From Random to Quasiperiodic Tilings

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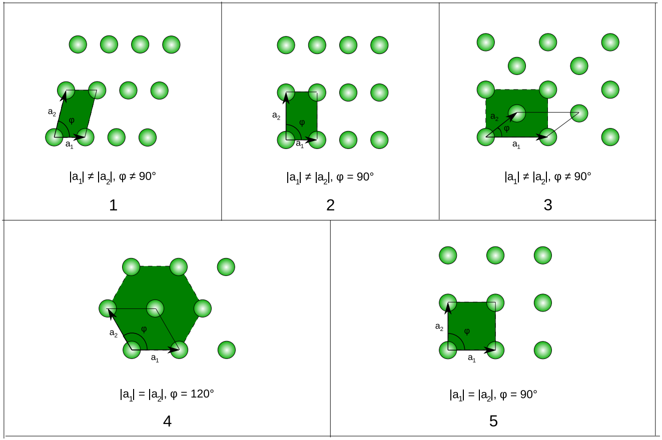
Inhomogeneous Random Systems
January 28, 2015

Outline

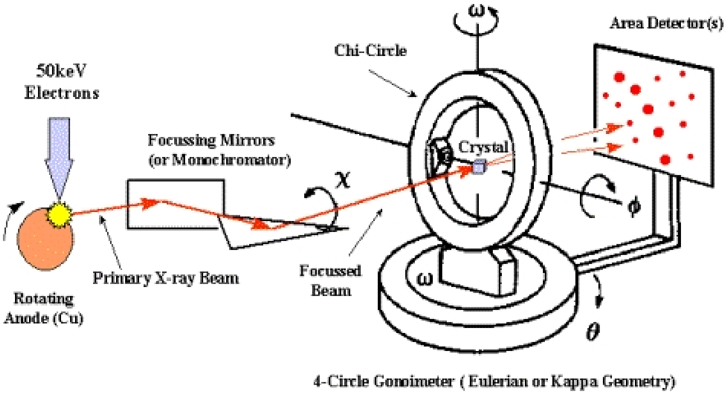
Outline

Crystals ($\sim$ 1900)



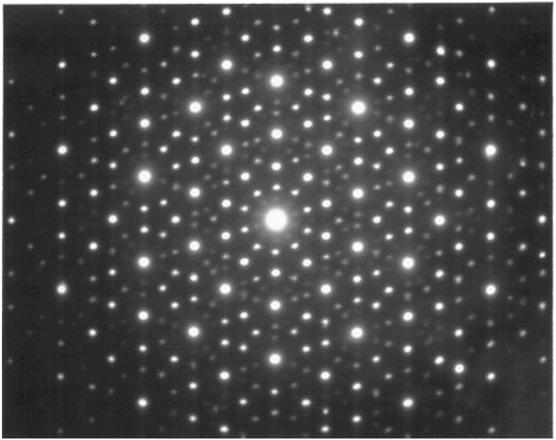
Crystal = ordered material = lattice + atomic pattern.

X-ray Diffraction (1912)



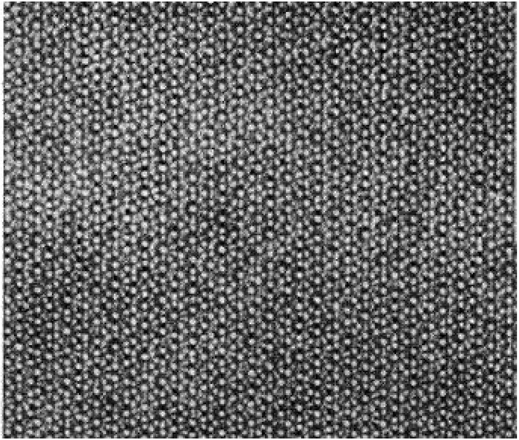
Crystal structure studied by X-ray diffraction.

Troubled times (1982–1992)



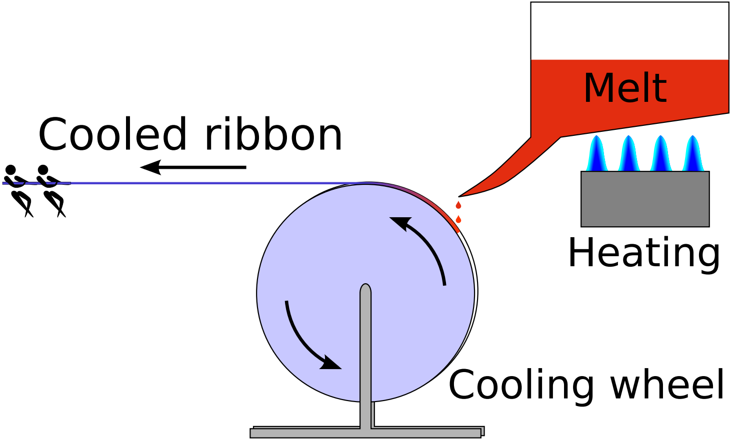
“Forbidden” ten-fold symmetry discovered $\rightsquigarrow$ quasicrystals.

HRTEM



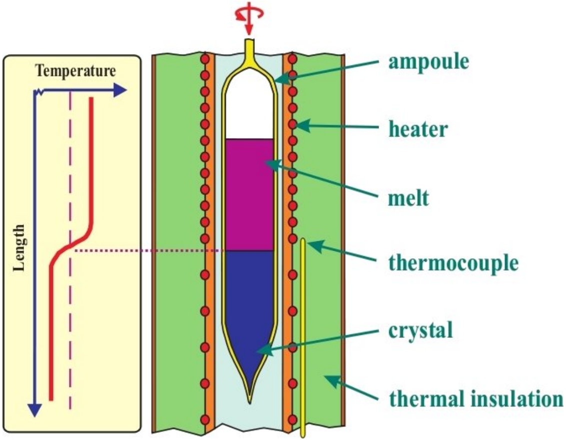
Transmission Electron Microscopy eventually showed the structure.

Quenching



First quasicrystals: rapid cooling from the melt. Many defects.

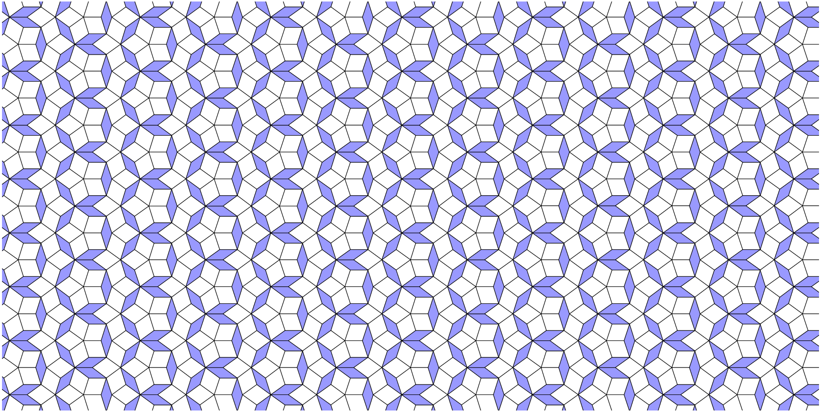
Bridgman-Stockbarger



Today quasicrystals: slow cooling from the melt. Less defects.

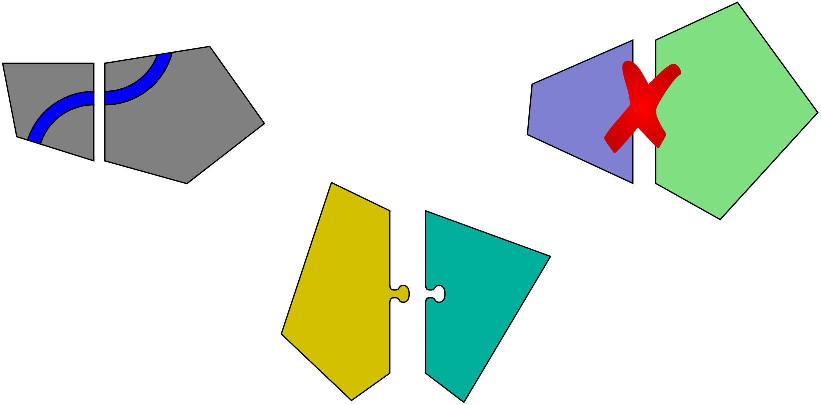
Outline

Tilings



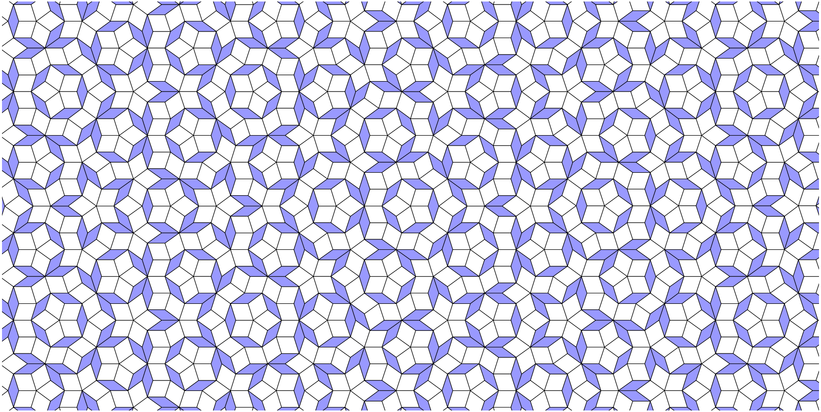
Covering of the space by interior-disjoint compacts called *tiles*.

Local rules



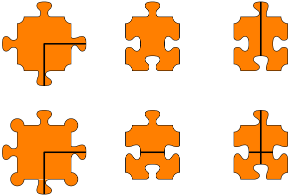
Inter-atomic energetic interaction $\rightsquigarrow$ constraints on neighbor tiles.

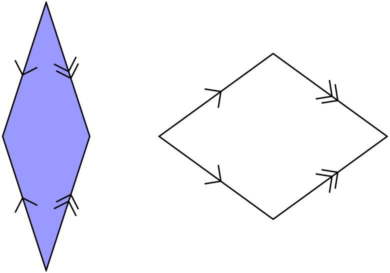
Quasiperiodic tilings



A pattern reoccurs at uniformly bounded distance from any point.

Aperiodic tilings (1964)

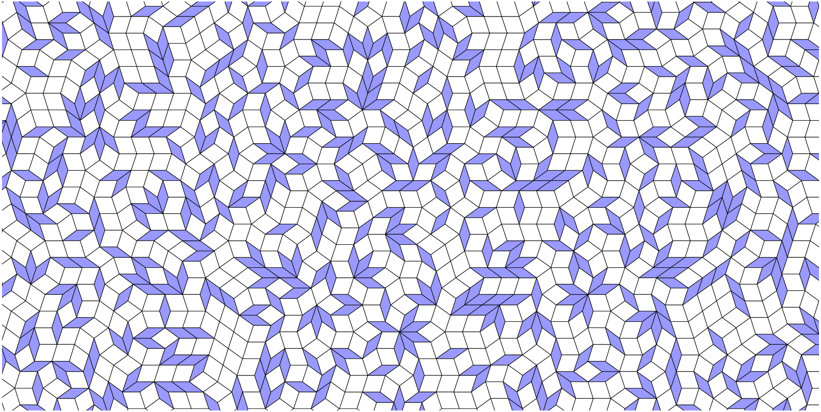




Aperiodic tile set: finite tile set that forms only non-periodic tilings.

It can always form quasiperiodic tilings.

Random tilings



At high T, entropy maximization supersedes energy minimization.

Cooling

General principle (*cf* Bridgman-Stockbarger method):

- start from an entropy maximizing tiling;
- perform local moves with proba. $\min(1, \exp(-\Delta E/T))$;
- progressively decrease T while still moving;
- hope to eventually minimize energy.

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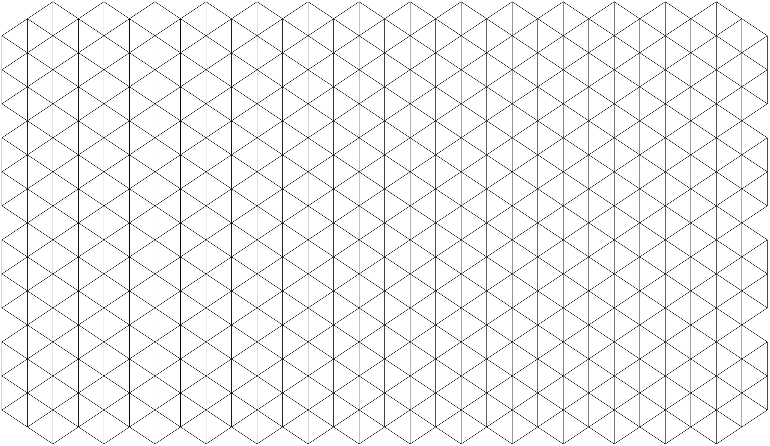
Directions:

- characterize entropy maximizing tilings;
- find suitable local moves;
- find a suitable cooling schedule;
- manage to prove something non-trivial. . .

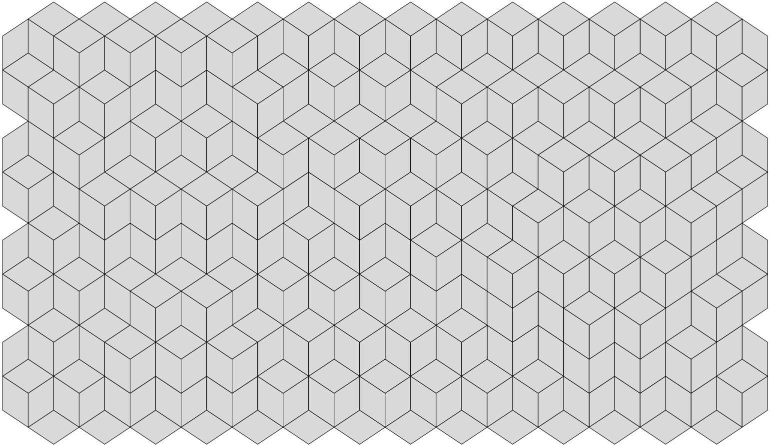
Outline

Dimer tilings



Perfect matchings of planar bipartite graphs.

Dimer tilings



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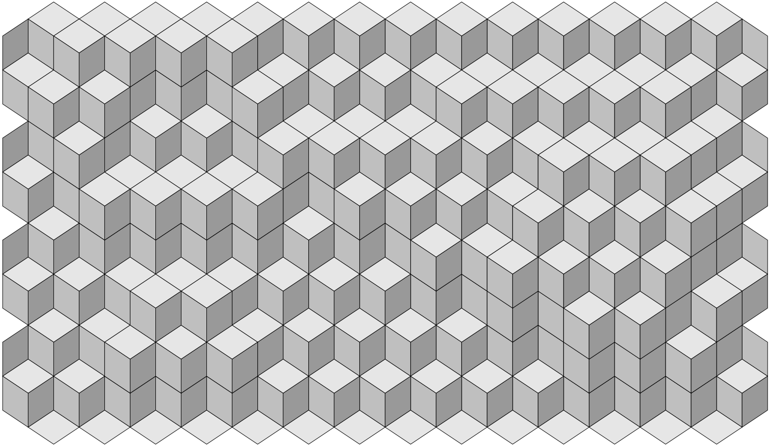

Perfect matchings of planar bipartite graphs. With height function.

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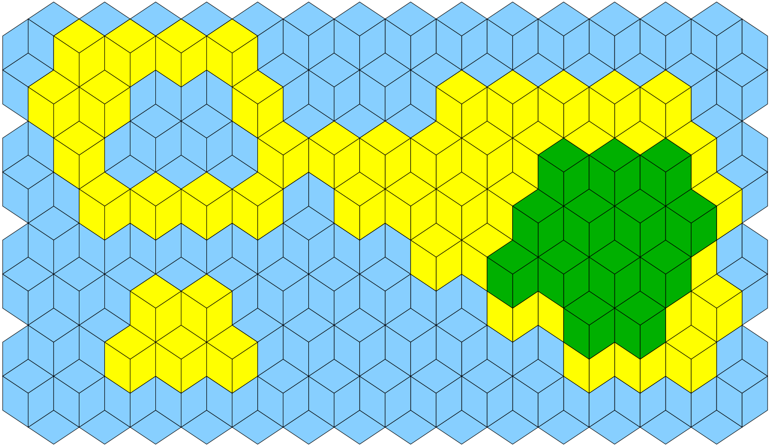

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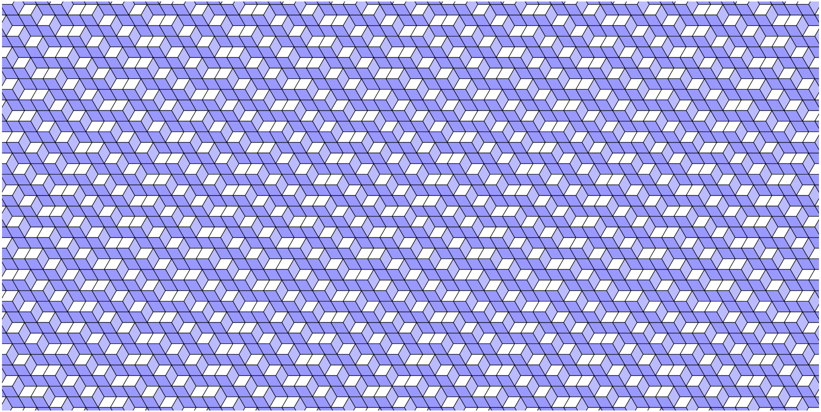
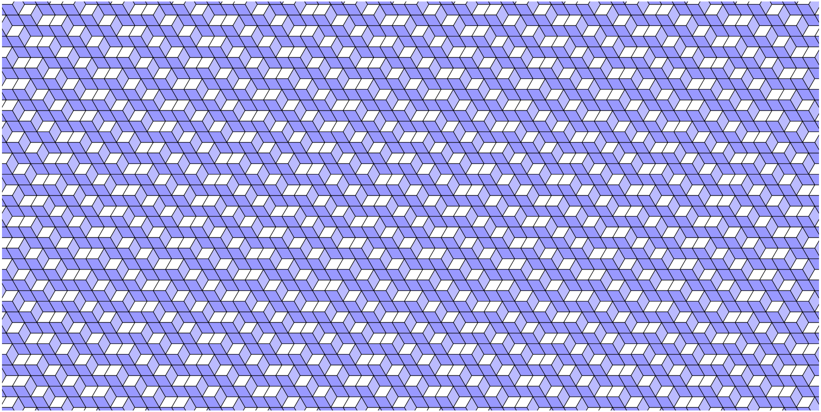
Quasiperiodic tilings



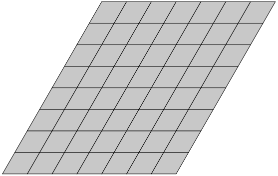
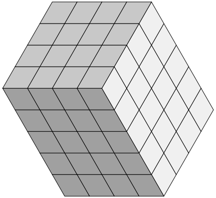
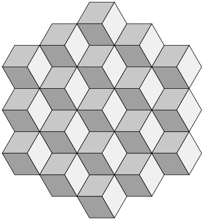
A tiling with lift in $P + [0, 1]^3$, P irrational plane, is quasiperiodic.

Aperiodic tilings



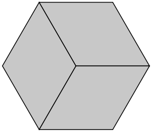
Tiles *must* be decorated (a thousand tiles seems not too much. . .)

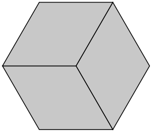
Random tilings

Entropy: $\dfrac{\log(\#\text{tilings})}{\#\text{tiles}}$. Maximum: Cohn-Kenyon-Propp, 2001.

Flip & local rules



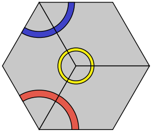


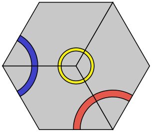
Classic local move on dimer tilings: *flip*.

Flip & local rules

But how potential decorations are affected by performing a flip?

Flip & local rules



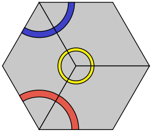


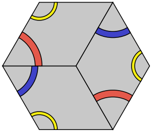
Is a flip a rotation?

Flip & local rules

Is a flip a rotation? A reflexion?

Flip & local rules



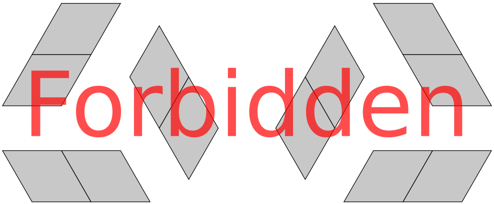


Is a flip a rotation? A reflexion? A tile exchange?

Flip & local rules

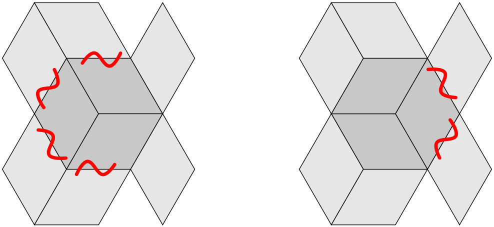
Is a flip a rotation? A reflexion? A tile exchange? A mix of this?

Flip & local rules



Problem avoided if the local rules are forbidden patterns.

Flip & local rules



To perform a flip, consider the forbidden patterns it is involved in.

Cooling

Perform flips which proba. $\min(1, \exp(-\Delta E/T))$. Mixing time τ?

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Mostly studied case: infinite temperature (no local rules).

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- $\tau = cn^2 \log(n)$ tower-flips (Wilson, 2004)
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- $\tau = O(n^2 \log^c(n))$ flips (Caputo-Martinelli-Toninelli, 2011)
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Other temperatures? Cooling schedule?

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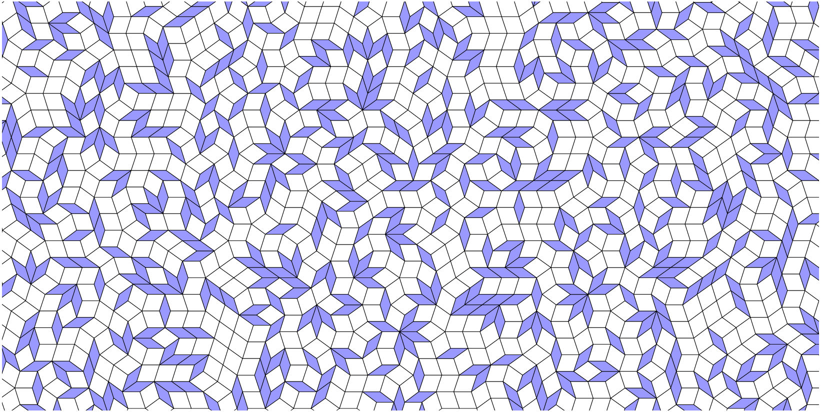
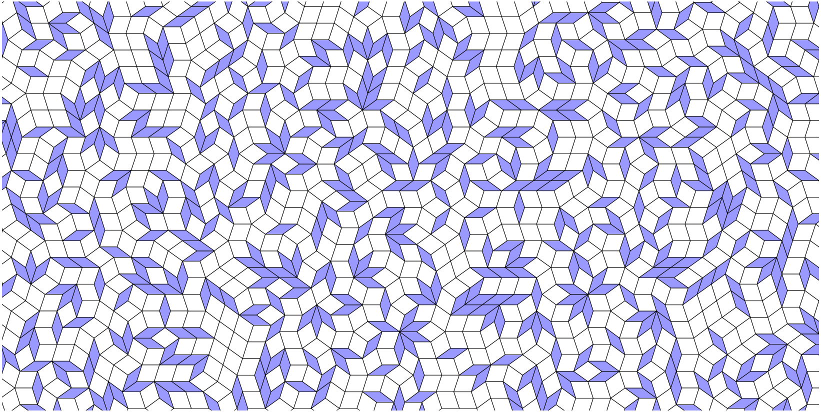
Rhombus tilings



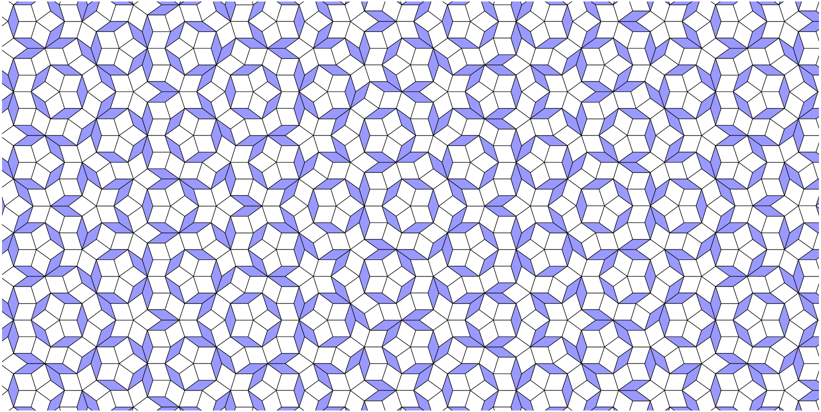
n non-colinear unit vectors in $\mathbb{R}^2 \rightsquigarrow \binom{n}{2}$ rhombi $\rightsquigarrow$ tiling of $\mathbb{R}^2$.

Rhombus tilings



Vectors as projections of a basis of $\mathbb{R}^n \rightsquigarrow$ tiling lifted in $\mathbb{R}^n$.

Rhombus tilings



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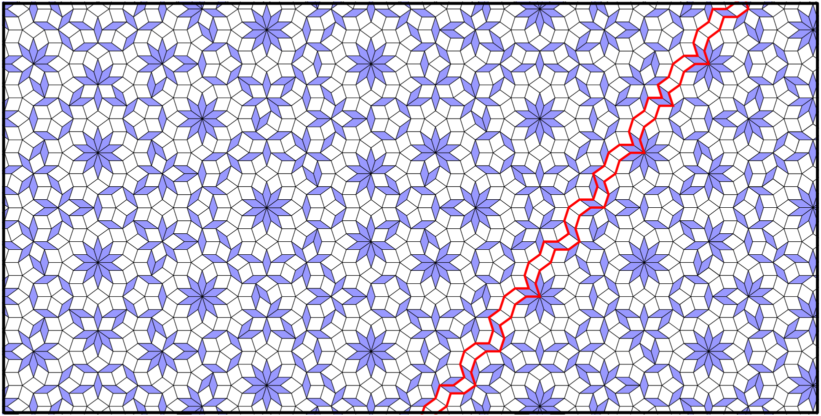
When only forbidden patterns are allowed:

- no local rules for non-algebraic planes (Le, 1995)
- sufficient conditions (Levitov, Le, Socolar, Bédaride-F.)
- local rules *iff* the plane is characterized by its *subperiods*?

Example 1: (generalized) Penrose tilings (1974)



Lift in $P+[0,1]^5$, $\vec{P}=\mathbb{R}\cos(2k\pi/5)_{0\leq k<5}+\mathbb{R}\sin(2k\pi/5)_{0\leq k<5}$.

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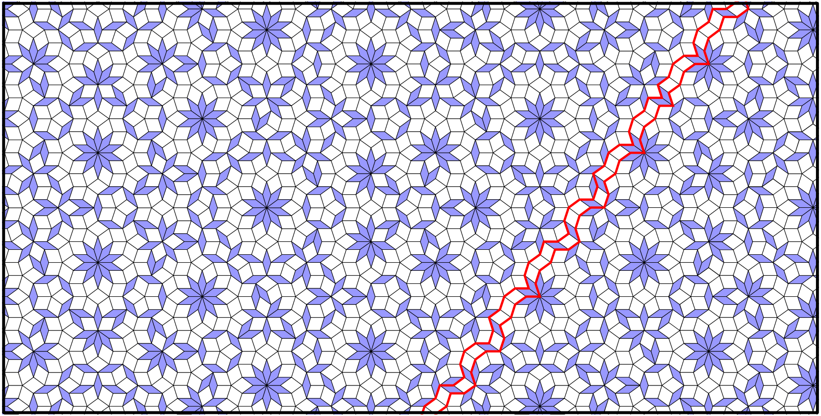

Lift in $P + [0,1]^5$, $\vec{P} = (\varphi, 1, -1, -\varphi, \varphi, 1, -1, \varphi, 1, \varphi)$.

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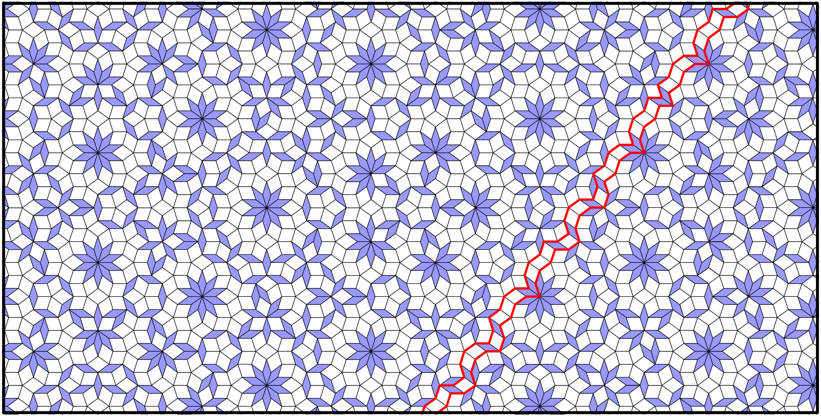

Characterized by an *alternation condition* (Socolar, 1990)

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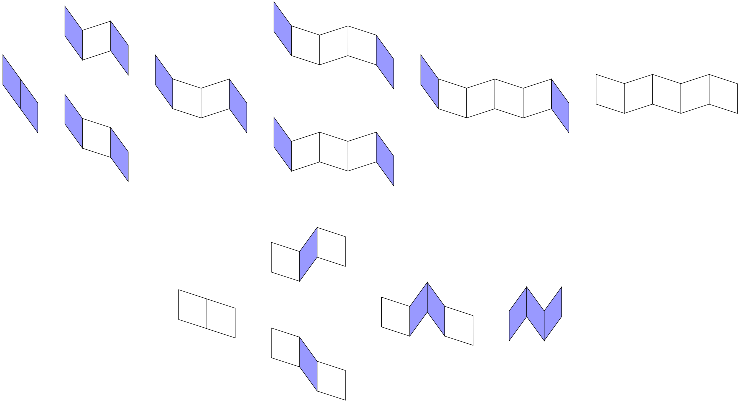
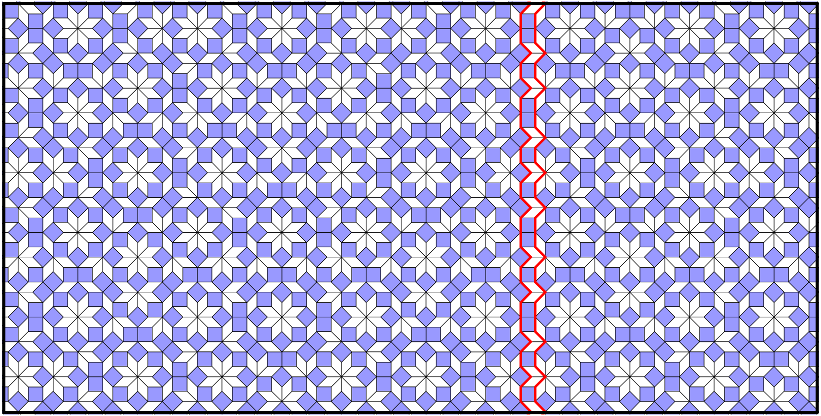


⇝ admits local rules defined by finitely many forbidden patterns.

Example 2: Ammann-Beenker tilings (1970's-1982)



Lift in $P + [0,1]^5$, $\vec{P} = (1, \sqrt{2}, 1, 1, \sqrt{2}, 1)$. Alternation condition?

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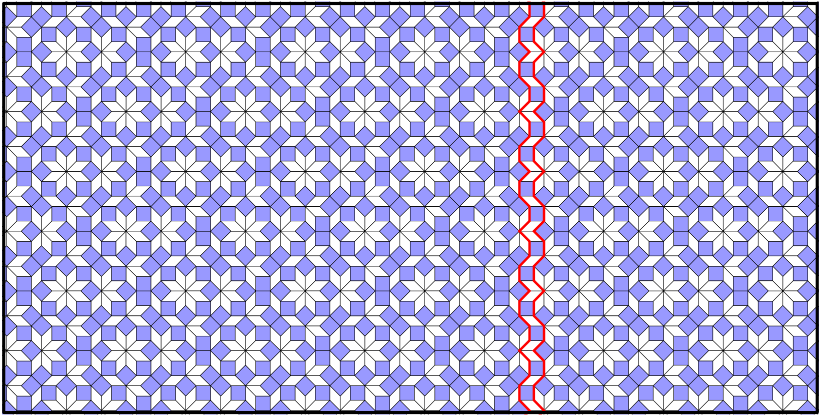

Enforces planarity, but allows any $\vec{P}_t = (1, t, 1, 1, 2/t, 1)$, $t > 0$.

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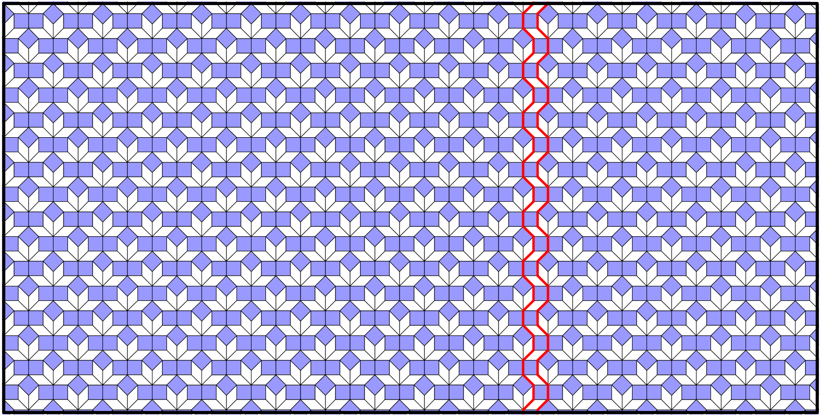

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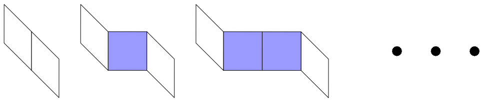

Forbidden patterns enforce planarity; boundary enforces the slope.

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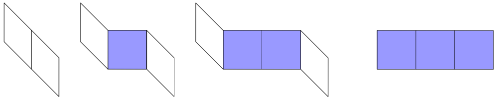

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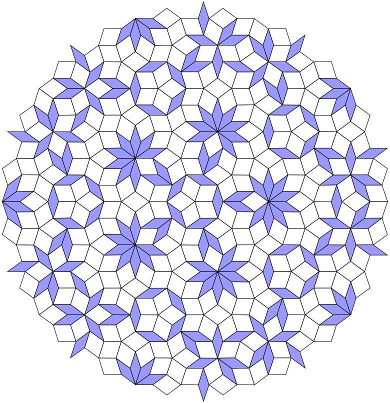
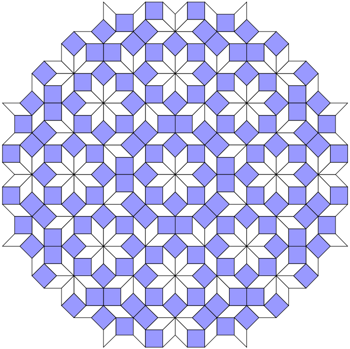
Random tilings





Do Penrose or Ammann-Beenker tilings maximize entropy?

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Simulations suggest that τ is the same as for dimers:

- $\tau = O(n^2 \log(n))$ at $T = \infty$ for Beenker (Destainville, 2006)
- $\tau = O(n^2 \log(n))$ at $T = 0$ for Penrose & Beenker (F., 2009)

But there is no rigorous result. Even ergodicity is open for $T = 0$.

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Other tempartures? Cooling schedule?

Conclusion

Aperiodicity if much more interesting beyond dimer tilings...
...but random tilings or cooling process seem much complicated!