# The Glauber dynamics on lozenge tilings and other dimer models

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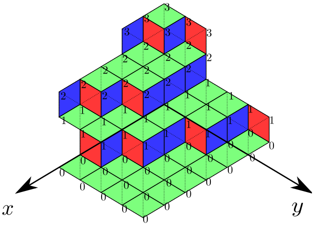
# The initial problem

We look at the time evolution of an interface between thermodynamical phases in $\mathbb{R}^3$, exactly when they are equally stable, for example :

- Ice and liquid water at 0°C.
- Solid salt in water saturated in salt.
- $+$ and - phases in low temperature Ising model.

We expect a slow evolution that minimize the size of the interface.

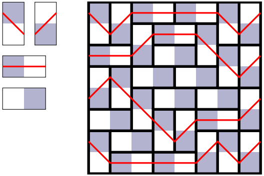
# Tiling, surface and height function

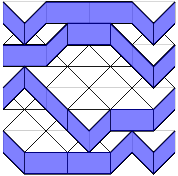



We represent the interface by a tiling of some region in the plane

It is parametrized by a height function written $h$.

By convention we use the $z$ coordinate as a function of the point in the picture.



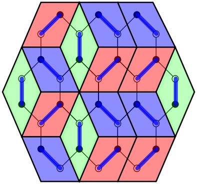


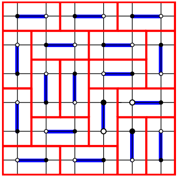
We draw a line in each domino depending on its orientation.

We see the resulting paths as level lines.

# Link with dimer model

We can see a tiling as a perfect matching of some graph : each vertex is connected to one and only one neighbour.

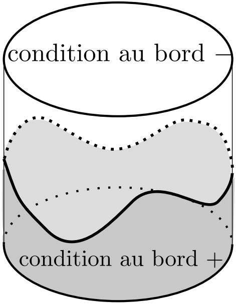




Many results are true for general underlying bipartite lattice.

# Dobrushin condition, boundary height

The boundary of the surface only depends on the domain and not on the actual tiling.



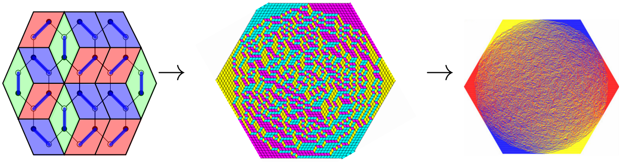

Lozenge tilings are equivalent to the Ising model at $T = 0$ on $\mathbb{Z}^3$ with the following boundary condition :

$+$ above some curve $-$ below.

We have to think about the boundary in $\mathbb{R}^3$ and not only its projection.

# Thermodynamic limit

We fix a curve in $\mathbb{R}^3$ and we use a grid of size $\frac{1}{L}$.



We write $\mu_{eq}^L$ for the uniform measure and $h_{eq}^L$ for a sample height function.
$h^L$ has increments $\pm\frac{1}{L}$ and a definition domain independent of $L$.

# Law of large numbers

With this normalization, $h^L_{eq}$ converges to a deterministic limit..

**Theorem (Cohn, Kenyon, Propp 2001)**

*Almost surely and for the infinite norm:*

$$h^L_{eq} \to h^\infty,$$

*where $h^\infty : U_\infty \to \mathbb{R}$ is deterministic.*

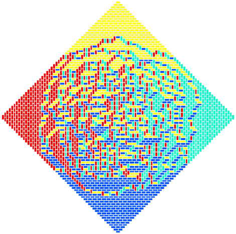
$h^\infty$ is the only argmin of

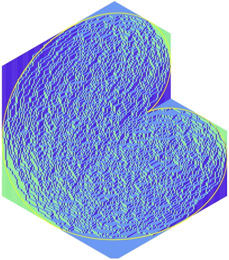
$$E(h) = \int_{U_\infty} \sigma(\nabla h(x)) d^2 x$$

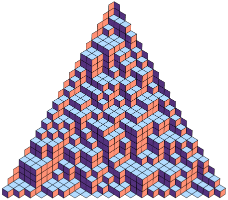
where $\sigma$ is known explicitly.

# Frozen and liquid phases

In $h^L_{eq}$, some region can be almost deterministic, we say they are "frozen".





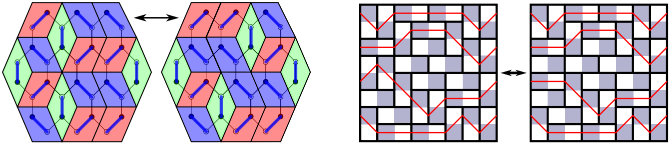


When the boundary curve is in a plane, then $h^\infty$ is linear. We call this the planar case.

# Definition

## Base move

Rotation of three lozenges / two dominoes.

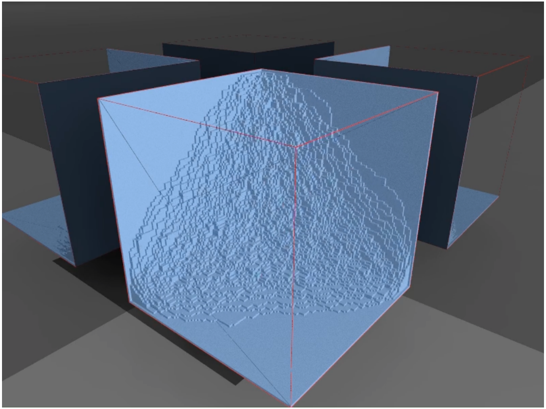


## "Glauber" dynamics

At each position, do a rotation at rate one if possible, independently of everything else.

The uniform measure $\mu_{\text{eq}}^{L}$ is reversible.

# Example



# Mixing time

- It is too difficult to study the large scale motion.
- We focus on the speed of convergence.
- We use the "*mixing time*" written $T_{\text{mix}}$.
  $\left(T_{\text{mix}} = \inf\{t \text{ such that } \|\mu_t - \mu_{\text{eq}}\|_{TV} \leq \frac{1}{2}\}\right)$.
- Estimating the mixing time is also useful to generate large random tilings.

# Historic

- Luby, Randall, Sinclair 2001 : $T_{\text{mix}} = O(L^6)$ for a well chosen non local dynamics.
- Randall, Tetali 2000 : $T_{\text{mix}} = O(L^{10})$ for the Glauber dynamics by comparisons.
- Wilson 2004 : $T_{\text{mix}} = cL^2 \log(L)$ for lozenges and non local. $\Rightarrow T_{\text{mix}} = O(L^6 \log L)$ for Glauber.
- Caputo, Martinelli, Toninelli 2011 (and with Simenhaus 2010) : $T_{\text{mix}} = O(L^2 \log^C L)$ for Glauber, lozenges and planar boundary.

# Dominoes theorem

Let $\mathcal{C}$ be a closed curve in $\mathbb{R}^3$ and let $h^\infty$ be the associated limit shape.

**Theorem (Laslier, Toninelli, PTRF, 2012)**

*For the Glauber dynamics on lozenge tilings or domino tilings and some other dimer models, if $h^\infty$ is linear (i.e. if $\mathcal{C}$ is in a plane) then*

$$cL^2 \leq T_{mix} \leq L^{2+o(1)}$$

- The lower bound is better that $\frac{L^2}{\log L}$ in CMT.
- The unit square has a planar boundary.

# Non planar theorem

## Theorem (Laslier, Toninelli, 2013)

*For the Glauber dynamics on lozenge tilings, if $h^\infty$ has no frozen phase then for all $\epsilon$ and $n$, there exists $T = L^{2+o(1)}$ such that*

$$\textit{except with proba} \leq L^{-n}, \|h^L_T - h^\infty\|_\infty \leq \epsilon$$

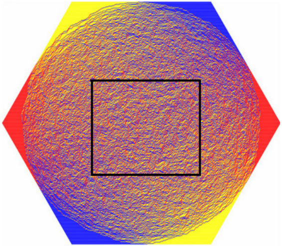
- The convergence is weaker than $T_{\text{mix}}$ because we have no information on the microscopic structure.
- The examples above have frozen regions be we can apply the result to sub-domains.
- We have no simpler way to characterize the lack of frozen region.

# Hexagonal theorem

**Theorem (Laslier, Toninelli, 2013)**

*For the Glauber dynamics on lozenge tilings, if $h^\infty$ is a subset of the liquid par in the limit shape for an hexagon, then*

$$T_{mix} = L^{2+o(1)}$$



- The theorem mainly shows that the method can give a mixing time in a non planar case.

# Hydrodynamic limit conjecture

In the limit, we expect a “law large numbers” for the dynamics, which is often called *hydrodynamics limit* :
When time is normalized by $L^2$, in a macroscopic scale we see a deterministic mouvement.

$$h^L_{L^2 t} \to h_t$$

where $h_t$ is the solution of a PDE

$$\partial_t h = \mu(\nabla h) \sum_{ij} (\partial^2_{ij}\sigma) \circ (\nabla h) . \partial^2_{ij} h.$$

The limit shape $h^\infty$ is characterized by $\partial_t h^\infty = 0$.

# Monotonicity

We define a partial order by :

$$h \leq h' \Leftrightarrow \forall x,\ h(x) \leq h'(x)$$

## Proposition

*For any pair of configurations $h_0 \leq h'_0$, there exists a coupling of the dynamics such that :*

$$\forall t,\ h_t \leq h'_t$$

*Remark :*
We are allowed to compare configurations with different boundary !

# Fluctuations

Let $D$ be a domain with either $\begin{cases} \text{a planar boundary} \\ \text{or} \\ \text{a diameter } \frac{L^\epsilon}{\sqrt{L}} \end{cases}$

## Lemma

*Height fluctuations at equilibrium in $D$ are logarithmic in $L$*

$$\forall \epsilon, \forall n, \mathbb{P}\left(|h^L_{eq}(x) - h^\infty(x)| \geq \frac{L^\epsilon}{L}\right) = O(L^{-n})$$

# Mixing time close to equilibrium

Let $D$ be a domain of size $L$ and let $h_0$ be a configuration in $D$ macroscopically close to equilibrium

$$\|h_0 - h^\infty\|_\infty \leq \frac{L^\epsilon}{L}.$$

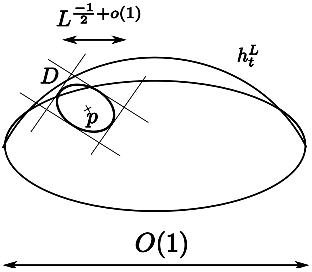
## Lemma

*The mixing time for the dynamics started in $h_0$ (and with an additional constraint) is of order $L^{2+o(1)}$.*

When we start very close to equilibrium (but still outside of the range of fluctuations), we know precisely the mixing time.

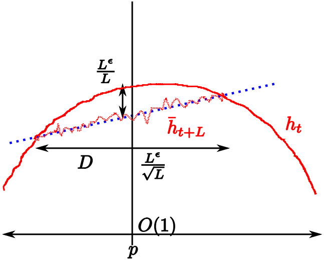
# Starting point

We do as if we wanted to show the hydrodynamic limit.




- We assume that at time $t$ the interface is almost equal to some smooth deterministic surface.
- We look at what happens in a small domain of radius $L^{-1/2+\epsilon}$ around some fixed point.

# Dynamics in a small domain




- We are in the domain of the lemmas
- Using lemma 2, after a time $L^{1+\epsilon}$ the small domain is at equilibrium.
- Using lemma 1, the point $p$ has really moved down by $\frac{L^\epsilon}{L}$.

# Conclusion

- We apply the estimate simultaneously around all points.
- After a time $L^{1+\epsilon}$, the surface has moved down by $\frac{L^\epsilon}{L}$.
- Therefore, after $L$ steps, which take a total time $L^{2+\epsilon}$, we arrive close to $h^\infty$.

# Rigorous version

- We are not precise enough to follow the "true" hydrodynamic limit.
- However we can let an upper bound evolve according to this scheme.
- Such a bound is enough for the mixing time.
- We have some freedom in the choice of the deterministic evolution.

# Fluctuations

## Lemma (Laslier, Toninelli, 2012)

*For any dimer model on a bipartite periodic lattice, let $D$ be a domain with planar boundary conditions. For any $\epsilon$ and $n$,*

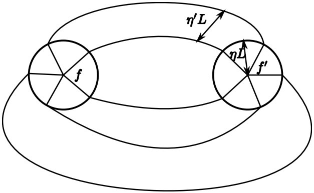
$$\mathbb{P}\left(\exists x: \ |h(x) - h^\infty(x)| \geq \frac{L^\epsilon}{L}\right) = O(L^{-n}).$$

## Lemma (Laslier, Toninelli, 2013)

*For lozenge tiling. Let $x$ be a point in the interior of the liquid domain of some limit shape $h^\infty$. Let $D$ be a domain, centred in $x$ and of radius $L^{-1/2+\delta}$. We fix the boundary height to be $h^\infty|_{\partial D}$. For any $\epsilon$ and $n$,*

$$\mathbb{P}\left(\exists x: \ |h(x) - h^\infty(x)| \geq \frac{L^\epsilon}{L}\right) = O(L^{-n}).$$

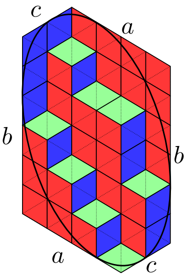
# Proof for planar fluctuations




- We know (Kenyon, Okounkov, Sheffield) the long range correlations for infinite planar measures.
- We compute the $k$th moment as sums along $k$ paths macroscopically far away.
- We compare finite and infinite domain by monotonicity.

# Proof for non planar fluctuations

- We know (Petrov 2012) the limit shape and the fluctuations in irregular hexagonal domains
- For a small domain, it is enough to know the Taylor development up to order 2.
- Any order 2 Taylor development can be found somewhere in an hexagon.
- We conclude by monotonicity.



# Convergence notion

- To prove a mixing time, we have to let the upper bound move until it is at distance $L^{o(1)}/L$ of the equilibrium.
- In the last steps, we need to look at bigger domains to see the vanishing curvature.
- For general non planar boundary, we cannot increase the size much because we only have second order Taylor development.
- For planar or hexagonal domains, we do not have such limitation.

# Mixing with floor/ceiling

**Lemma (Laslier, Toninelli, 2012)**

*For the dynamics in a domain at scale $L$, constrained to stay within a floor and a ceiling at distance $H/L$ of $h^\infty$,*

$$T_{mix} \leq CL^2 H^9 \log^4(L)$$

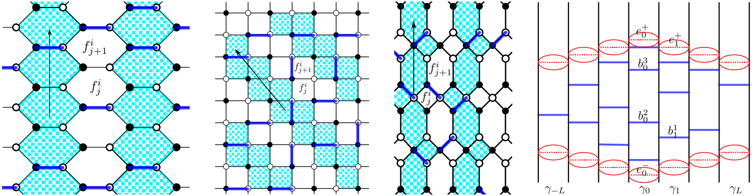
The floor and ceiling constraint is not very strict as long as $H$ is bigger than equilibrium fluctuations.

# Sketch of the proof

- By monotonicity, it is enough to look at the volume between the dynamics started from the maximal and minimal configurations.
- We define and auxiliary dynamics for which the volume between configurations is a super-martingale.
  Very model dependent !
- We lower bound the variance of the volume.
  $\Rightarrow$ mixing time for the auxiliary dynamics.
- Comparison between the original and auxiliary dynamics via Peres-Winkler censoring.

# Auxiliary dynamics

We can forget about some of the dimers.




The interaction is that they have to stay in staggered row.

*Dynamics* : At rate one, re-sample half the columns uniformly conditioned on the others.

*Trick* : It is enough to compute the drift between configurations that differ by a single move.

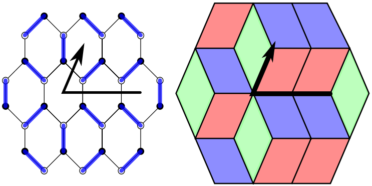
# Variance of the volume

## Gradient descent

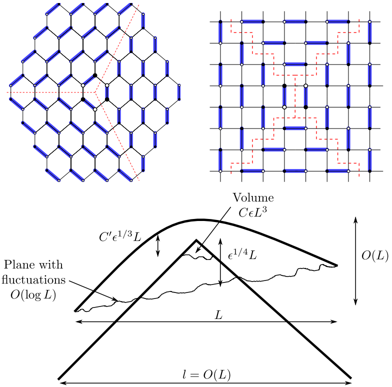
### Question

Find places where the interfaces are different and one of them can move.

- We do a gradient descent.
- Such a path has length $O(H)$.
- We find $O(V/H^3)$ points where a move is possible, and each one contributes $O(1)$ to the variance.



# Lower bound




- For some domains, the volume drift is $\geq -\frac{1}{L^2}$
- After a time $cL^2$, these domains have barely evolved.
- By monotonicity, we get the general case.

Thank you for your attention.