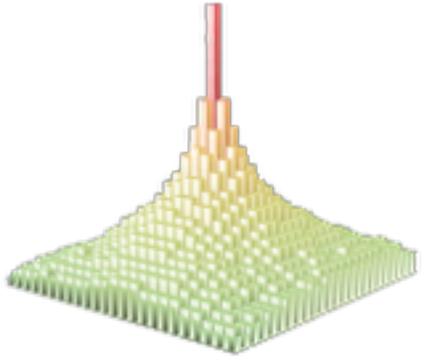
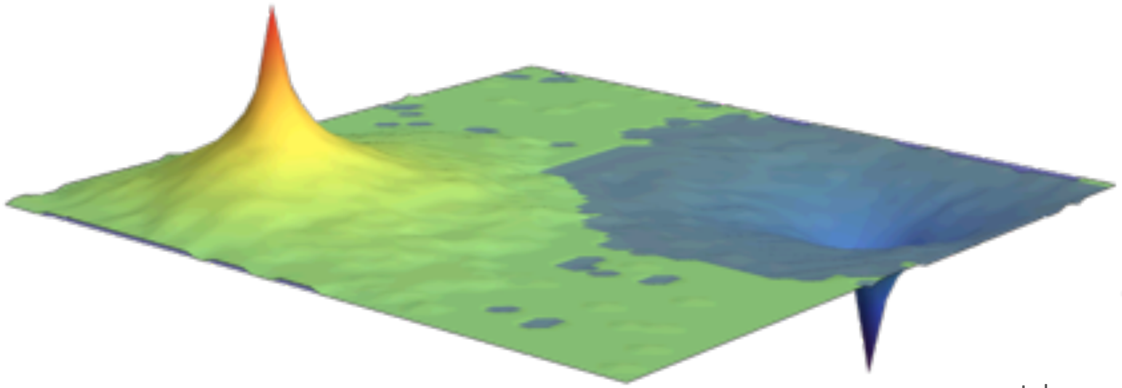


Harmonic Pinnacles in the Discrete Gaussian Model



JOINT WORK WITH
E. LUBETZKY AND A. SLY

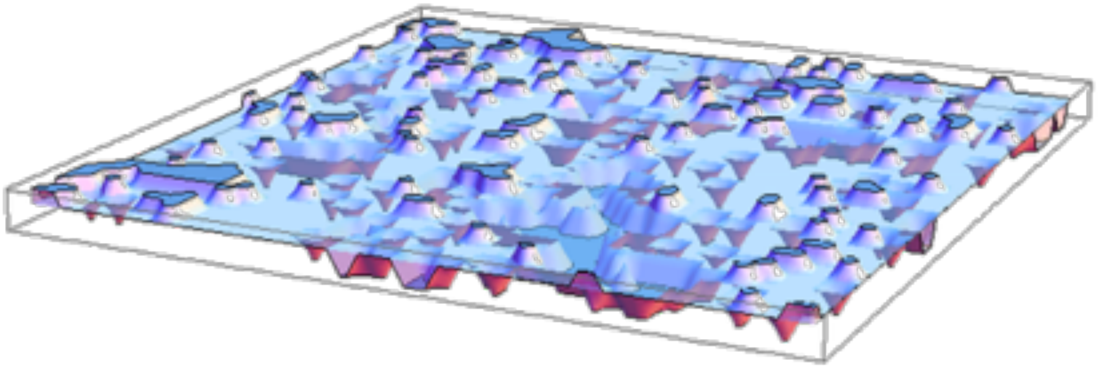
The Discrete Gaussian model

- <u>DEFINITION:</u> (*2D DG model*) probability measure on $\eta : \Lambda \to \mathbb{Z}$ for $\Lambda = \{1, \dots, L\}^2$ given by

$$\pi_\Lambda(\eta) = \frac{1}{Z_{\beta,\Lambda}} \exp\left(-\beta \sum_{x\sim y} |\eta_x - \eta_y|^2\right)$$

  where $\eta_x = 0$ for $x \notin \Lambda$ (0 boundary condition).

  - $\beta \geq 0$: inverse temperature
  - $Z_{\beta,\Lambda}$: partition function
  - $\pi = \lim_{L\to\infty} \pi_\Lambda$: ∞-volume DG



The Discrete Gaussian model

- Family of surfaces models in the '50
- Dubbed Discrete Gaussian Model by [Chui-Weeks '76]
- Dual of the Villain XY model [Villain '75]
- Related by duality to the Coulomb gas model
- its $\mathbb{R}$-valued analogue: β scales out $\Rightarrow$ Discrete Gaussian Free Field

DG surface: basic questions

- ***Height profile:***
  I. What are the height fluctuations at the origin (say), e.g., what is $\mathbb{E}\left[\eta_0^2\right]$? Does it diverge with L?
  II. What is the maximum height $X_L = \max_x \eta_x$?

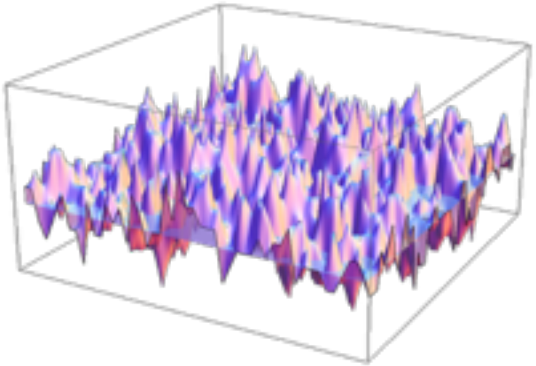
- ***The effect of a floor:***
  III. How are these affected by conditioning that $\eta \geq 0$?

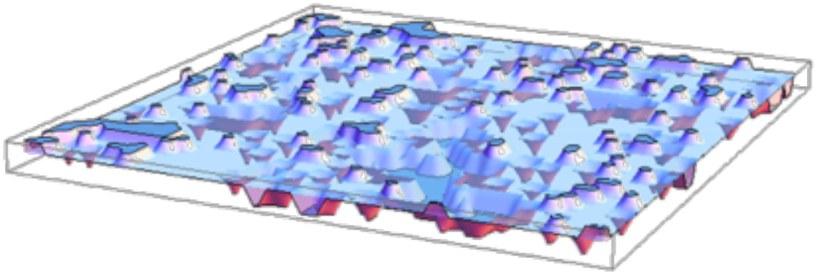
- rigorously studied in breakthrough papers from the 80's
  [Fröhlich, Spencer '81a, '81b, '83], [Brandenberger, Wayne '82], [Bricmont, Fontaine, Lebowitz '82], [Bricmont, El-Mellouki, Fröhlich '86], ...

DG surface: predicted behavior

- *Roughening phase transition* at a critical $\beta_{\mathrm{R}} \approx 0.665$:

$\beta < \beta_R$ $\qquad\qquad$ $\beta > \beta_R$





- Transition exclusive to dimension $d = 2$: surface is rough for $d = 1$ and rigid for $d \geq 3$ [Temperley '52, '56] [Bricmont, Fontaine, Lebowitz '82] via [Fröhlich, Simon, Spencer '75]

High temperature DG vs. the DGFF

- **DGFF profile:** What is $\mathbb{E}\left[\eta_0^2\right]$? What is $X_L = \max_x \eta_x$?
  - $\mathrm{Var}(\eta_0) \sim \frac{2}{\pi}\log L$, $\mathbb{E}X_L \sim 2\sqrt{2/\pi}\log L$, concentration
    [Bolthausen, Deuschel, Giacomin '01],
    [Bolthausen, Deuschel, Zeitouni '11], [Bramson, Zeitouni '12], ...

- **DGFF above a floor:** (conditioning that $\eta \geq 0$)
  - Surface bulk concentrates around $\mathbb{E}X_L$ and behaves ≈ shifted DGFF:
    $\mathbb{E}[\, X_L \mid \eta \geq 0 \,] \sim 2\,\mathbb{E}X_L \sim 4\sqrt{2/\pi}\log L$, concentration
    [Bolthausen, Deuschel, Giacomin '01]
    Analogue for $\mathbb{Z}^3$ due to [Bolthausen, Deuschel, Zeitouni '95]

- **DG for small enough β:**
  - Indeed $\mathrm{Var}(\eta_0) \asymp \log L$ [Fröhlich, Spencer '81a,'81b]
    (proof via Coulomb gas model analysis)

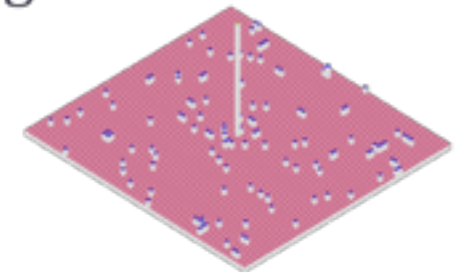
Low temperature DG

- Large enough β: surface is *rigid* by a Peierls argument ([Gallavotti, Martin-Lōf, Miracle-Solé '73] [Brandenberger, Wayne '82])
- [Bricmont, El-Mellouki, Fröhlich '86]:
  - maximum: $\mathbb{E}[X_L] \asymp \sqrt{\beta^{-1}\log L}$
  - average with floor: $\mathbb{E}\left[\frac{1}{|\Lambda|}\sum_x \eta_x \,\middle|\, \eta \geq 0\right] \asymp \sqrt{\beta^{-1}\log L}$
  - analogous results for the Absolute-Value SOS model (Hamiltonian: $\mathcal{H}(\eta) = \sum_{x\sim y}|\eta_x - \eta_y|$) with order $\beta^{-1}\log L$

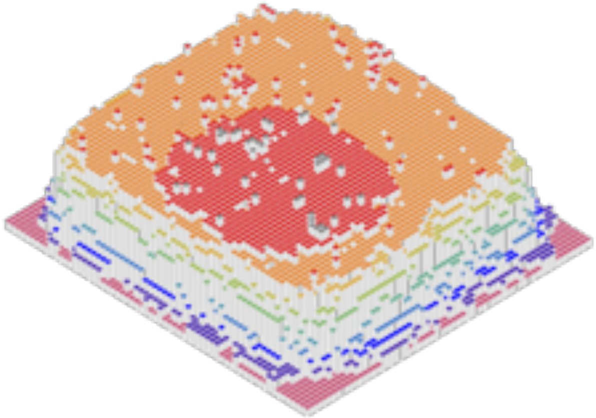
- Quoting *Phase Transitions & Critical Phenomena, Vol.* 10 :
  "The origin of this apparently paradoxical result is that 'spikes' grow downwards from the surface; if any spike touches the surface, such a configuration does not contribute to the entropy. This drives the surface away to 'infinity'."

Intuition to the BEF'86 results

- [Bricmont, El-Mellouki, Fröhlich '86]:
  - maximum: $\mathbb{E}[X_L] \asymp \sqrt{\beta^{-1}\log L}$
  - average with floor: $\mathbb{E}\left[\frac{1}{|\Lambda|}\sum_x \eta_x \,\middle|\, \eta \geq 0\right] \asymp \sqrt{\beta^{-1}\log L}$
- Proof ideas:
  - maximum: LD governed by isolated spikes; a spike of height h costs $\exp(-c\beta h^2)$.
  - surface height above a floor is at most $2\mathbb{E}[X_L]$
  - lower bound on this height: Pirogov-Sinaï theory

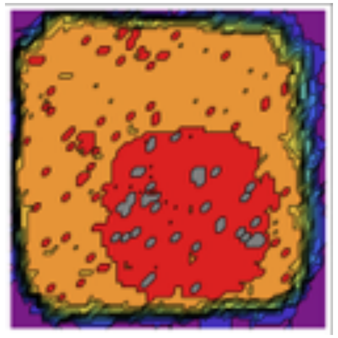


Progress for SOS in recent years



[Caputo, Lubetzky, M, Sly, Toninelli '12, '13, '15] :

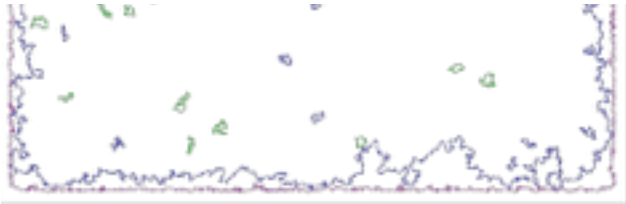
- maximum concentrates on $\frac{1}{2\beta}\log L$;
- average height above a floor concentrates on $\frac{1}{4\beta}\log L$;
- deterministic scaling limit of the level lines loop ensemble;
- $L^{1/3}$ fluctuations exponent of the largest loop around its limit.



[Caputo, M, Toninelli '14] : Large deviations without the floor:

$$\pi_\Lambda\,(\eta_\Lambda > 0) \asymp \exp(-\tau L \log L)$$

surface tension



Low temperature DG:

Previous work: [Bricmont, El-Meloukki, Frohlich '86]:

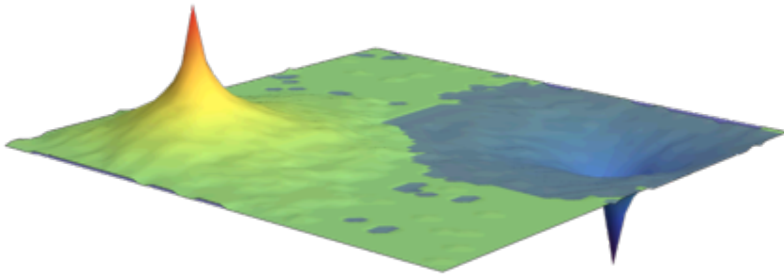
- Maximum X_L : $\mathbf{E}[X_L] \asymp \sqrt{\beta^{-1} \log L}$

- **Theorem** [Lubetzky,M. Sly]:

∃ M=M(L) ~ $\sqrt{(1/2\pi\beta) \log L \log\log L}$ such that $X_L \in$ {M,M+1} w.h.p.

Remark

- for a.e. L (log density) X_L=M w.h.p.
- Extra $\sqrt{\log\log L}$ due to nature of large deviations.
- in DG: "harmonic pinnacles" preferables to spikes.

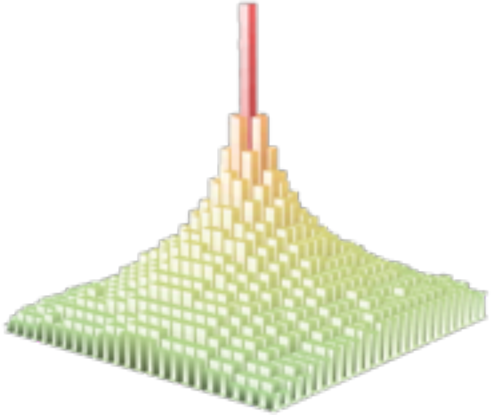


Low temperature DG

Central ingredient: Large Deviation estimate on ∞-volume DG:

Proposition [Lubetzky, M., Sly] :

$$\pi(\eta_0 \geq h) = \exp(-(2\pi\beta + o(1))\frac{h^2}{\log h})$$



(cf. $\exp(-ch^2)$ for a spike of height h.)

- M:= max integer such that $\pi(\eta_0 \geq M \geq L^{-2}\log^5 L)$

Low temperature DG

- Previous work [Bricmont-El Meloukki, Frohlich] : av. height with floor ~ $\sqrt{\beta^{-1}\log L}$

- **Theorem** [Lubetzky, M, Sly] : conditioned on η > 0 $\exists H \ \sim \sqrt{(1/4\pi\beta)\log L \ \log\log L}$

  such that :

  (i) $\#\{x : \ \eta_x \in \{H, H+1\}\ \} \geq (1-\epsilon(\beta))L^2$

  (ii) For any h < H : single macroscopic loop with area $L^2(1\text{-}o(1))$;

  (iii) Height H : single macroscopic loop with area $(1-\epsilon(\beta))L^2$

  (iv) No (H+2) macroscopic loop;

  (v) No negative macroscopic loops

Low temperature DG

Roughly put: with the floor, w.h.p. :

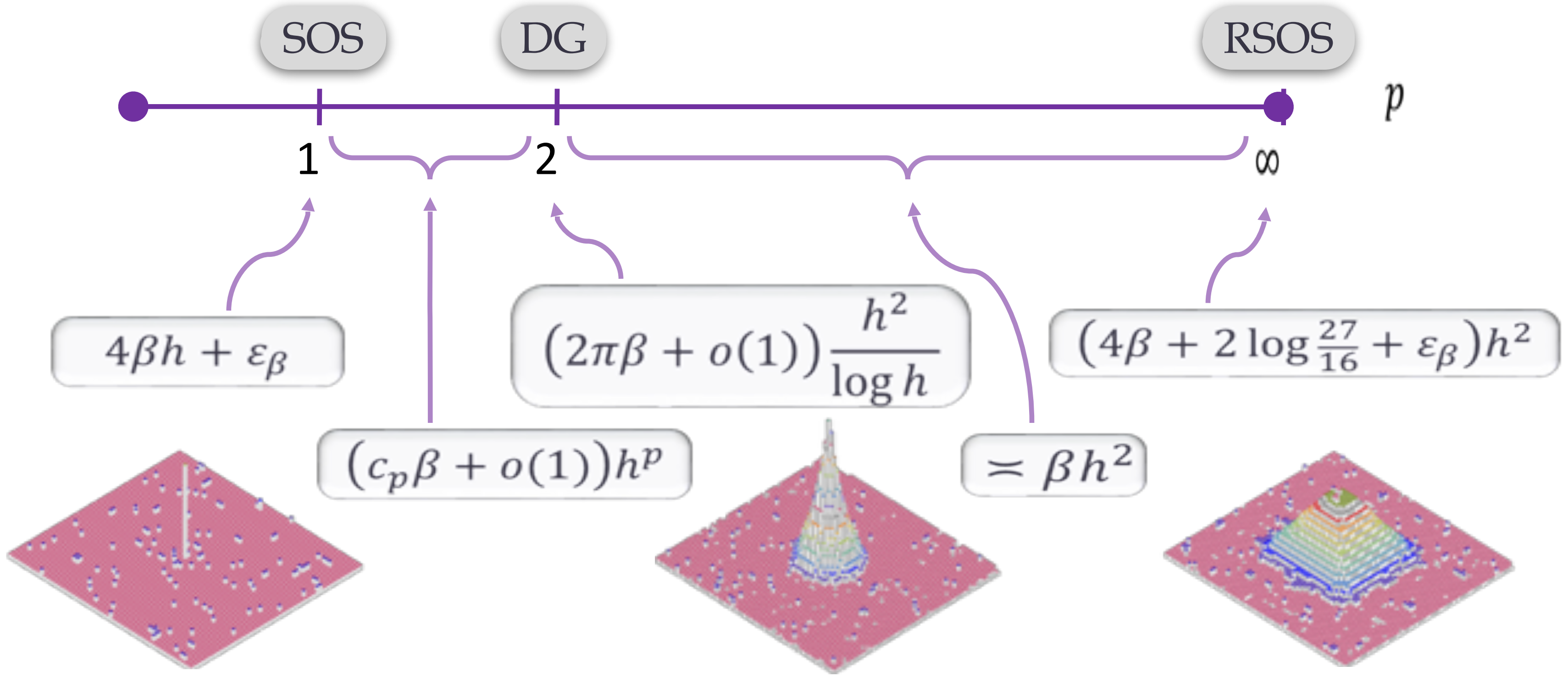
- DG surface is a plateau at height H ~ 1/√2 M
- Plateau is approx a raised (by H) version of the surface *without* floor
- the floor raises the maximum by a factor (1+1/√2).

Theorem [Lubetzky, M, Sly] Conditioned on η>0 :

$$\exists M^* \sim (1+\frac{1}{\sqrt{2}})M \ \ \text{such that, w.h.p.} \ X_L \in \{M^*, M^*+1, M^*+2\}$$

Generalizations to p-Hamiltonians

Extensions to surface models with $H(\eta) = \sum_{x \sim y} |\eta_x - \eta_y|^p \,, \quad p \in [1, +\infty]$



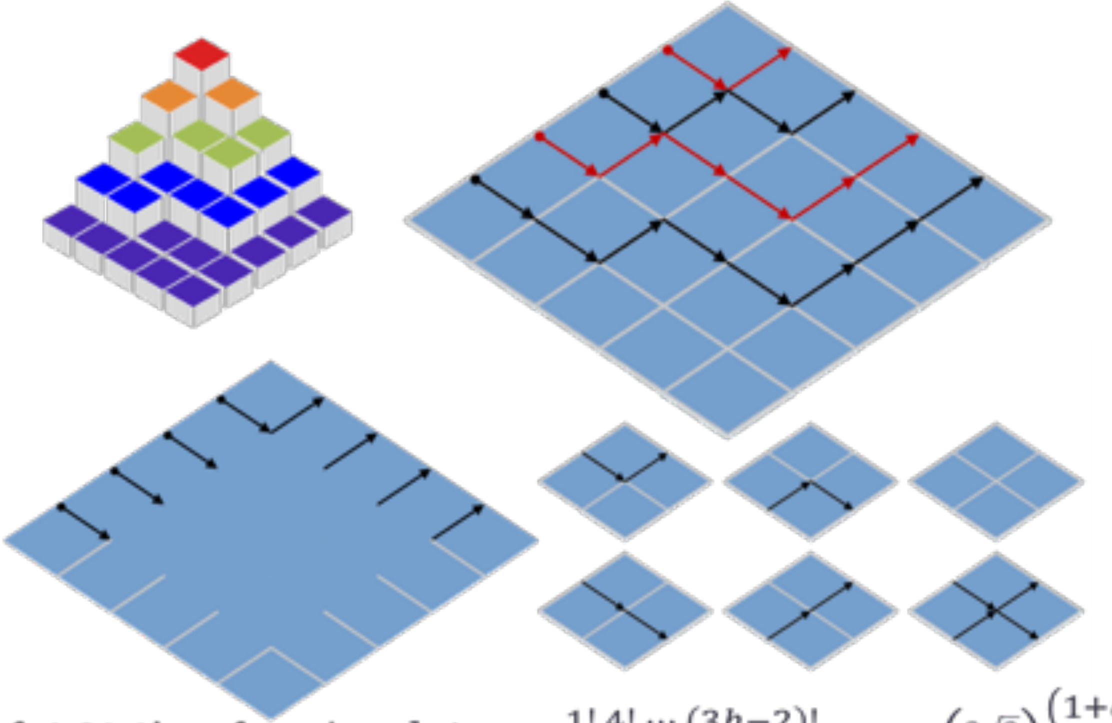
Example: LD in infinite volume : $-\log\left(\pi(\eta_o \geq h)\right)$

Generalizations to p-Hamiltonians

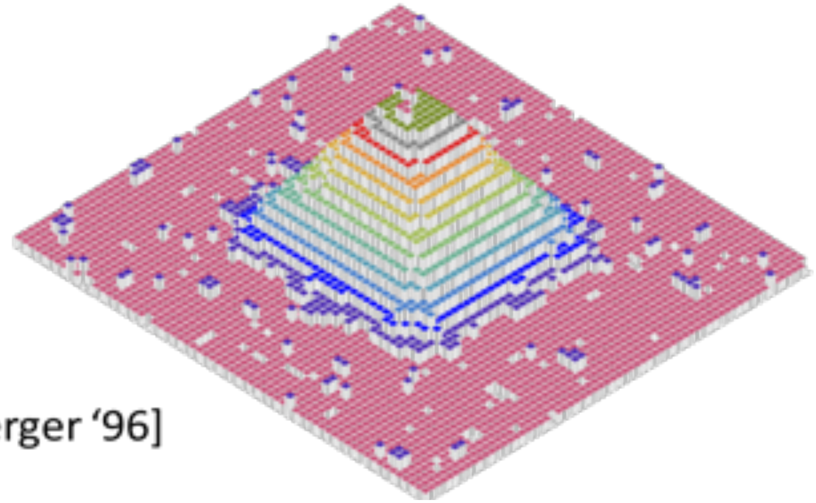
Model	Large deviation $-\log \pi(\eta_0 \geq h)$	Maximum center (M)	Maximum window	Height above floor center (H)	Height above floor window
$p = 1$ (SOS)	$4\beta h + \varepsilon_\beta$	$\frac{1}{2\beta} \log L$	$O(1)$	$\lceil \frac{1}{4\beta} \log L \rceil$	± 1
$1 < p < 2$	$(c_p\beta + o(1))\, h^p$	$\left(\frac{2+o(1)}{c_p\beta} \log L\right)^{\frac{1}{p}}$	± 1	$\left(\frac{1+o(1)}{2}\right)^{\frac{1}{p}} M$	± 1
$p = 2$ (DG)	$(2\pi\beta + o(1))\frac{h^2}{\log h}$	$\sqrt{\frac{1+o(1)}{2\pi\beta} \log L \log\log L}$	± 1	$\frac{1+o(1)}{\sqrt{2}} M$	± 1
$2 < p < \infty$	$\asymp \beta h^2$	$\asymp \sqrt{\frac{1}{\beta} \log L}$	± 1	$\frac{1+o(1)}{\sqrt{2}} M$	± 1
$p = \infty$ (RSOS)	$\left(4\beta + 2\log \frac{27}{16} + \varepsilon_\beta\right) h^2$	$(1 \pm \varepsilon_\beta)\sqrt{\frac{2}{c_\infty} \log L}$	± 1	$\frac{1+o(1)}{\sqrt{2}} M$	± 1

From LD in RSOS to the # of alternating sign matrices (ASMs)

- Correspondence between RSOS optimal-energy surfaces, edge-disjoint walks and (via the *square ice* model) ASMs:



AMS = $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$



- # of ASM's of order h is $\frac{1!\,4!\cdots(3h-2)!}{h!\,(h+1)!\cdots(2h-1)!} = \left(\frac{3\sqrt{3}}{4}\right)^{(1+o(1))h^2}$ [Zeilberger '96]
- translates into an entropy term of $\exp\left[2\log\left(\frac{27}{16}\right)h^2\right]$

Height large deviations in the DG

$$\log \pi(\eta_0 \geq h) \sim -2\pi\beta \frac{h^2}{\log h}$$

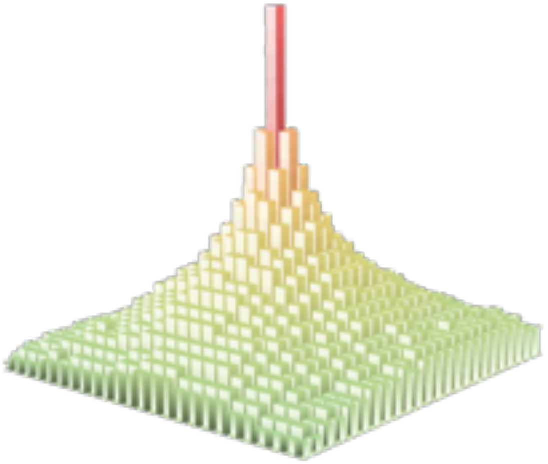
LD dominated by "harmonic pinnacles", integer approximations to the Dirichlet problem :

$$I_r(h) = \inf\{ \sum_{x\sim y} (\varphi_x - \varphi_y)^2 \ : \ \varphi \restriction_{B_r^c} = 0,\ \varphi_0 = h\}$$

- real solution: harmonic function ϕ:

$$\phi_x = \mathbf{P}_x(\tau_0 < \tau_{\partial B_r})h = \left(1 - \frac{\log|x| + O(1)}{\log r}\right)$$

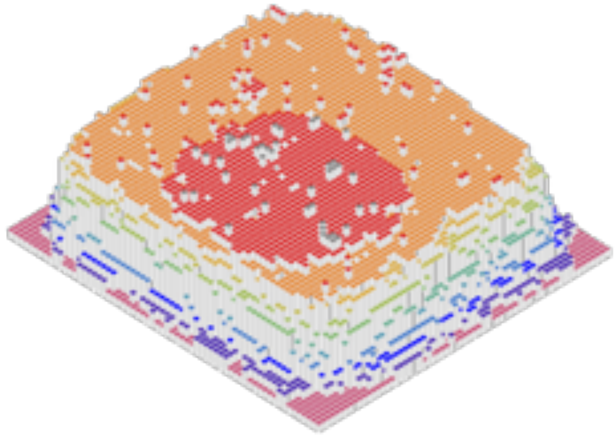
$$I_r(h) = 4h^2 \frac{\sum_x \mathbf{P}_x(\tau_0 < \tau_{\partial B_r})}{\mathbf{E}_0(\tau_{\partial_{B_r}}} \sim 2\pi \frac{h^2}{\log r}$$



From the real to the discrete Dirichlet problem

- Real solution: $\phi_x \approx \left(1 - \frac{\log|x|}{\log r}\right) h$, $I_r(h) \sim 2\pi \frac{h^2}{\log r}$
- Discrete approximation (rounding) ends once $\phi_x < 1$:
  - Solving $\phi_x = 1$ for $|x| = r - 1$ gives $r \sim h/\log h$
  - Substituting in $I_r(h)$ gives $2\pi \frac{h^2}{\log h}$ (the sought LD rate).
- The volume of B_r is $O(h^2/\log^2 h)$, so the rounding cost (even when charging 2β per bond in B_r) is negligible.
  - exploit exact formulas (ϕ harmonic)
  - main part: there is *no benefit from larger domains*.
- Additional ingredients: control $\frac{\pi(\eta_0=h)}{\pi(\eta_0=h-1)}$ and $\pi(\eta_z = h \mid \eta_0 = h)$.

DG with floor

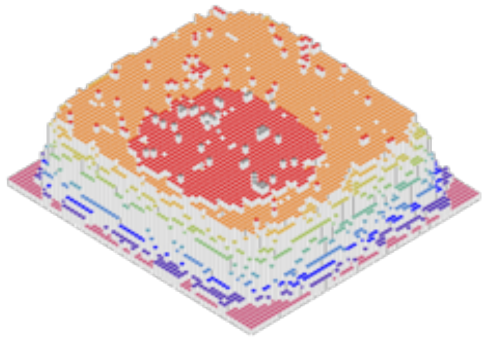


- **Main building block** :
  - Fix two integers (h, k) such that

$$4\beta + 2 \ \leq \ k\pi(\eta_0 \geq h) \ \leq \ 4\beta + 4$$

  - Then, w.h.p. the DG with ***floor & boundary height (h-1)*** on a square of side ***k*** will contain a ***h-level loop*** whose interior fills almost everything.
  - The relation between (h, k) embodies the *"entropic repulsion"*.

DG with floor



- Use the building block to recursively raise the surface from height 0 ➞ 1 ➞ 2 ➞ ……. ➞ H where

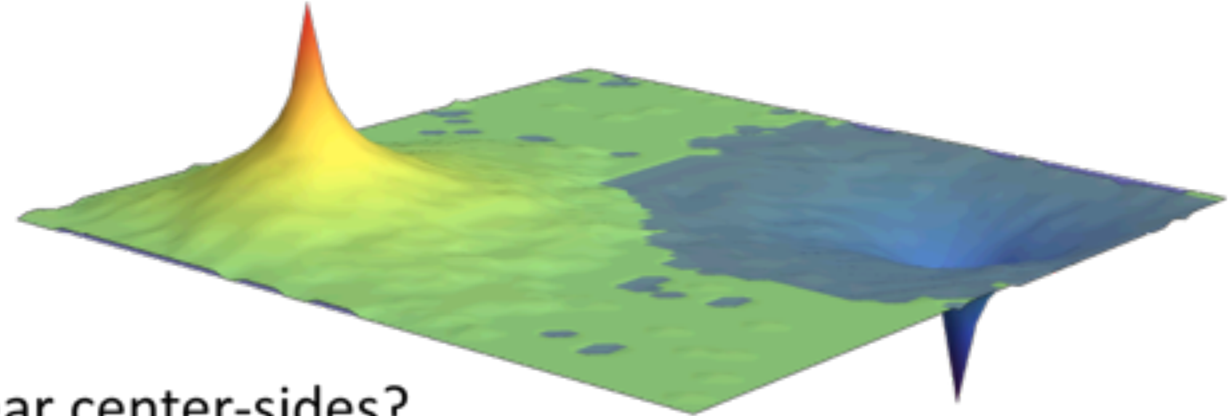
$$L\ \pi(\eta_0 \geq H) \approx 4\beta$$

by tiling the original L x L box with boxes of side

$$k = k(h) \approx \frac{4\beta}{\pi(\eta_0 \geq h)}, \quad h = 1, 2, \dots$$

- An important point is to check that in this process the loss of area near the boundary is negligible.

Open problems



- Low temperature:
  - $L^{1/3+o(1)}$ fluctuations near center-sides?
  - Critical behavior (exceptional L's):
    - Wulff-shape scaling limit?
    - $L^{1/2+o(1)}$ fluctuations near corners?
- High temperature:
  - DG ≈ DGFF…; tightness of maximum? asymptotics?
- Understand β near β_R…

Thank you