\zeta+\lambda-\eta))}{\vartheta_1(\gamma(\zeta+\lambda-\eta))} ,
$$

(Anti-ferroelectric regime, $-\infty < \Delta < -1$)

NB: $(\Delta, b, z) \longrightarrow (\eta, \lambda, \zeta)$

Red curves: disordered regime [FC-Pronko'10]
Green curves: anti-ferroelectric regime [FC-Pronko-Zinn-Justin'10]



NB: $\Delta = 0 \implies$ Arctic Circle

$\Delta = -\infty \implies$ straight line

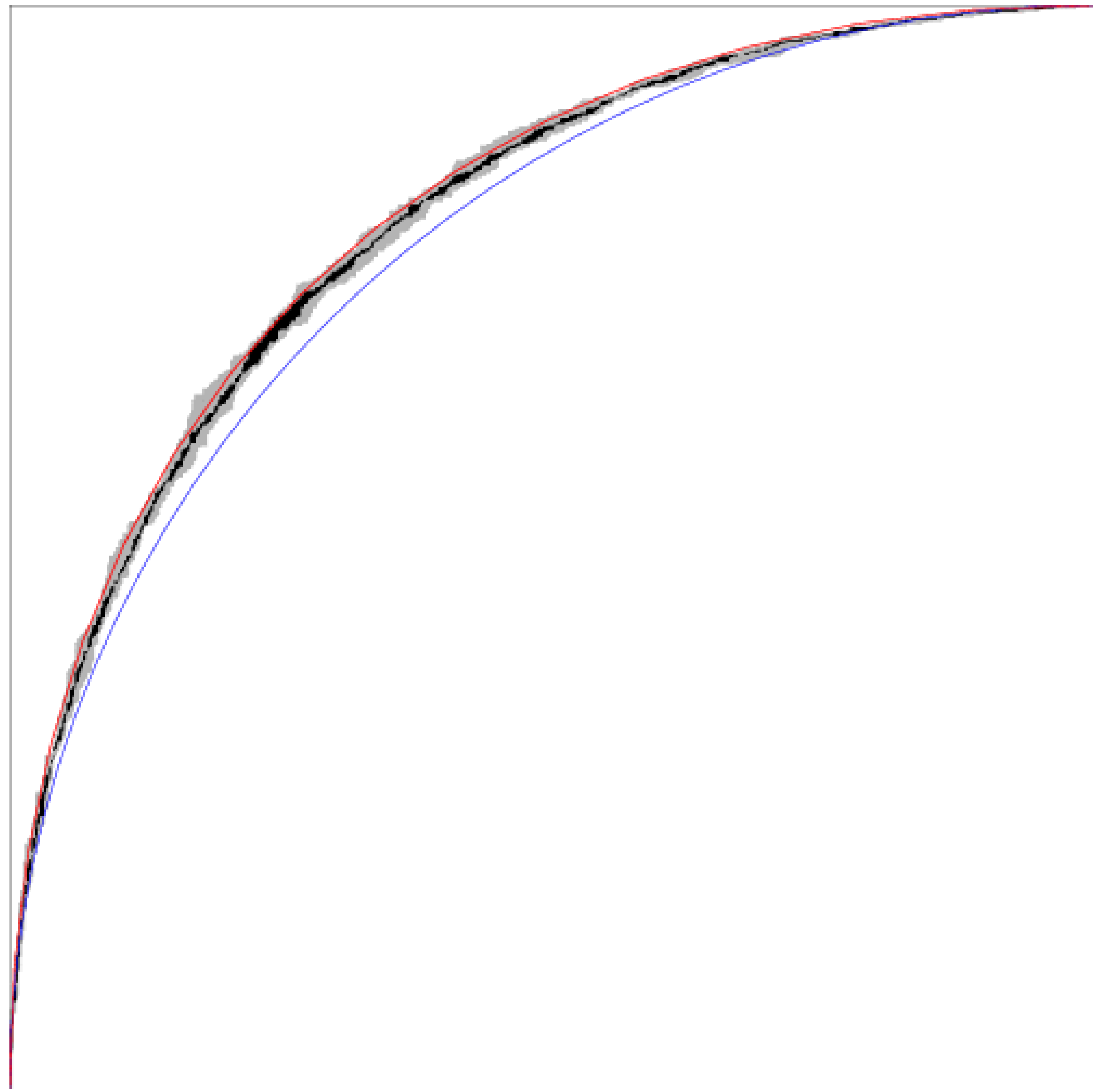
$\Delta = 1/2$

$(2x-1)^2 + (2y-1)^2 - 4xy = 1, \qquad x, y \in \left[0, \frac{1}{2}\right].$

N=1500

10 samples

[FC-Pronko'10]



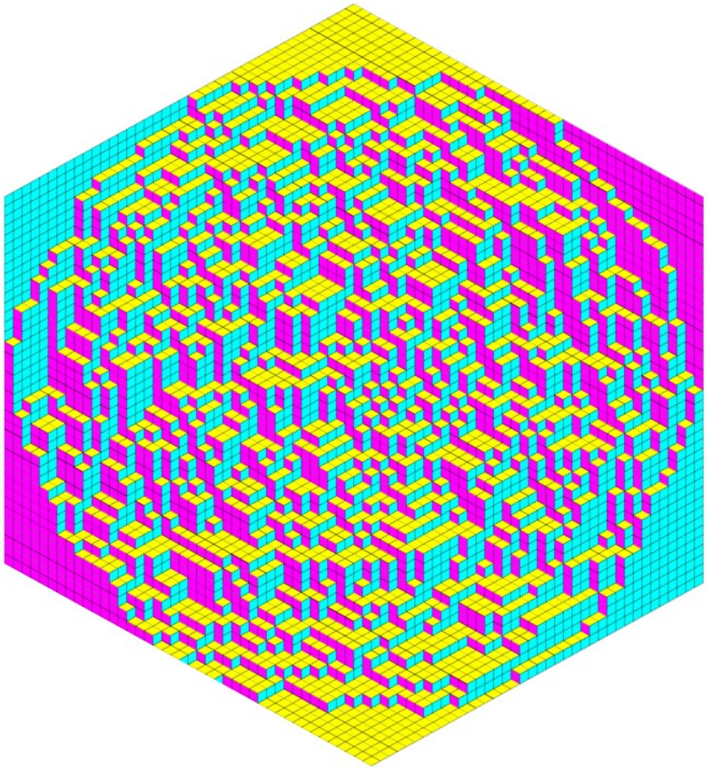
Ben Wieland (April 2008)

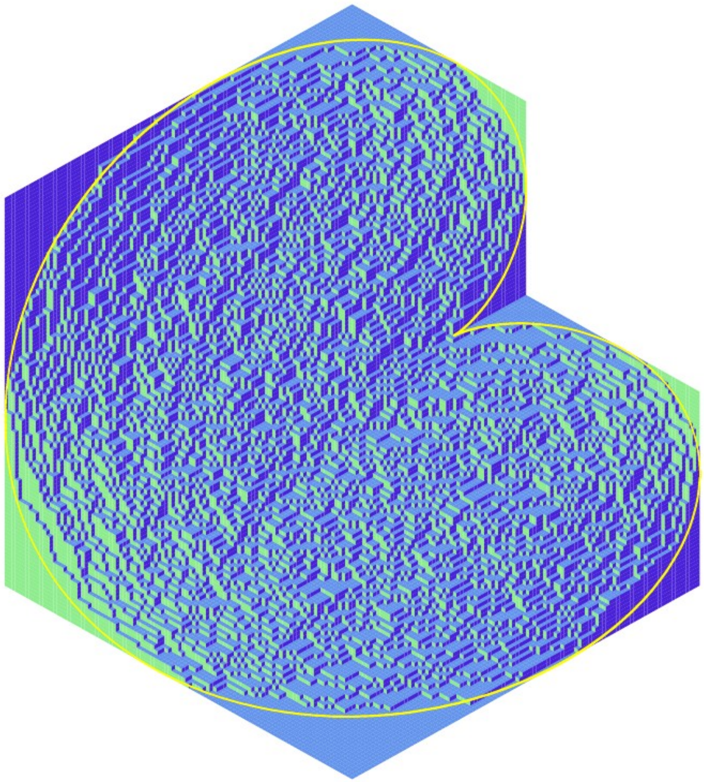
http://www.math.brown.edu/~wieland

Criticisms

- The present derivation of Arctic curves is based on an assumption (the `condensation hypothesis') which is rather bold and probably hard to prove.

Extension of the result to more generic domains?

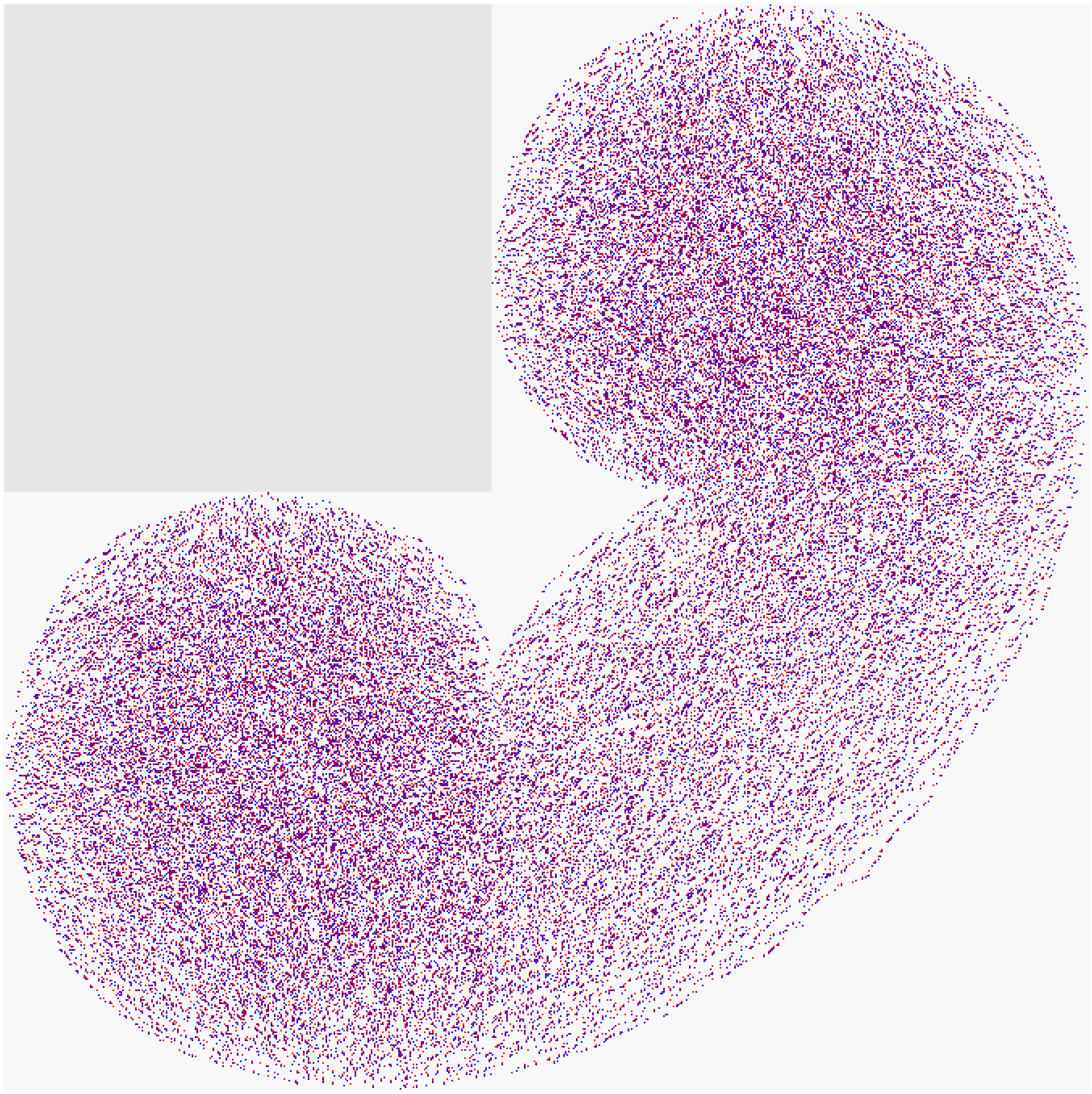




For usual dimer models, use the theory provided by [Kenyon-Sheffield-Okounkov]

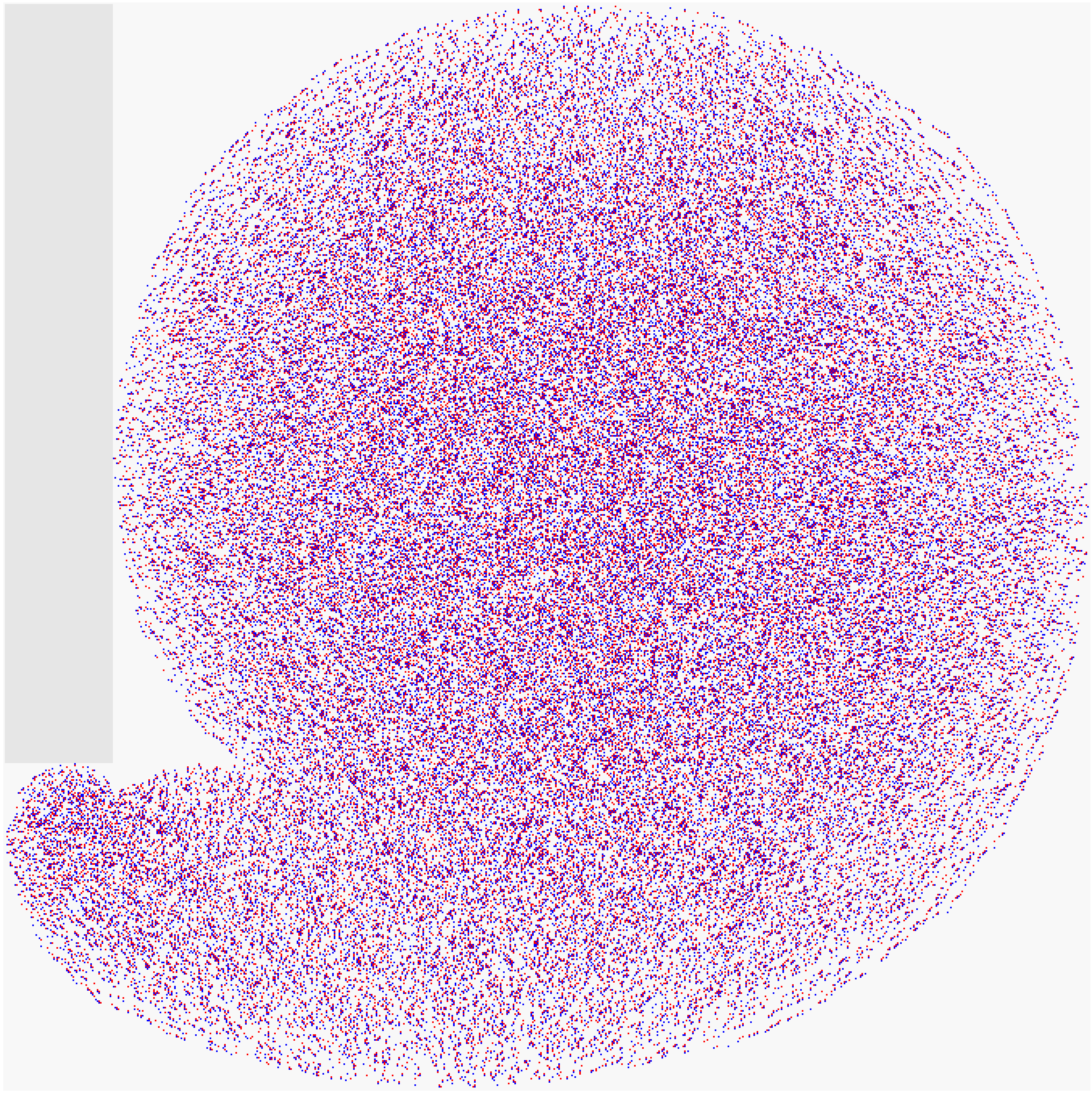
6VM

$\Delta = \frac{1}{2}$



$N = 1000$
$r, s = 450$

[FC-Sportiello, *to appear*]



$N = 1000$
$r = 100$
$s = 700$

[FC-Sportiello, *to appear*]

Six-vertex model on generic domains

[FC-Sportiello, *to appear*]

Our previous result on the Arctic curve in a square domain can be rephrased as follows:

The arctic curve is the geometric caustic (envelope) of the family of straight lines:

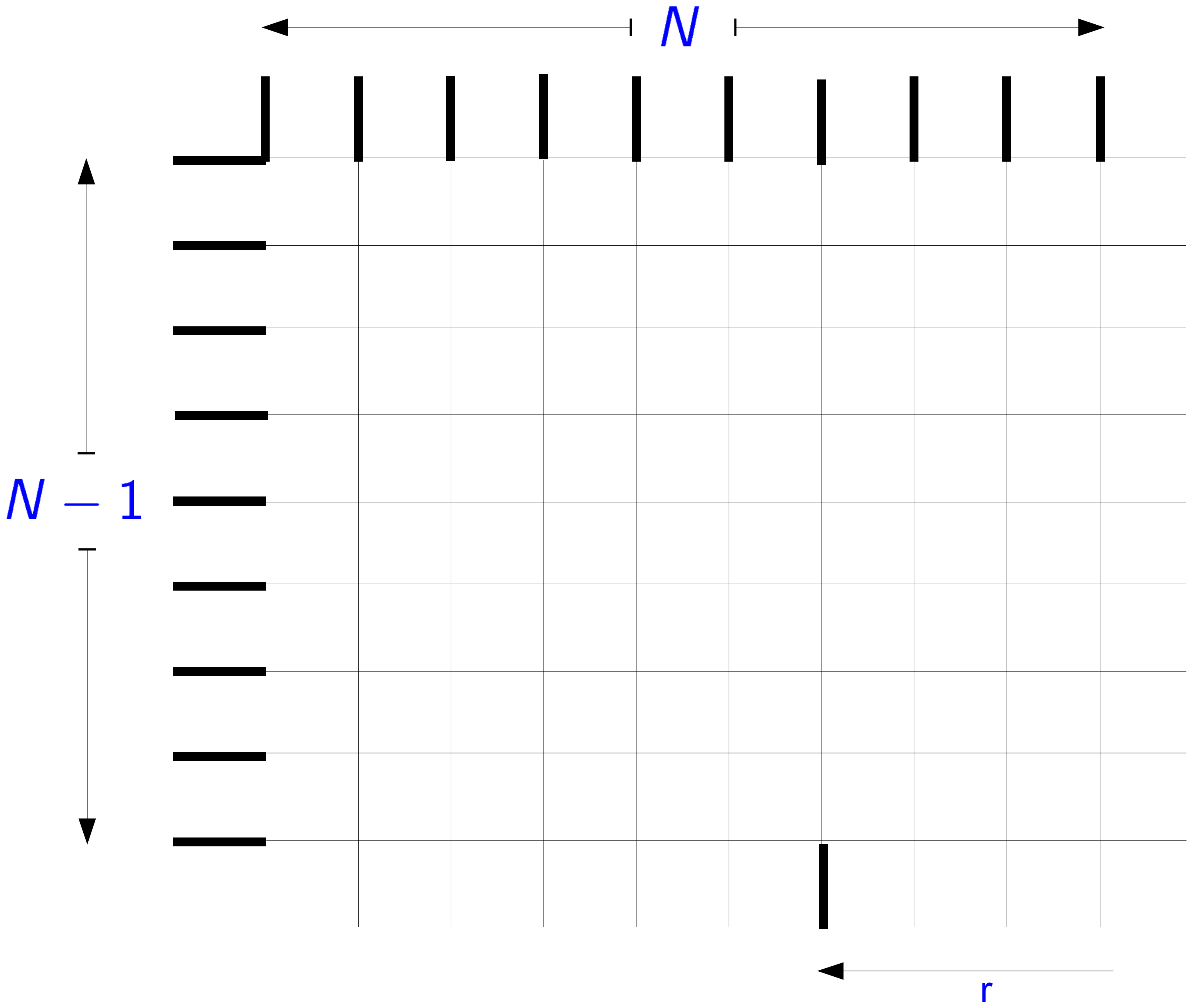
$$-x\frac{1}{z}+y\,\frac{(t^2-2\Delta t+1)}{(z-1)(t^2z-2\Delta t+1)}+\lim_{N\to\infty}\frac{1}{N}\partial_z \ln h_N(z)=0\,,\qquad z\in[1,+\infty)$$

<u>Questions:</u>

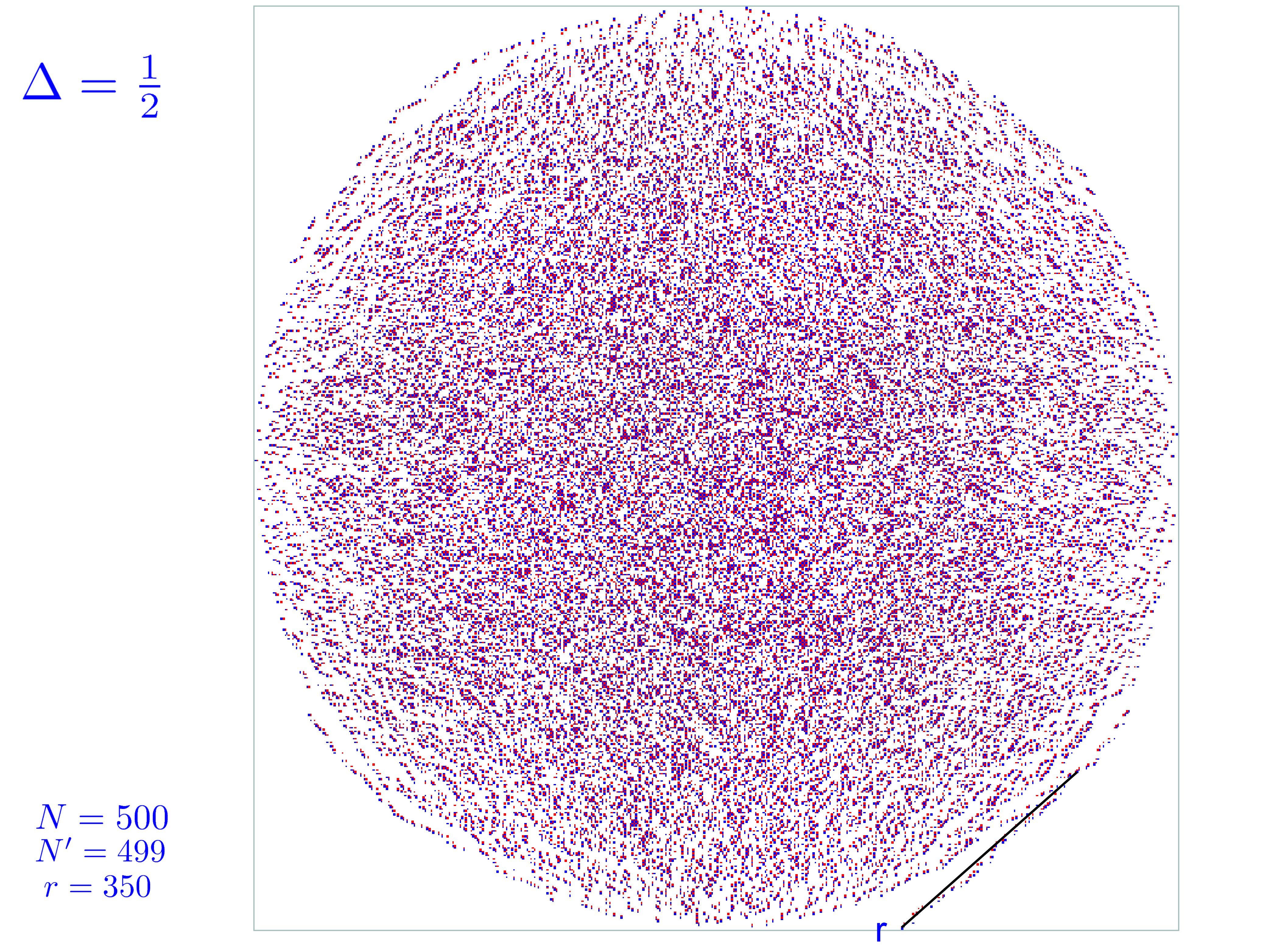
- What is the geometrical meaning of this family of straight line?
  - why the constant term is determined by the boundary correlator $h_N(z)$?
  - what determines the angular coefficient of these lines?

Understanding this would provide:

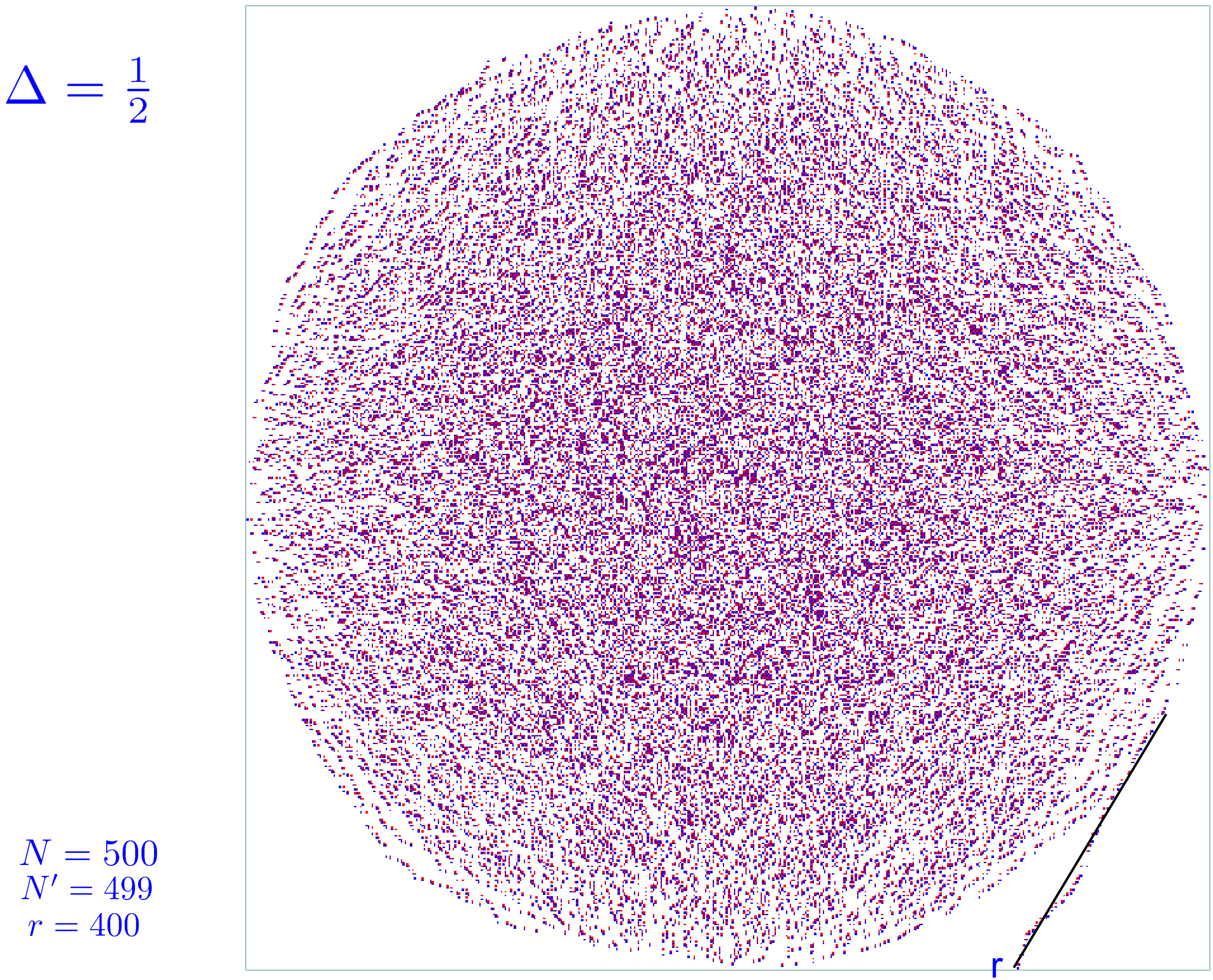
- an alternative (geometrical) derivation of the Arctic curve;
- a geometrical strategy to attack the problem of Arctic curves in generic domains.



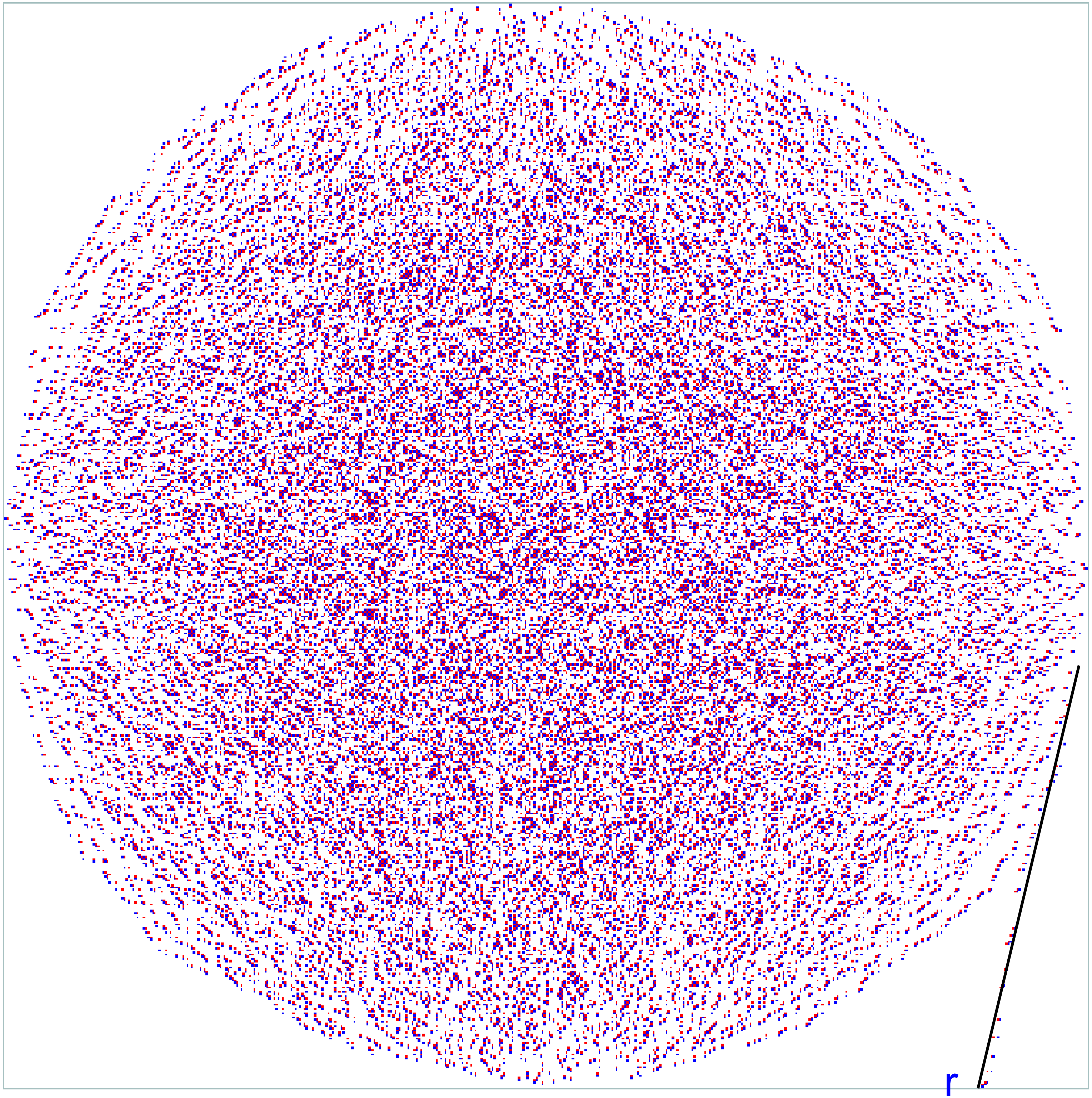
N
$N-1$
r



$\Delta = \frac{1}{2}$
$N = 500$
$N' = 499$
$r = 350$
r

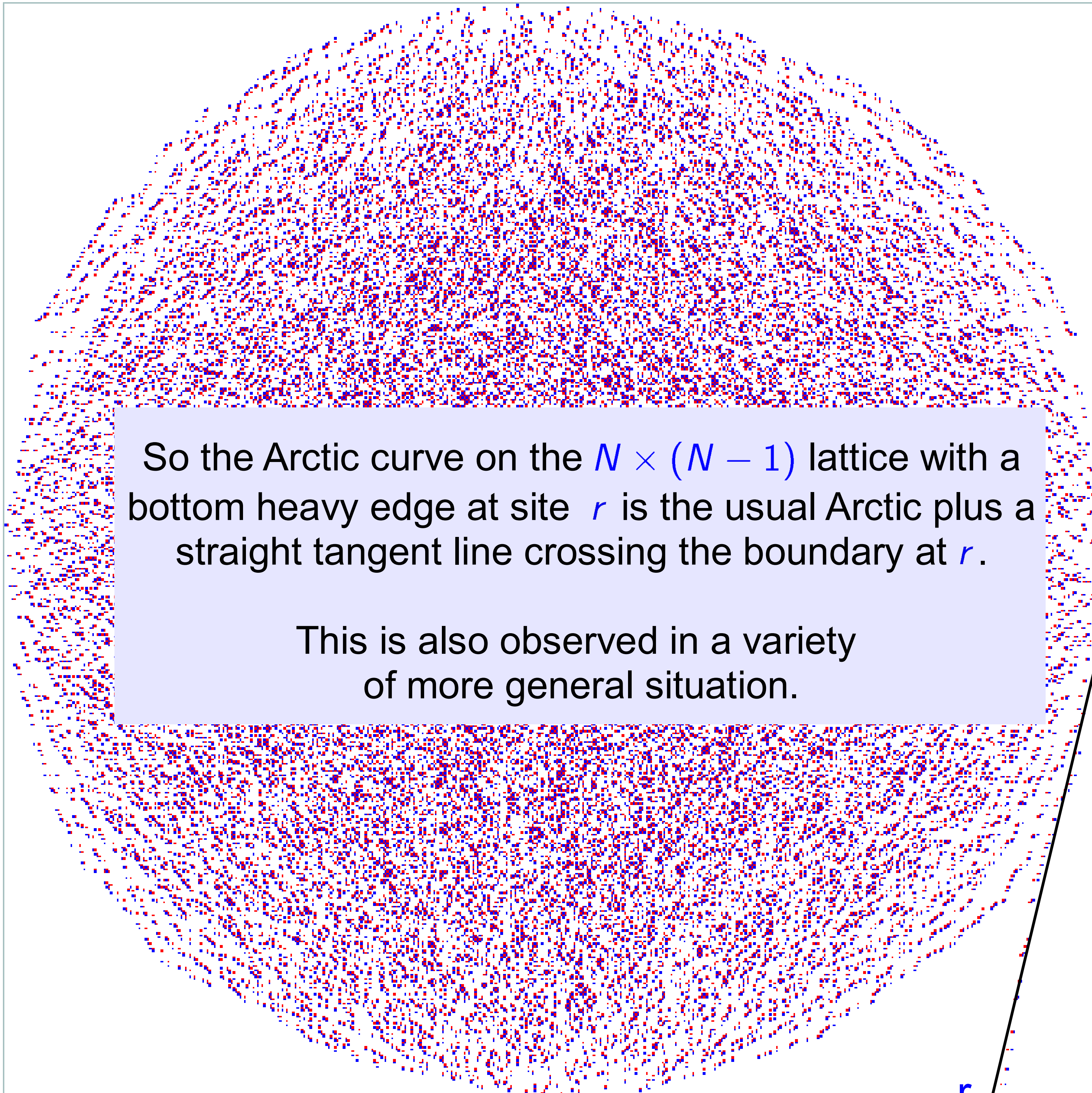

$\Delta = \frac{1}{2}$
$N = 500$
$N' = 499$
$r = 400$
r

$\Delta = \frac{1}{2}$



$N = 500$
$N' = 499$
$r = 450$

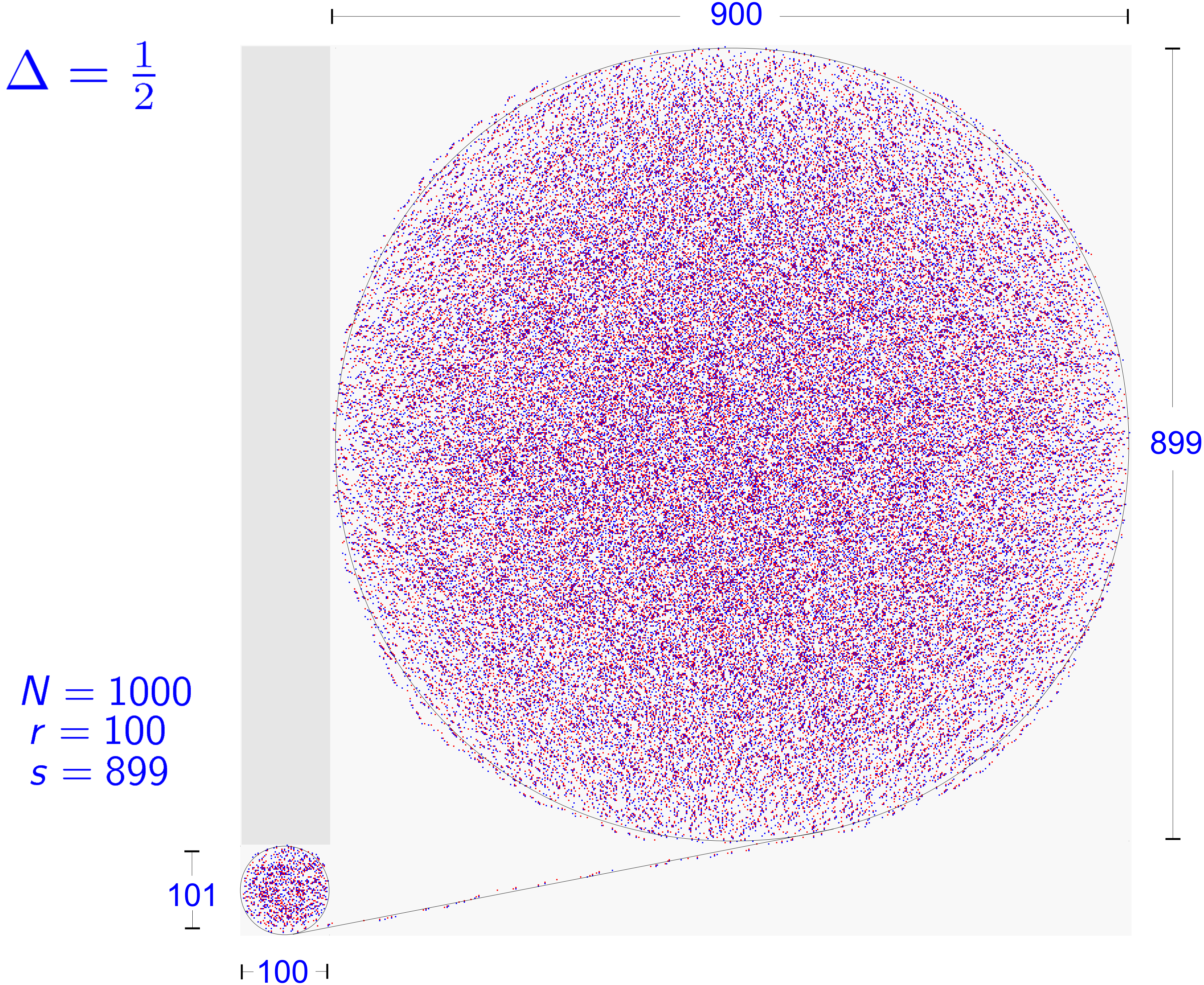
$\Delta = \frac{1}{2}$

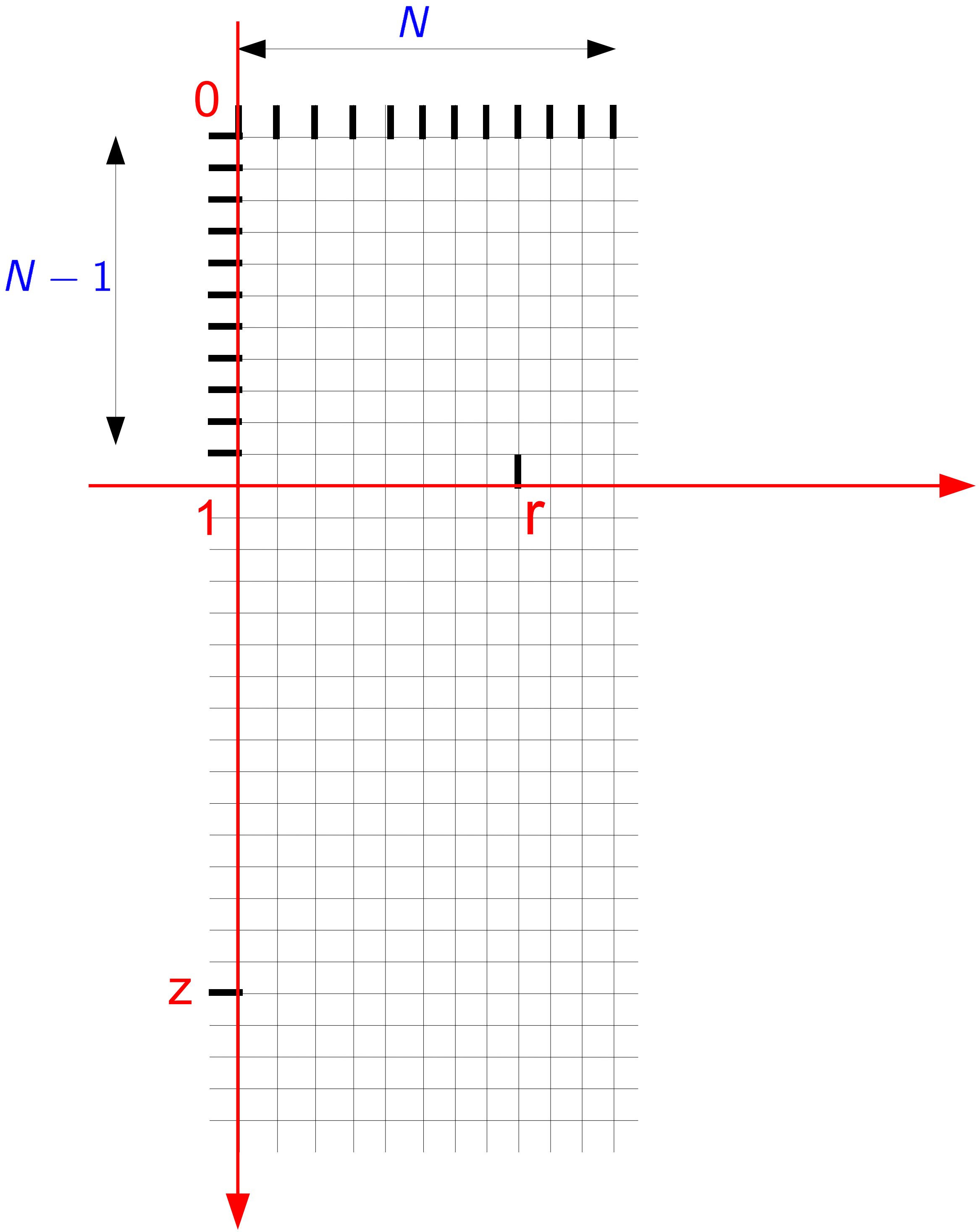


So the Arctic curve on the $N \times (N-1)$ lattice with a bottom heavy edge at site r is the usual Arctic plus a straight tangent line crossing the boundary at r.

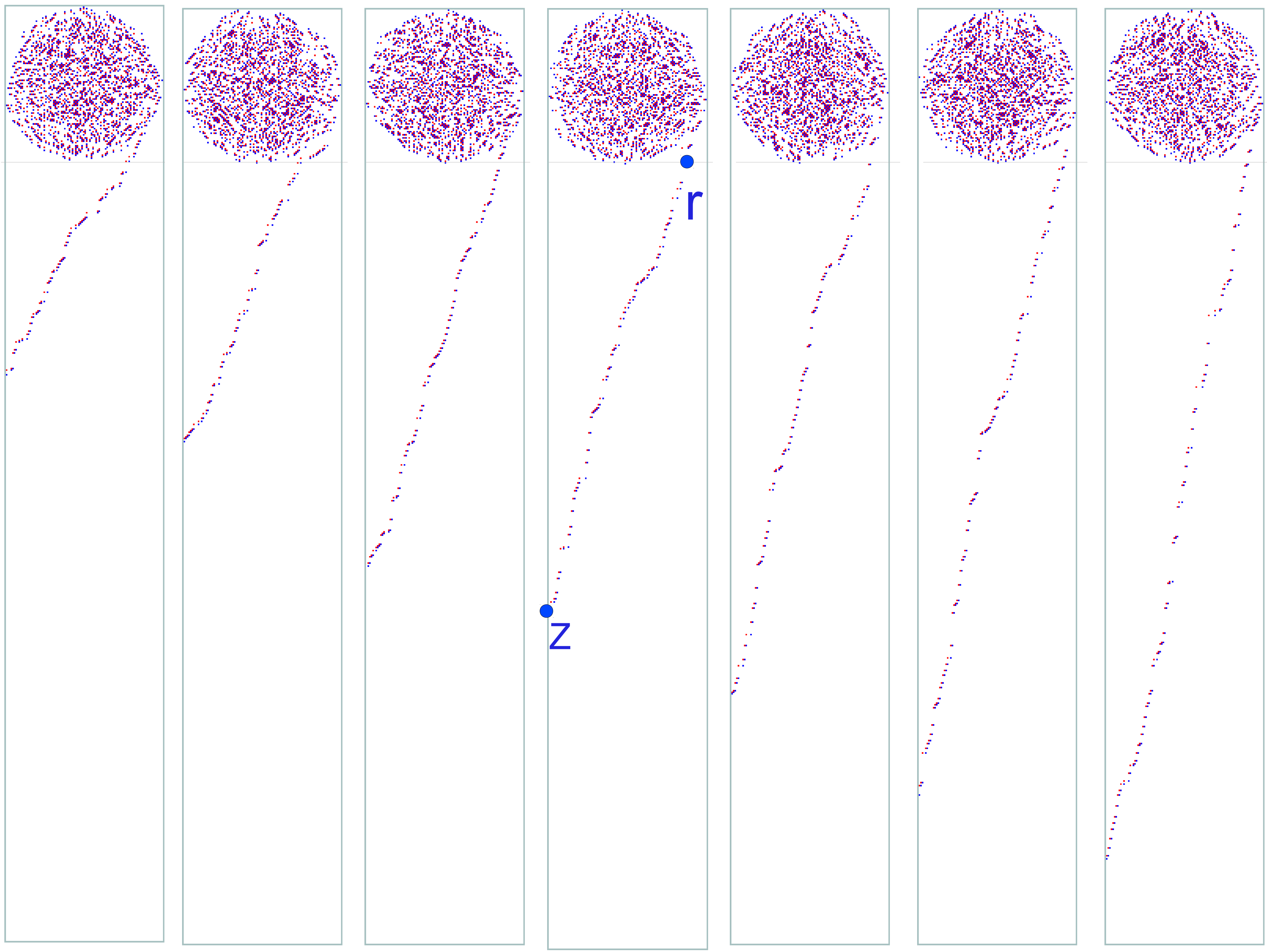
This is also observed in a variety of more general situation.

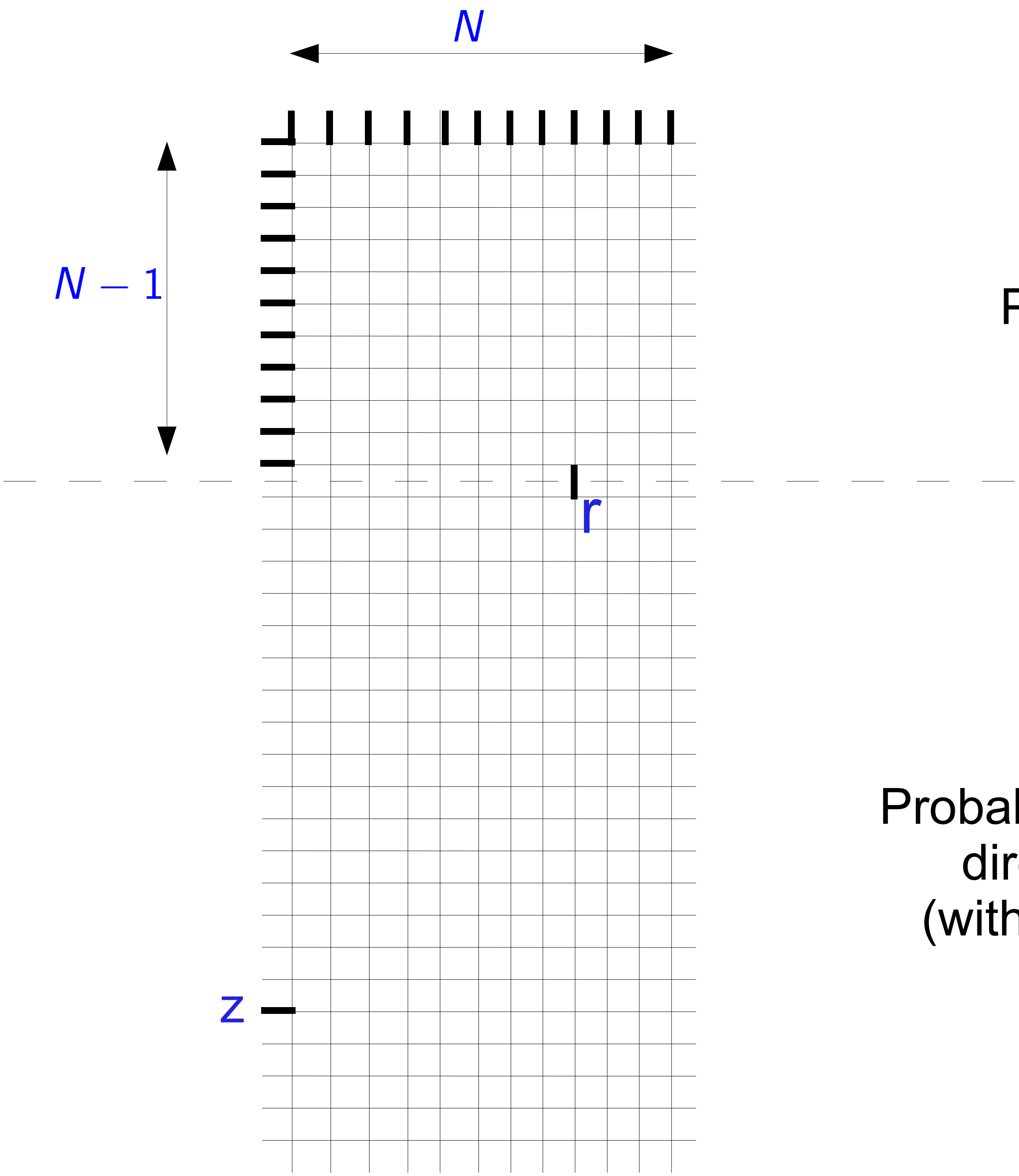
$N = 500$
$N' = 499$
$r = 450$


$\Delta = \frac{1}{2}$
900
899
101
100
$N = 1000$
$r = 100$
$s = 899$



N
0
N − 1
1
r
z


r
z



Probability $\propto$ $H_N^{(r)}$

Probability of having a weighted directed path from z to r (with weights given by 6VM)

Maximizing the above probability with respect to X , one obtains a family of straight lines, parameterized by z :

$$-x\frac{1}{z}+y\frac{(t^2-2\Delta t+1)}{(z-1)(t^2 z-2\Delta t+1)}+\lim_{N\to\infty}\frac{1}{N}\partial_z \ln h_N(z)=0\,, \qquad z\in[1,+\infty)$$

which we immediately recognize! The point is that this \`geometrical' construction interpretation holds for generic domains!

Thus on generic domains the problem of computing the Arctic curve is reduced to the evaluation of the (generating function of the) boundary correlation function, $h_N(z)$.

[FC-Sportiello, *to appear*]

NB: The above result has been checked in all possible cases where the explicit form of the function $h_N(z)$ is known, and appear to work also for other lattices (e.g., rhombi tilings of an hexagon, etc).

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$$-x\frac{1}{z}+y\frac{(t^2-2\Delta t+1)}{(z-1)(t^2z-2\Delta t+1)}+\lim_{N\to\infty}\frac{1}{N}\partial_z \ln h_N(z)=0\,, \qquad z\in[1,+\infty)$$

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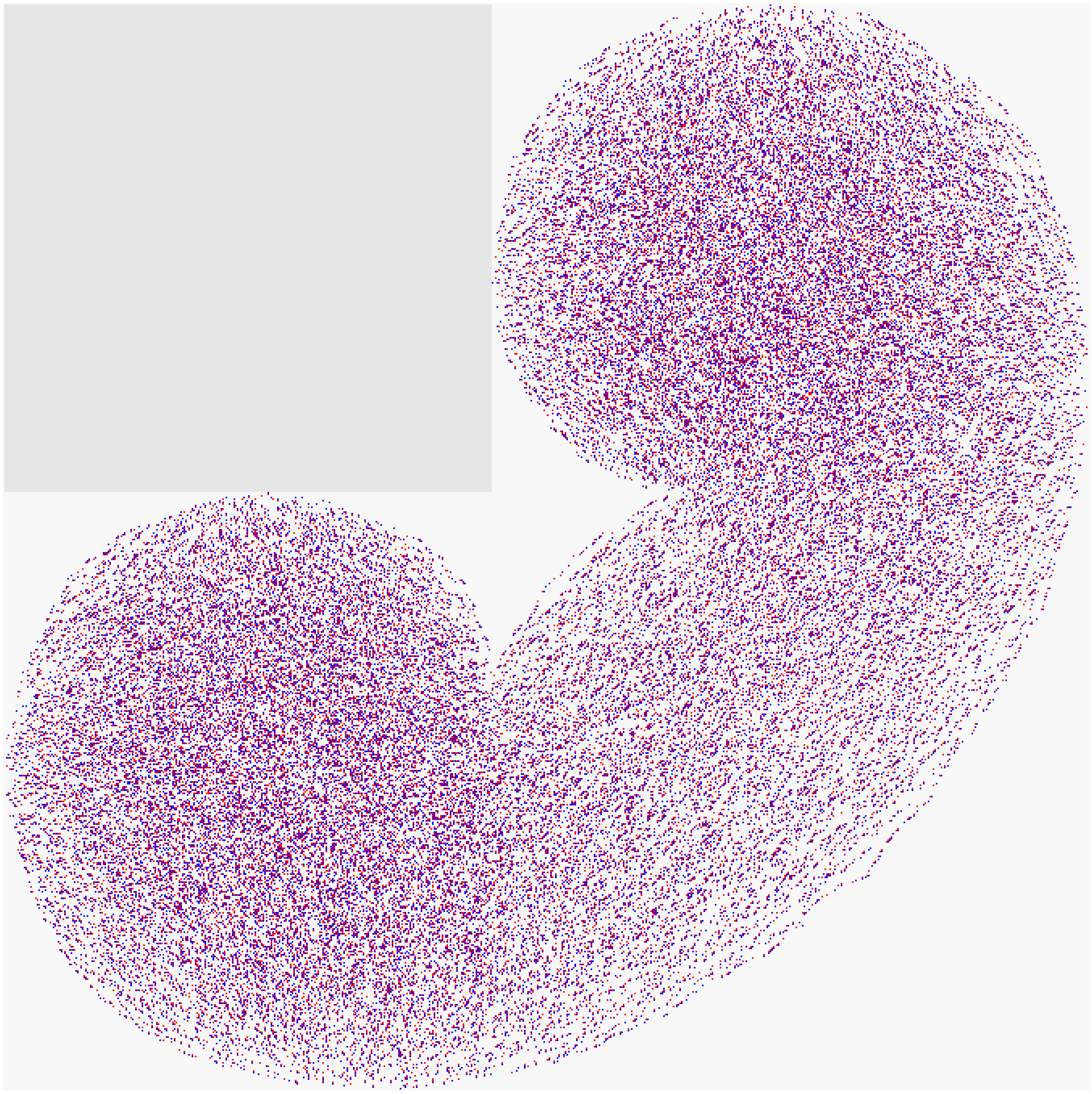
Thus on generic domains the problem of computing the Arctic curve is reduced to the evaluation of the (generating function of the) boundary correlation function, $h_N(z)$.

Conjecture:

For the six-vertex model with DWBC, for generic domains as above, the portion of Arctic curve adjacent to a given boundary of the domain is given simply by the Legendre transform of the corresponding one point boundary correlation function.

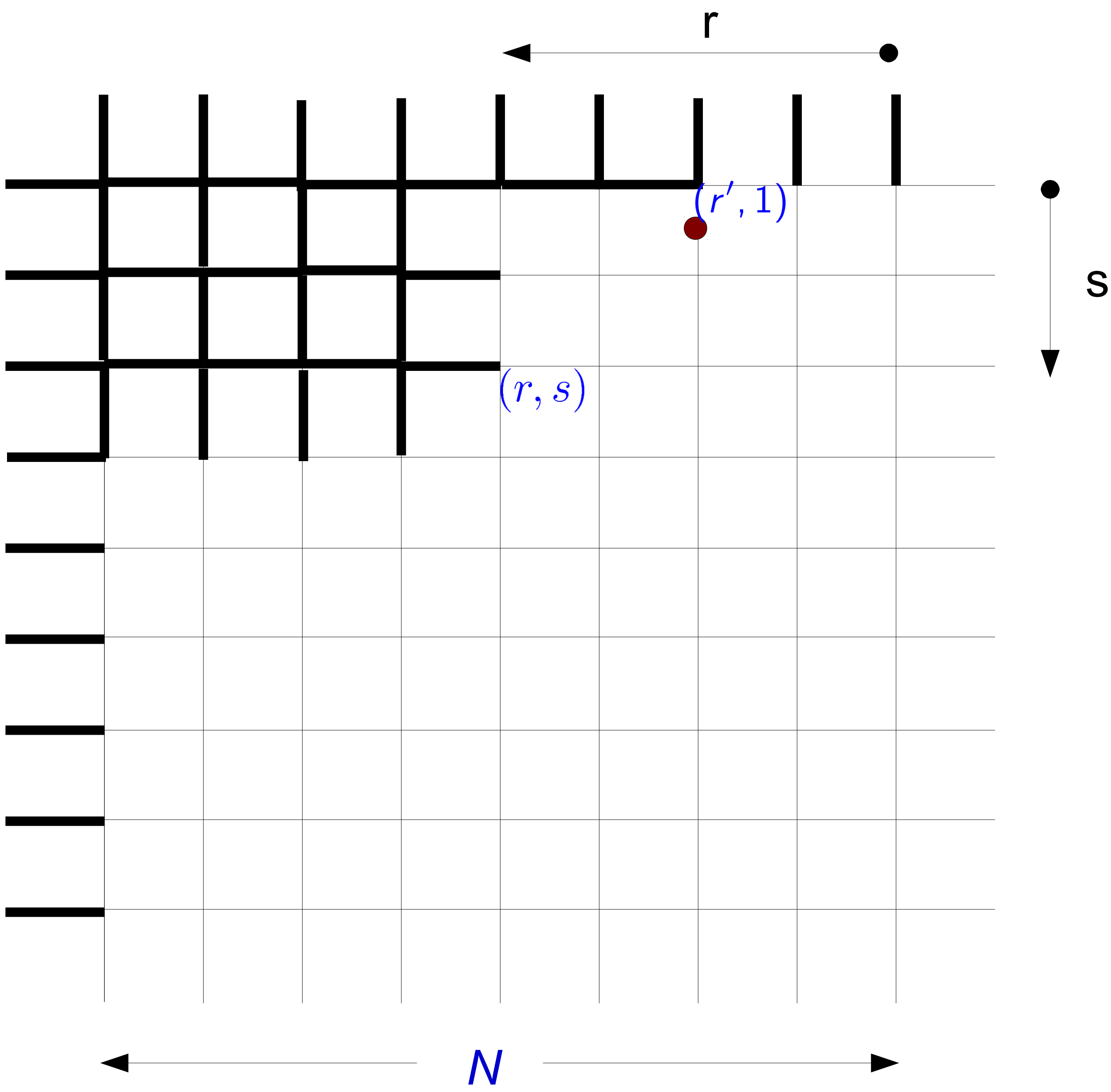
6VM

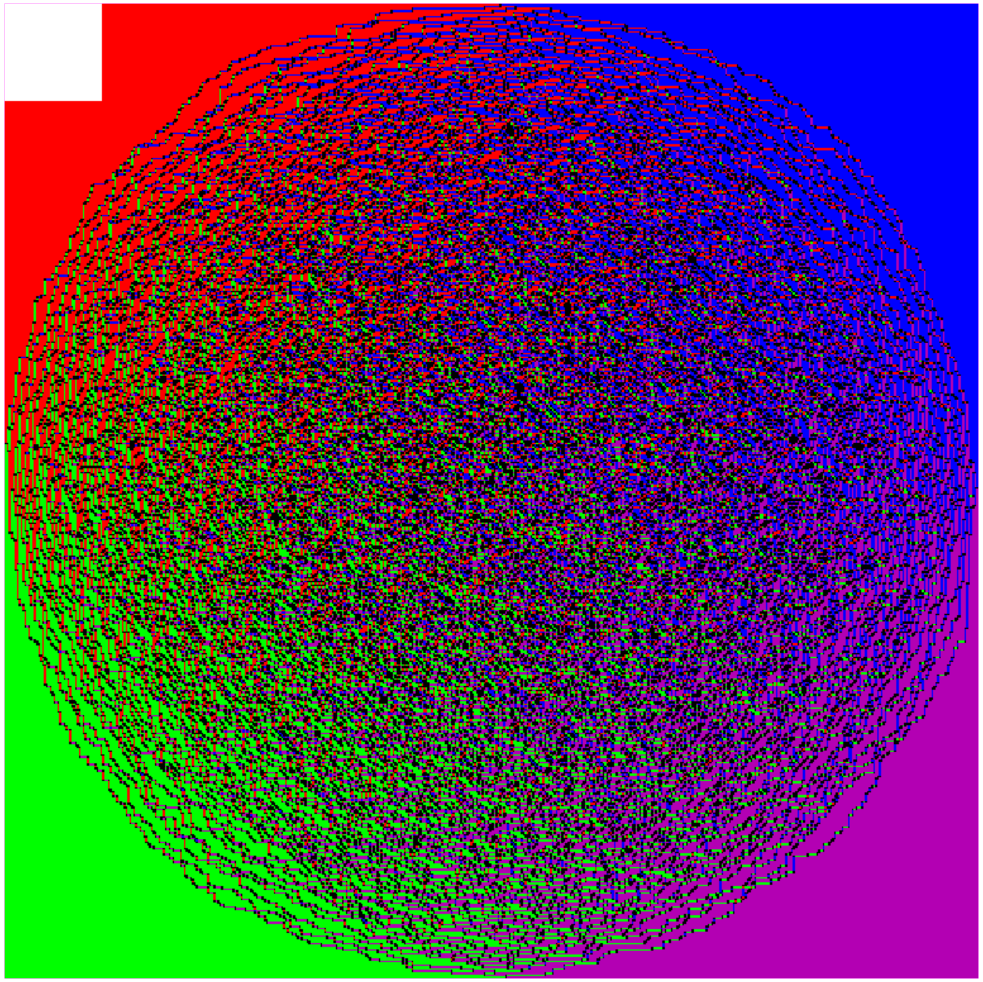
$\Delta = \frac{1}{2}$



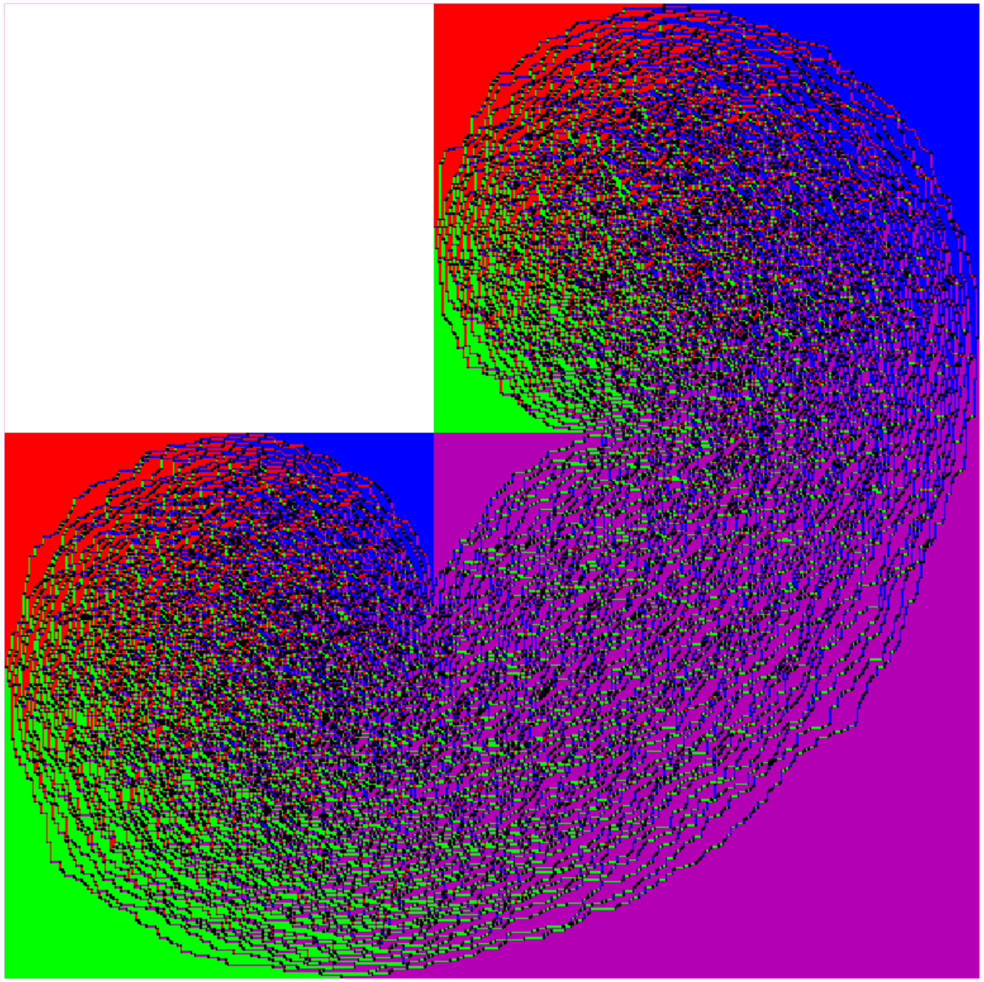
$N = 1000$
$r, s = 450$

Need for more sophisticated correlation functions





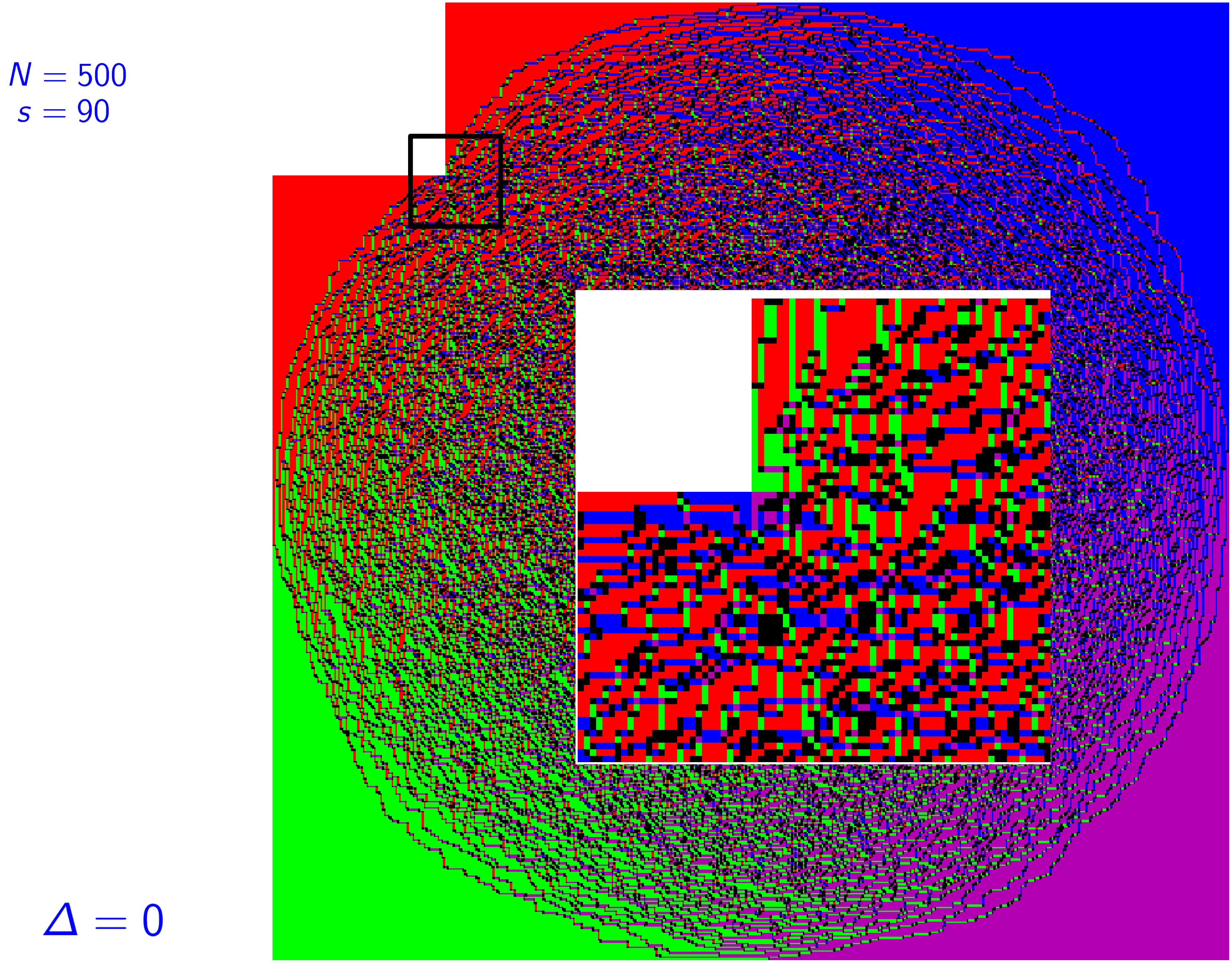
$N = 500$
$s = 50$



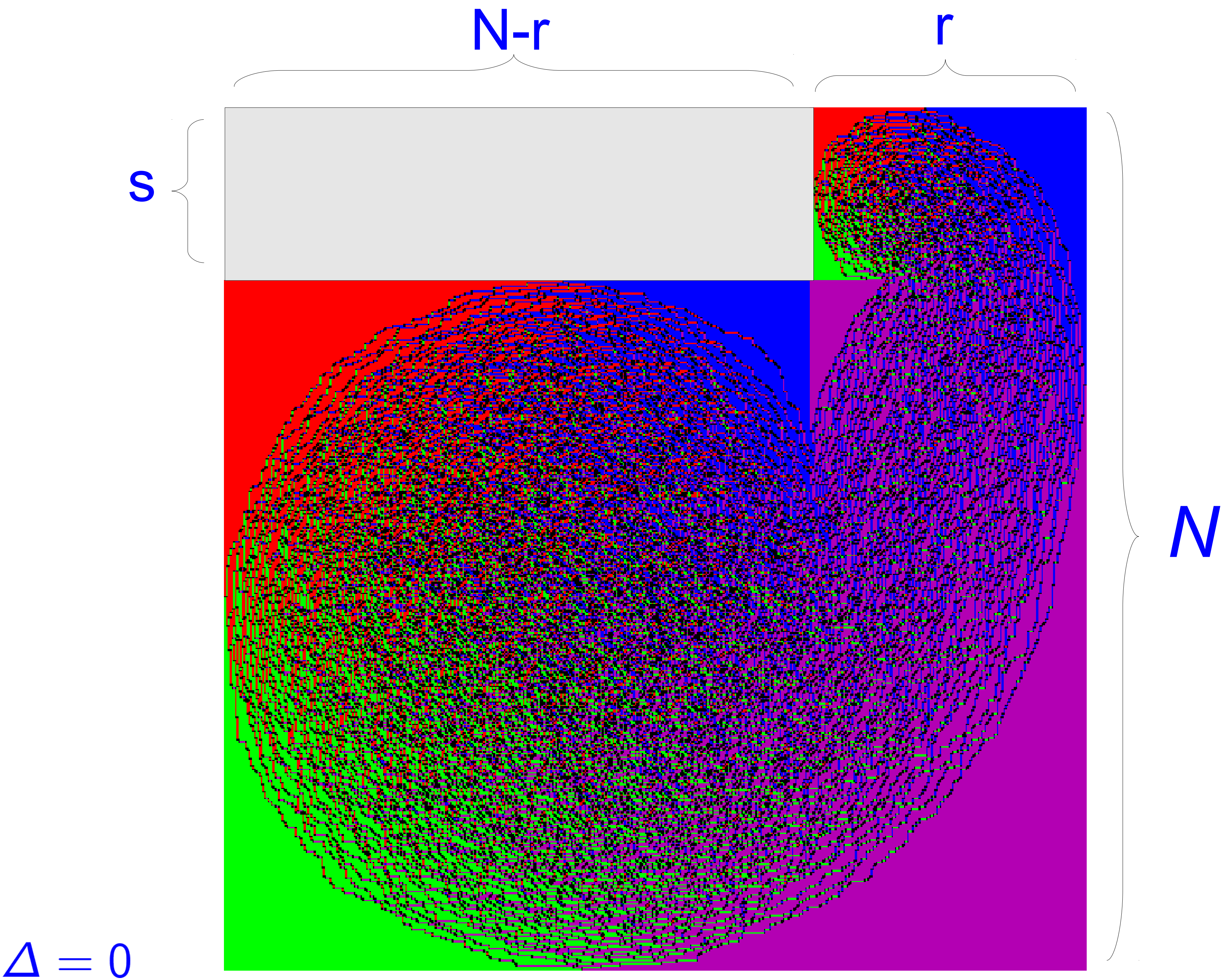
$N = 500$
$s = 220$

$\Delta = 0$

[FC-Sportiello]



$N = 500$
$s = 90$
$\Delta = 0$


N-r
r
s
N
$\Delta = 0$

Theorem: [FC-Pronko'13]

At $\Delta = 0$, in the scaling limit: [FC-Pronko'15]

$$r, s, N \to, \qquad \frac{N-r}{N} = x, \qquad \frac{s}{N} = y$$

the free energy per site of the model is:

$\sigma(x, y) = 0$ outside the original arctic curve

$\sigma(x, y) = \tilde{\sigma}(x, y)$ inside the original arctic curve

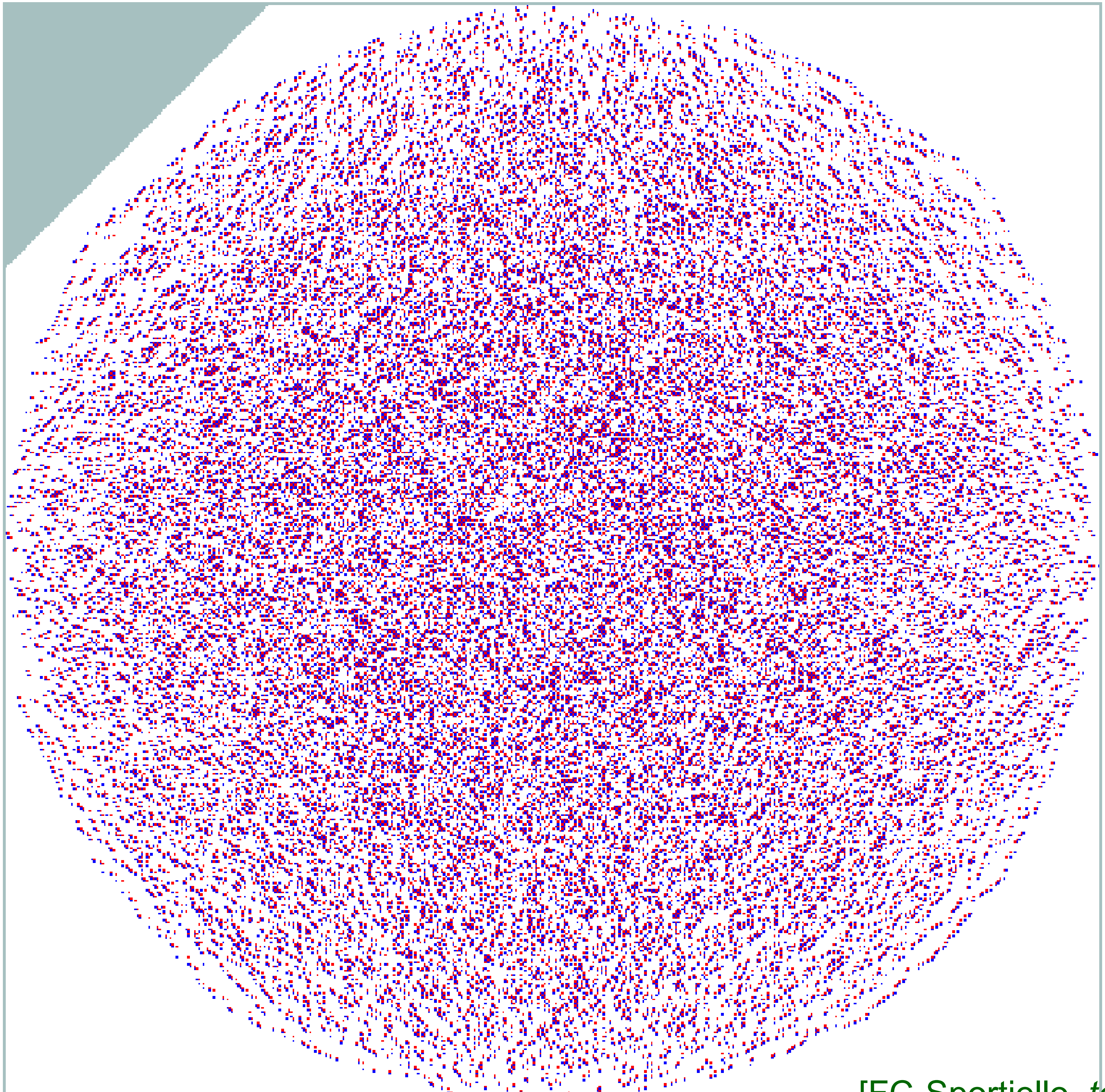
Moreover, near the Arctic curve, the function $\tilde{\sigma}(x, y)$ vanishes as ϵ^3, where ϵ is the distance from the ellipse (from inside)

Thus the model undergoes a third-order phase transition.

$\tilde{\sigma}(x, y)$ is the lower tail rate function [Johansson'00]

The calculation is based on some discrete Coulomb gas representation. The observed phase transition is somewhat related to [Gross-Witten'80], [Douglas-Kazakov'92], [Dean-Majumdar'06], but here the actual mechanism is different mechanism

$\Delta = 1/2$

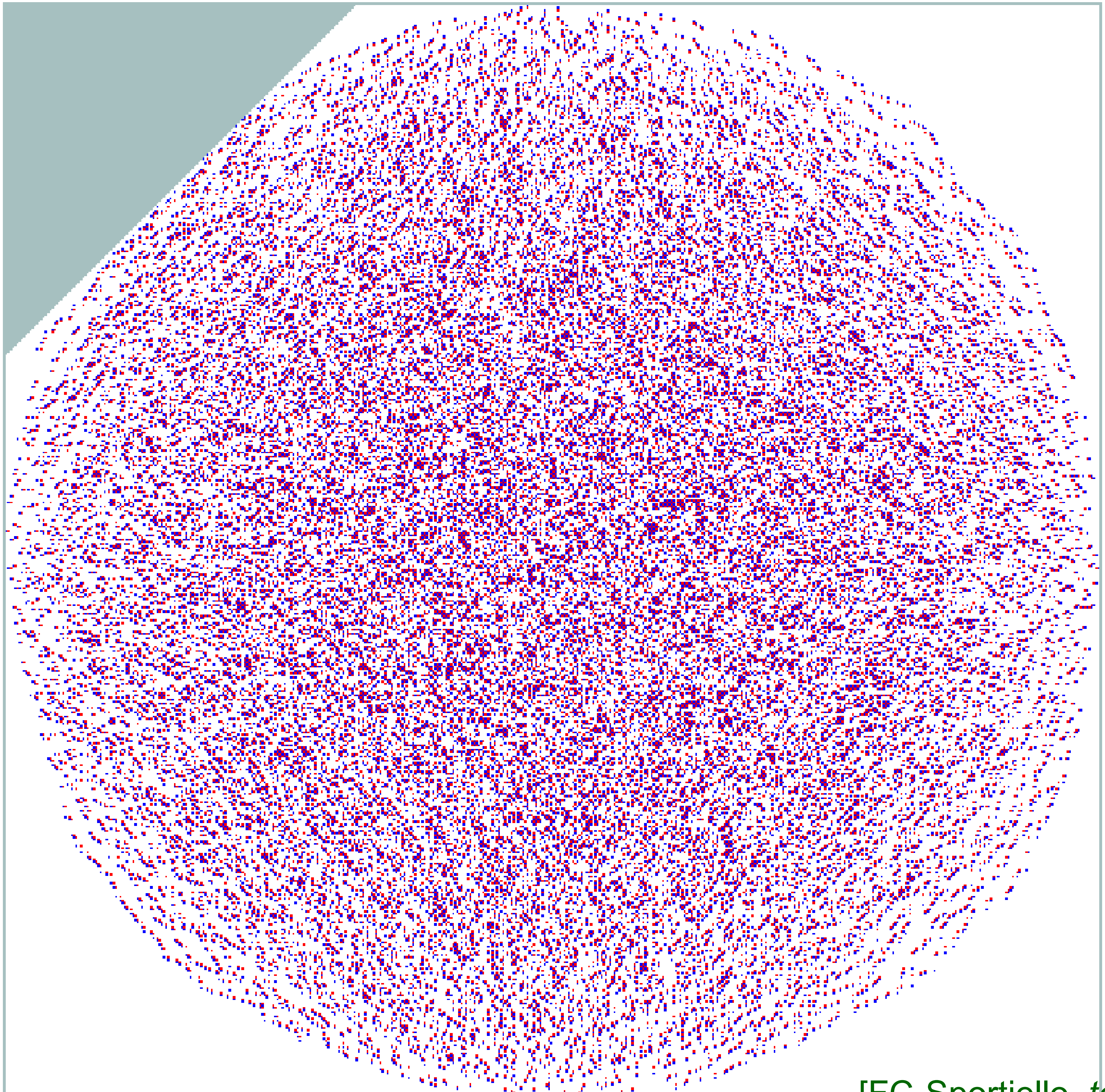


N=500
r=120

[FC-Sportiello, *to appear*]

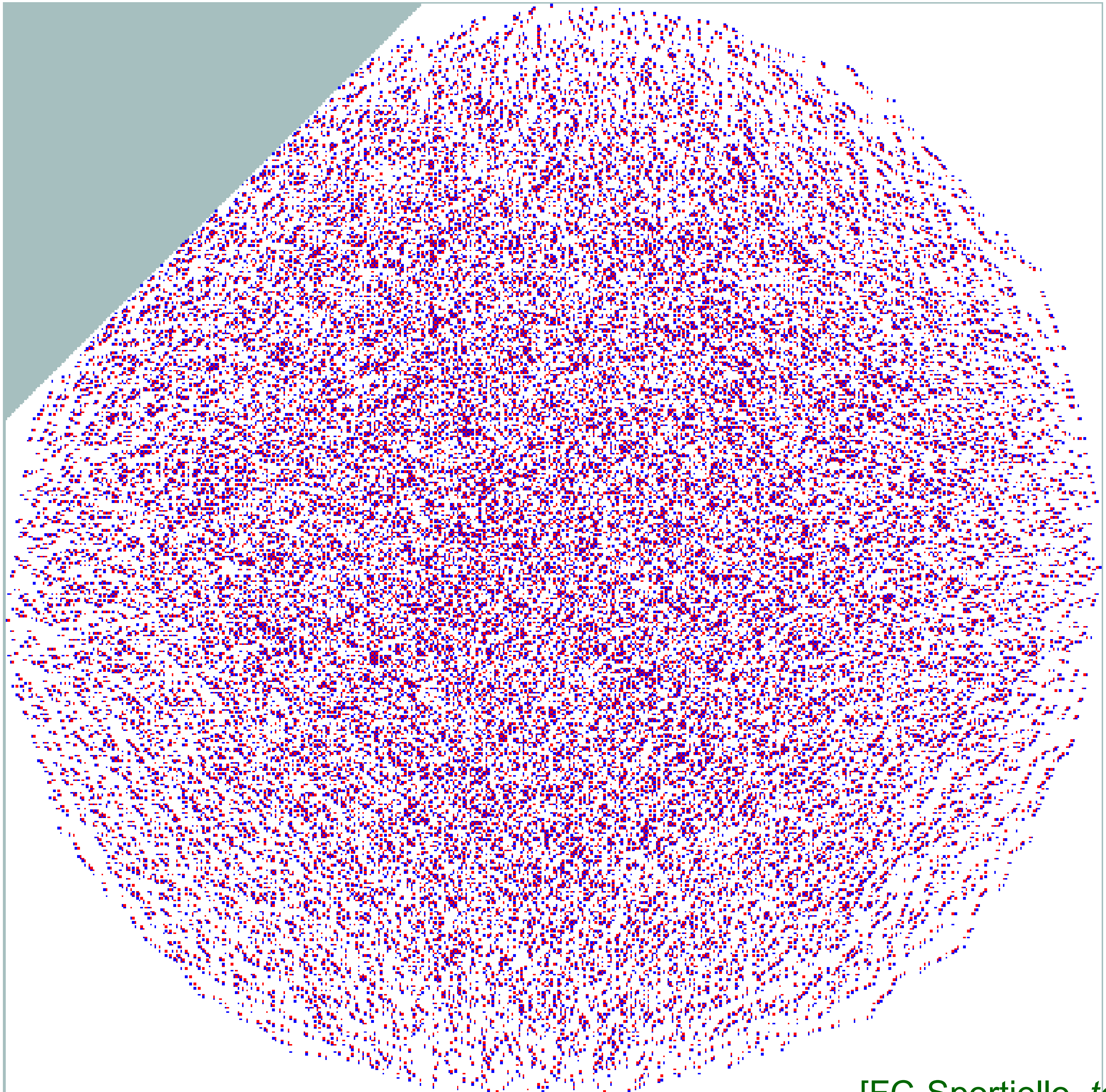
$\Delta = 1/2$

N=500
r=160



[FC-Sportiello, *to appear*]

$\Delta = 1/2$



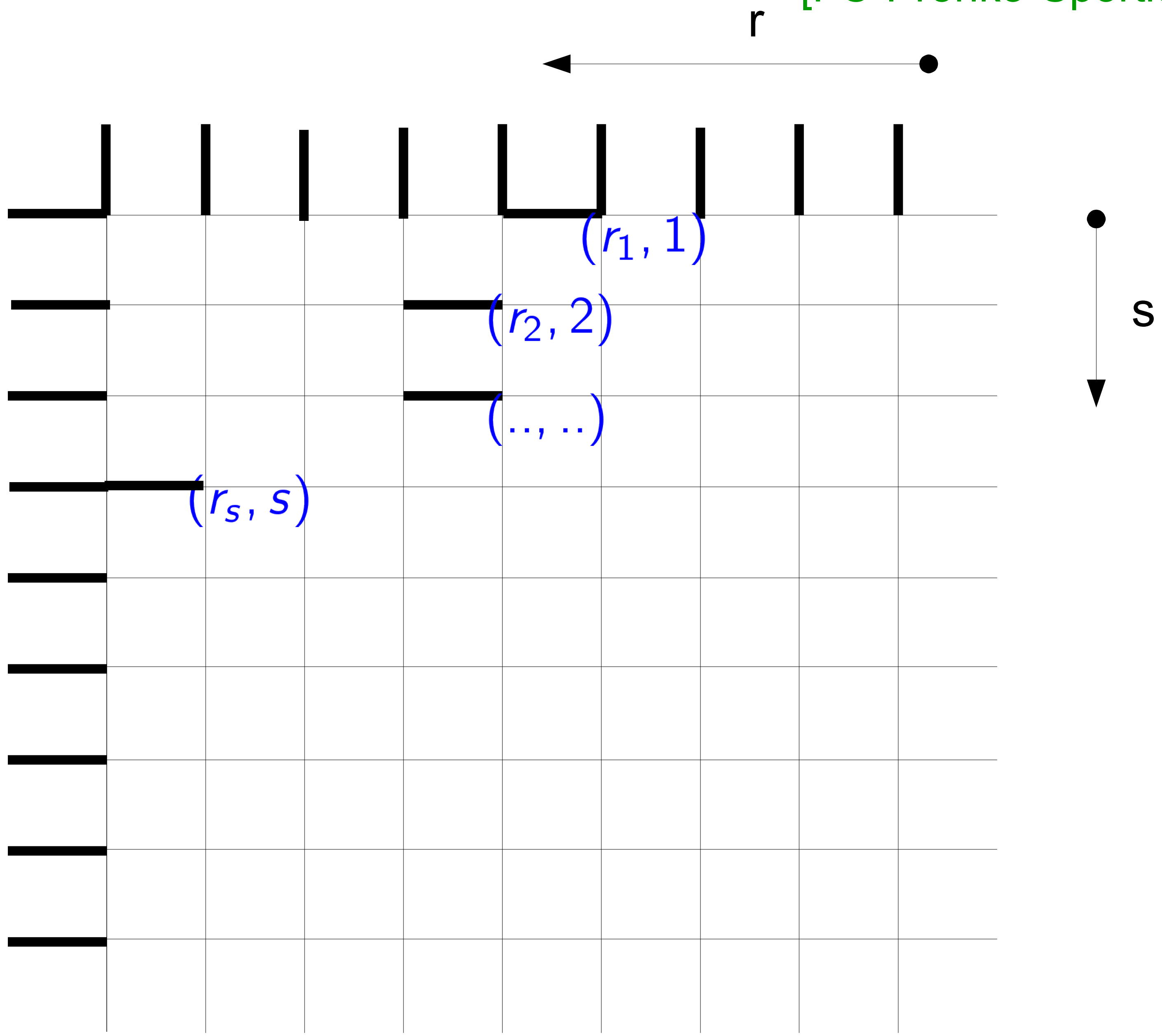
N=500
r=190

[FC-Sportiello, *to appear*]

Generalized Emptiness Formation Probability (GEFP)

[FC-Pronko-Sportiello, *to appear*]

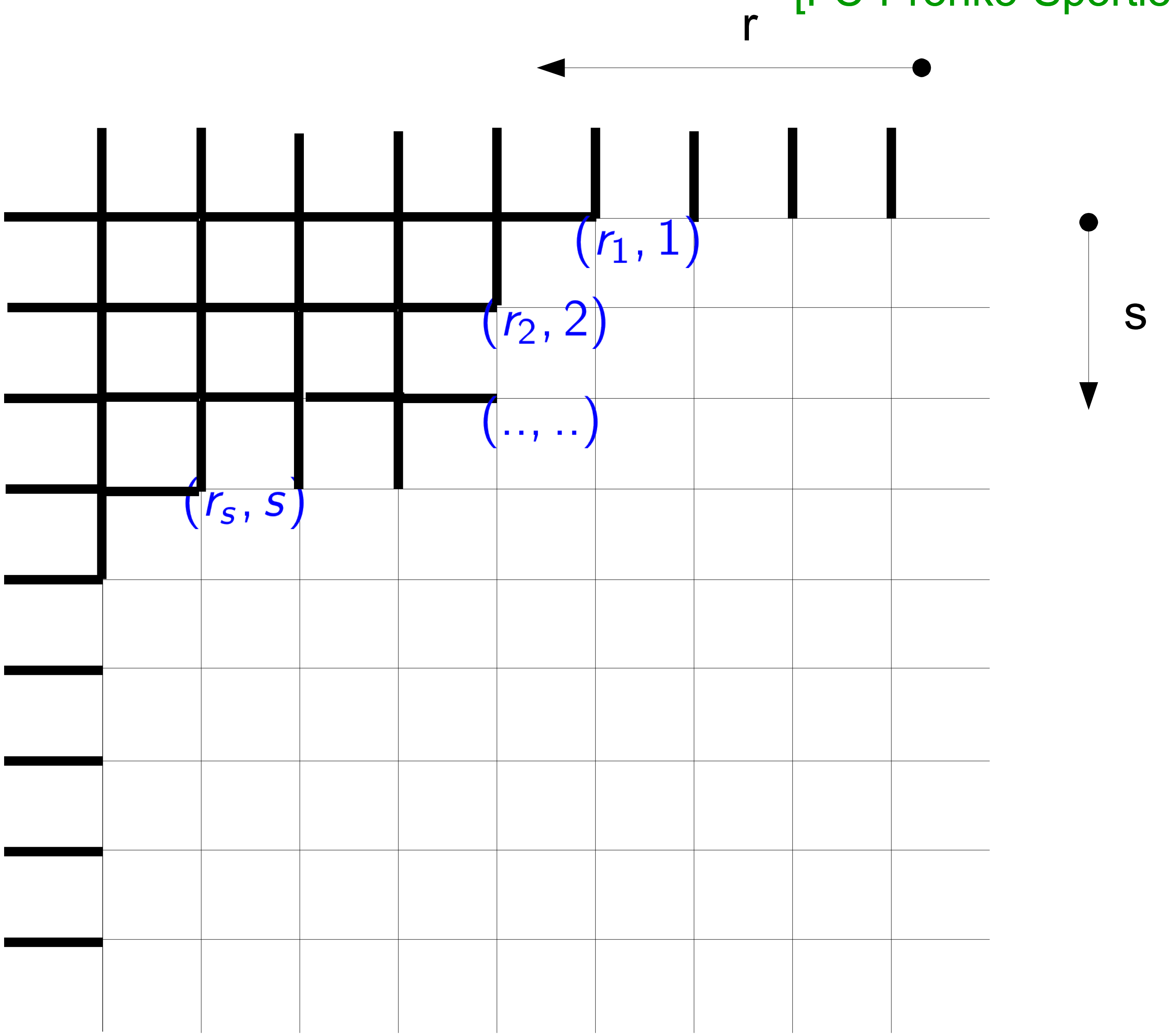
$G_{N,s}^{(r_1,\dots,r_s)}$



Generalized Emptiness Formation Probability (GEFP)

[FC-Pronko-Sportiello, *to appear*]

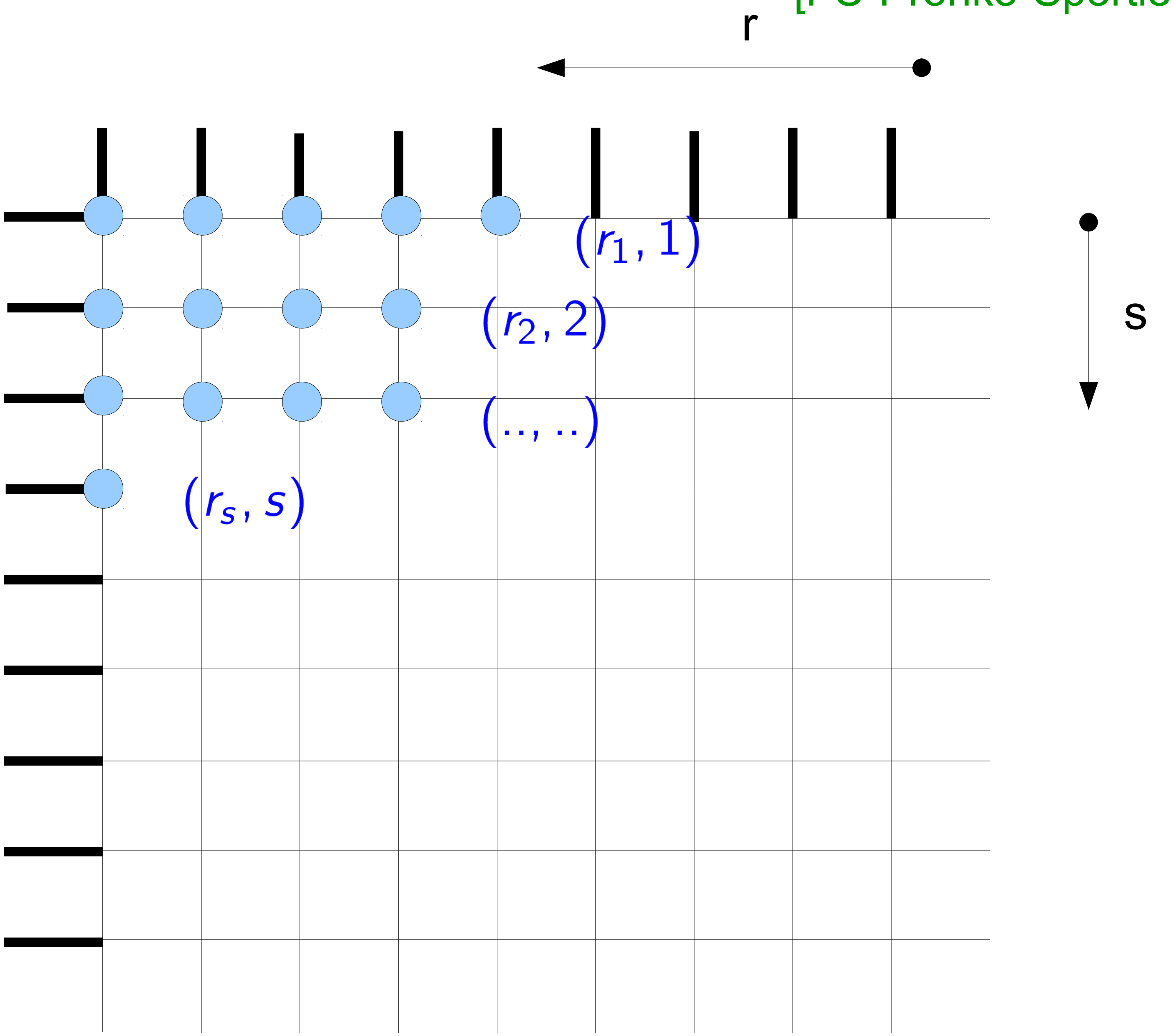
$G_{N,s}^{(r_1,\ldots,r_s)}$



Generalized Emptiness Formation Probability (GEFP)

[FC-Pronko-Sportiello, *to appear*]

$G_{N,s}^{(r_1,\dots,r_s)}$



Multiple Integral representation for GEFP

[FC-Pronko-Sportiello, *to appear*]

Theorem:

For $1_1 \leq r_2 \leq \cdots \leq r_s \leq N, \quad s = 1, 2, \ldots, N$, the following Multiple Integral Representation holds: :

$$G_{N,s}^{(r_1,\ldots,r_s)} = \frac{(-1)^s}{2\pi \mathrm{i}} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{[(t^2 - 2\Delta t) z_j + 1]^{s-j}}{z_j^{r_j} (z_j - 1)^{s-j+1}}$$

$$\times \prod_{1 \leq j < k \leq s} \frac{z_j - z_k}{t^2 z_j z_k - 2\Delta t z_j + 1} \, h_{N,s}(z_1, \ldots, z_s) \, \mathrm{d}z_1 \cdots \mathrm{d}z_s .$$

To be compared with that for EFP (corresponding to $r_1 = r_2 = \cdots = r_s \equiv r$):

$$G_{N,s}^{(r_1,\ldots,r_s)} = \frac{(-1)^s}{2\pi \mathrm{i}} \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{[(t^2 - 2\Delta t) z_j + 1]^{s-j}}{z_j^{r} (z_j - 1)^{s-j+1}}$$

$$\times \prod_{1 \leq j < k \leq s} \frac{z_j - z_k}{t^2 z_j z_k - 2\Delta t z_j + 1} \, h_{N,s}(z_1, \ldots, z_s) \, \mathrm{d}z_1 \cdots \mathrm{d}z_s .$$