Height representation of XOR-Ising loops via bipartite dimers

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Inhomogeneous Random Systems, IHP
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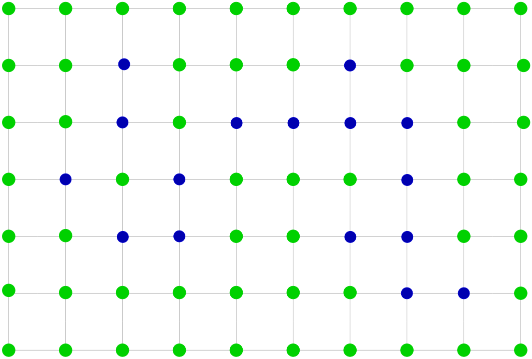
The Ising model

- Let $G = (V, E)$ be a finite graph embedded in the plane.
- A spin configuration σ assigns to every vertex x a spin $\sigma_x \in \{-1, 1\}$.



+1/-1 are represented by green/blue dots.

Set of spin configurations : $\{-1, 1\}^V$.

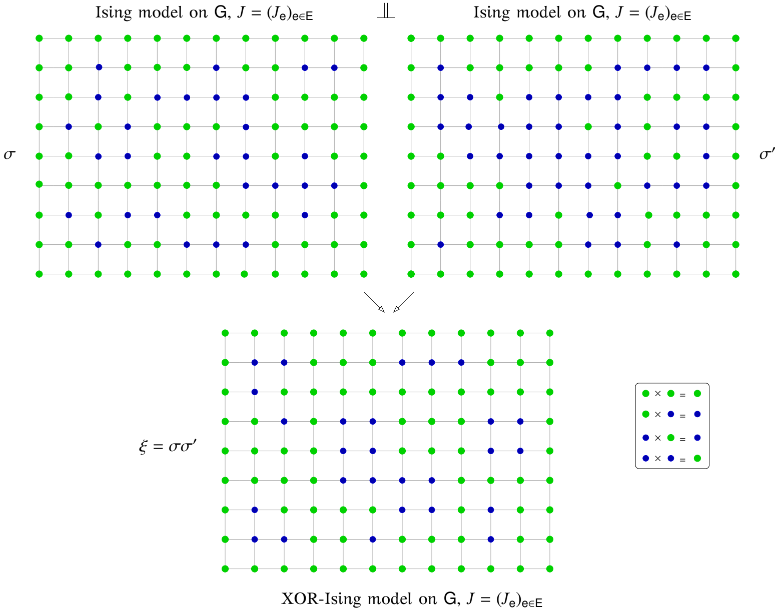
The Ising model

- Edges of G are assigned positive coupling constants: $J = (J_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}}$.
- Ising Boltzmann measure:

$$\forall\, \sigma \in \{-1,1\}^{\mathsf{V}}, \quad \mathbb{P}_{\text{Ising}}(\sigma) = \frac{\exp\left(\sum_{\mathsf{e}=\mathsf{xy}\in\mathsf{E}} J_{\mathsf{xy}}\sigma_{\mathsf{x}}\sigma_{\mathsf{y}}\right)}{Z_{\text{Ising}}(\mathsf{G},J)},$$

where $Z_{\text{Ising}}(\mathsf{G},J) = \sum_{\sigma\in\{-1,1\}^{\mathsf{V}}} \exp\left(\sum_{\mathsf{e}=\mathsf{xy}\in\mathsf{E}} J_{\mathsf{xy}}\sigma_{\mathsf{x}}\sigma_{\mathsf{y}}\right)$ is the Ising partition function.

The XOR-Ising model



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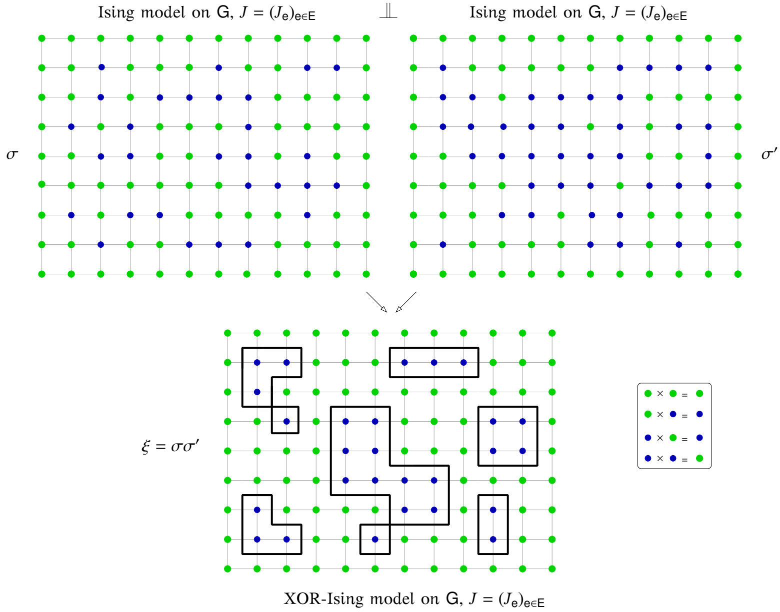


Conjecture for the XOR-Ising model

Conjecture (Wilson (11), Ikhlef, Picco, Santachiara)

The scaling limit of polygon configurations separating ± 1 clusters of the critical XOR-Ising model are contour lines of the Gaussian free field, with the heights of the contours spaced $\sqrt{2}$ times as far apart as they are for [...] the double dimer model on the square lattice.

Result

Theorem (Boutillier, dT)

- *Polygon configurations of the XOR-Ising model have the same law as a family of contours in a bipartite dimer model.*
- *This family of contours are the level lines of a restriction of the height function of this bipartite dimer model.*

Remark

Proved when the graph G *is embedded in a surface of genus* g, $g \geq 0$, *or when* G *is planar, infinite.*

- When the XOR-Ising model is *critical*, so is the bipartite dimer model.
- Using results of [dT] on the convergence of the height function, this gives partial proof of Wilson's conjecture.

Low temperature expansion [Kramers & Wannier]

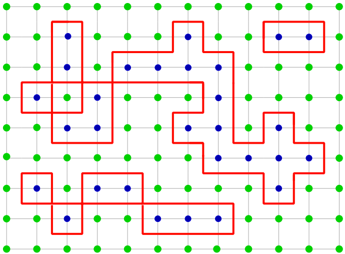
- Polygon configuration: subset of edges s.t. each vertex is incident to an even number of edges.

- Write, $$e^{J_{\mathsf{e}}\sigma_{\mathsf{x}}\sigma_{\mathsf{y}}} = e^{J_{\mathsf{e}}}(\delta_{\{\sigma_{\mathsf{x}}=\sigma_y\}} + e^{-2J_{\mathsf{e}}}\delta_{\{\sigma_{\mathsf{x}}\neq\sigma_y\}}).$$

  The partition function is then equal to (LTE):

$$Z_{\text{Ising}}(\mathsf{G},J) = \sum_{\sigma\in\{-1,1\}^{\mathsf{V}}} \prod_{\mathsf{e}=\mathsf{xy}\in\mathsf{E}} e^{J_{\mathsf{e}}\sigma_{\mathsf{x}}\sigma_{\mathsf{y}}} = \mathcal{C} \sum_{\mathsf{P}^*\in\mathcal{P}(\mathsf{G}^*)} \prod_{\mathsf{e}^*\in\mathsf{P}^*} e^{-2J_{\mathsf{e}}}.$$

- Geometric interp: polygon config. separate clusters of ±1 spins.



High temperature expansion [Kramers & Wannier]

- Write, $e^{J_e \sigma_x \sigma_y} = \cosh(J_e)(1 + \sigma_x \sigma_y \tanh(J_e)).$

  The partition function is then equal to (HTE):

$$Z_{\text{Ising}}(G, J) = \sum_{\sigma \in \{-1,1\}^V} \prod_{e=xy \in E} e^{J_e \sigma_x \sigma_y} = \mathcal{C}' \sum_{P \in \mathcal{P}(G)} \prod_{e \in P} \tanh(J_e).$$

- No geometric interpretation using spin configurations.

Double Ising model

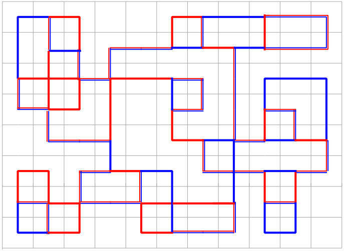
- Take 2 independent copies (red/blue) of an Ising model on G, with coupling constants J.
- Using the LTE, consider the probability measure $\mathbb{P}_{\text{2-Ising}}$:

$$\forall\,(\mathsf{P}^*,\mathsf{P}^*)\in\mathcal{P}(\mathsf{G}^*)^2,\quad \mathbb{P}_{\text{2-Ising}}(\mathsf{P}^*,\mathsf{P}^*)=\frac{\mathcal{C}^2\big(\prod_{\mathsf{e}^*\in\mathsf{P}^*}\mathsf{e}^{-2J_{\mathsf{e}}}\big)\big(\prod_{\mathsf{e}^*\in\mathsf{P}^*}\mathsf{e}^{-2J_{\mathsf{e}}}\big)}{Z_{\text{2-Ising}}(\mathsf{G},J)},$$

where $Z_{\text{2-Ising}}(\mathsf{G},J)=Z_{\text{Ising}}(\mathsf{G},J)^2$.

Double Ising model

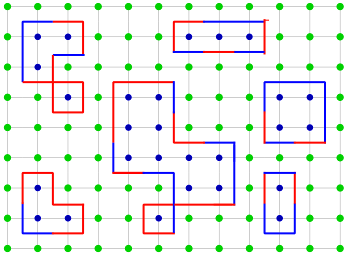
- Let $(\mathsf{P}^*, \mathsf{P}^*)$ be two polygon configurations.
- Consider the superimposition $\mathsf{P}^* \cup \mathsf{P}^*$.



Superimposition $\mathsf{P}^* \cup \mathsf{P}^*$

- Define two new edge configurations:
  - $\mathrm{Mono}(\mathsf{P}^*, \mathsf{P}^*)$: monochromatic edges.
  - $\mathrm{Bi}(\mathsf{P}^*, \mathsf{P}^*)$: bichromatic edges.

Monochromatic edges



Monochromatic edge configuration of $\mathsf{P}^* \cup \mathsf{P}^*$.

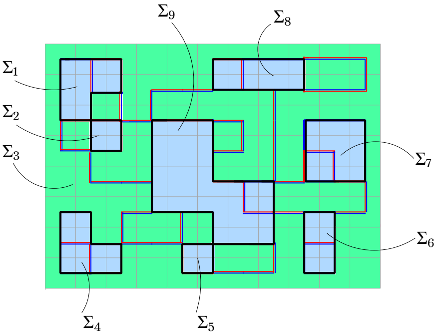
Lemma

Mono($\mathsf{P}^*, \mathsf{P}^*$) *is the polygon configuration separating* ± 1 *clusters of the corresponding XOR-Ising spin configuration.*

Goal: *understand the law of monochromatic edge configurations.*

Bichromatic edge configurations

- Let $(\mathsf{P}^*, \mathsf{P}^*)$ be two polygon configurations.
- $\mathrm{Mono}(\mathsf{P}^*, \mathsf{P}^*)$ separates the surface into connected comp. $(\Sigma_i)_i$.



Lemma

For every i, the restriction of $\mathrm{Bi}(\mathsf{P}^*, \mathsf{P}^*)$ to Σ_i is the LTE of an Ising configuration on G_{Σ_i}, with coupling constants $(2J_{\mathsf{e}})$.

Probability of monochromatic configurations

Lemma

Let P^* be a polygon configuration, separating the surface into n_{P} connected components. For every i, let P_i^* be a polygon configuration of $\mathsf{G}_{\Sigma_i}^*$.

Then, there are $2^{n_{\mathsf{P}}}$ pairs of polygon configurations $(\mathsf{P}^*, \mathsf{P}^*)$ having P^* as monochromatic edges, and $\mathsf{P}_1^*, \cdots, \mathsf{P}_{n_{\mathsf{P}}}^*$ as bichromatic edges.

Denote by $W(\mathsf{P}^*)$ the contribution to $Z_{\text{2-Ising}}(\mathsf{G}, J)$ of the pairs of polygon configurations $(\mathsf{P}^*, \mathsf{P}^*)$ such that $\mathrm{Mono}(\mathsf{P}^*, \mathsf{P}^*) = \mathsf{P}^*$.

Corollary

- $W(\mathsf{P}^*) = \mathcal{C}\left(\prod_{\mathsf{e}^* \in \mathsf{P}^*} e^{-2J_{\mathsf{e}}}\right) \prod_{i=1}^{n_{\mathsf{P}}} \left(2Z_{\mathrm{LT}}(\mathsf{G}_{\Sigma_i}^*, 2J)\right)$.
- $Z_{\text{2-Ising}}(\mathsf{G}, J) = \sum_{\mathsf{P}^* \in \mathcal{P}(\mathsf{G}^*)} W(\mathsf{P}^*)$.

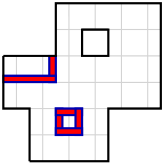
$$\boxed{\mathbb{P}_{\text{2-Ising}}(\mathrm{Mono} = \mathsf{P}^*) = \frac{W(\mathsf{P}^*)}{Z_{\text{2-Ising}}(\mathsf{G}, J)}.}$$

Mixed contour expansion

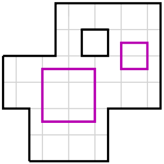
$$\boxed{W(\mathsf{P}^*) = \mathcal{C}\big(\textstyle\prod_{\mathsf{e}^*\in\mathsf{P}^*} e^{-2J_{\mathsf{e}}}\big) \prod_{i=1}^{n_{\mathsf{P}}}\Big(2Z_{\mathrm{LT}}(\mathsf{G}^*_{\Sigma_i}, 2J)\Big).}$$

Idea [Nienhuis]: use Kramers and Wannier high temperature expansion in each connected component Σ_i.

$$Z_{\mathrm{LT}}(\mathsf{G}^*_{\Sigma_i}, 2J) = \mathcal{C}(\Sigma_i) Z_{\mathrm{HT}}(\mathsf{G}_{\Sigma_i}, 2J).$$



Low temp. expansion on $\mathsf{G}^*_{\Sigma_i}$



High temp. expansion on G_{Σ_i}.

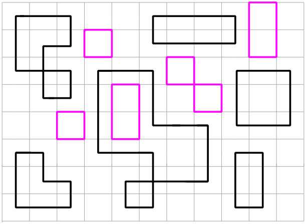
Mixed contour expansion

Combining the terms:

Proposition

For every polygon configuration P^*,

$$W(\mathsf{P}^*) = \mathcal{C} \prod_{\mathsf{e}^* \in \mathsf{P}^*} \left(\frac{2e^{-2J_\mathsf{e}}}{1 + e^{-4J_\mathsf{e}}} \right) \sum_{\{\mathsf{P} \in \mathcal{P}(\mathsf{G}) :\, \mathsf{P}^* \cap \mathsf{P} = \emptyset\}} \prod_{\mathsf{e} \in \mathsf{P}} \left(\frac{1 - e^{-4J_\mathsf{e}}}{1 + e^{-4J_\mathsf{e}}} \right)$$



$$\mathbb{P}_{\text{2-Ising}}(\text{Mono} = \mathsf{P}^*) = \frac{\prod_{\mathsf{e}^* \in \mathsf{P}^*} \left(\frac{2e^{-2J_\mathsf{e}}}{1+e^{-4J_\mathsf{e}}} \right) \sum_{\{\mathsf{P} \in \mathcal{P}(\mathsf{G}) :\, \mathsf{P}^* \cap \mathsf{P} = \emptyset\}} \prod_{\mathsf{e} \in \mathsf{P}} \left(\frac{1-e^{-4J_\mathsf{e}}}{1+e^{-4J_\mathsf{e}}} \right)}{\sum_{\mathsf{P}^* \in \mathcal{P}(\mathsf{G}^*)} \cdots}$$

Higher genus

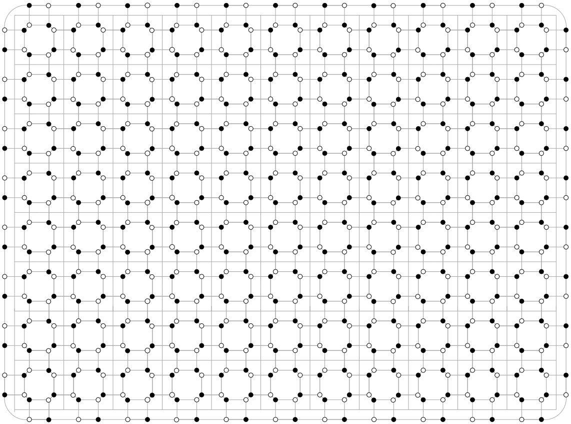
If the graph is embedded in a surface Σ of genus $g \geq 0$.

- Consider $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \simeq \{0,1\}^{2g}$.
- Family of Ising models, indexed by $\varepsilon \in \{0,1\}^{2g}$.
- The double Ising model partition function is defined as:

$$Z_{2\text{-Ising}}(\mathsf{G}, J) = \sum_{\varepsilon \in \{0,1\}^{2g}} Z^{\varepsilon}_{\text{Ising}}(\mathsf{G}, J)^2.$$

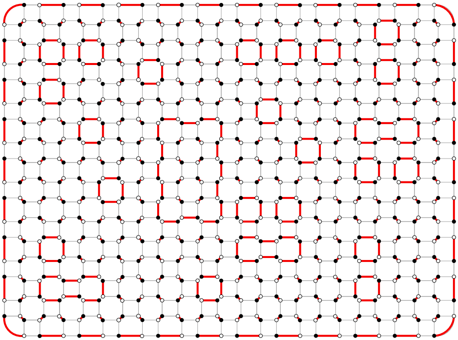
From pairs of non-intersecting primal/dual polygon config. to the dimer model on the bipartite graph G^{Q}

- The bipartite graph G^{Q}:



The dimer model on G^Q

- A dimer configuration of G^Q is a subset of edges M such that each vertex is incident to exactly on edge of M.



- Positive weight function ν on the edges.
- Dimer Boltzmann measure: $\forall\, M \in \mathcal{M}(G^Q),\ \mathbb{P}_{\text{dimer}}(M) = \frac{\prod_{e \in E^Q} \nu_e}{Z_{\text{dimer}}(G^Q, \nu)}$.

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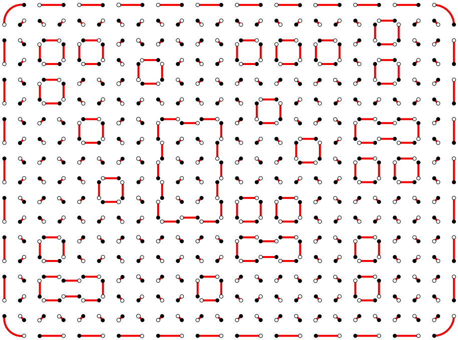

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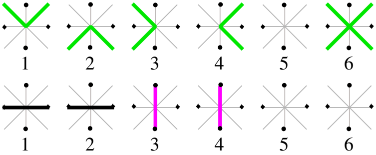
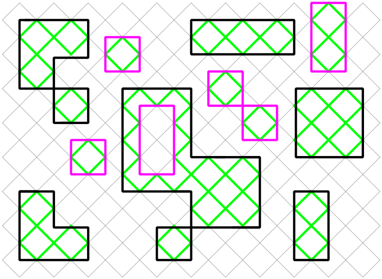
First step of the mapping

- From pairs of polygon configurations to the 6-vertex model on the medial graph [Nienhuis]

Local mapping





- Weights: $\omega_{12} = \frac{2e^{-2J\mathrm{e}}}{1+e^{-4J\mathrm{e}}}$, $\omega_{34} = \frac{1-e^{-4J\mathrm{e}}}{1+e^{-4J\mathrm{e}}}$, $\omega_{56} = 1$.

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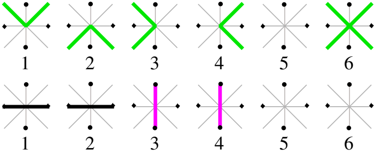

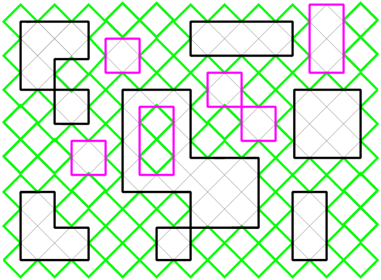

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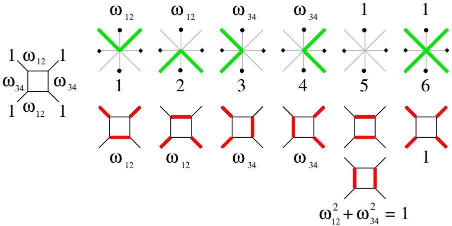
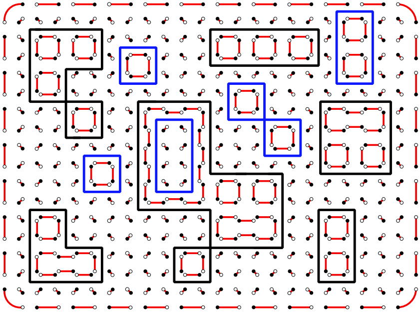
Second step of the mapping

- From the 6-vertex model to the dimer model [Wu-Lin, Dubédat]

Local mapping





Conclusion

- To every dimer configuration M of G^{Q}, assign

$$\mathrm{Poly}(\mathsf{M}) = (\mathrm{Poly}_1(\mathsf{M}), \mathrm{Poly}_2(\mathsf{M})),$$

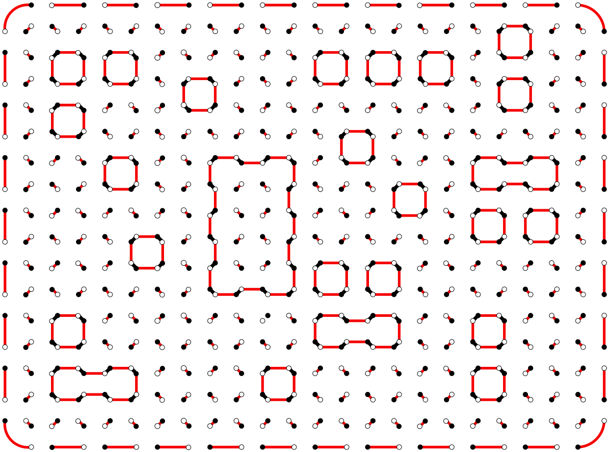
the pair of polygon configurations given by the mappings.

Theorem

For every polygon configuration P^* of G^*,

$$\boxed{\mathbb{P}_{\text{2-Ising}}(\mathrm{Mono} = \mathsf{P}^*) = \mathbb{P}_{\text{dimer}}(\mathrm{Poly}_1 = \mathsf{P}^*)}$$

Height function for bipartite dimers (Thurston)



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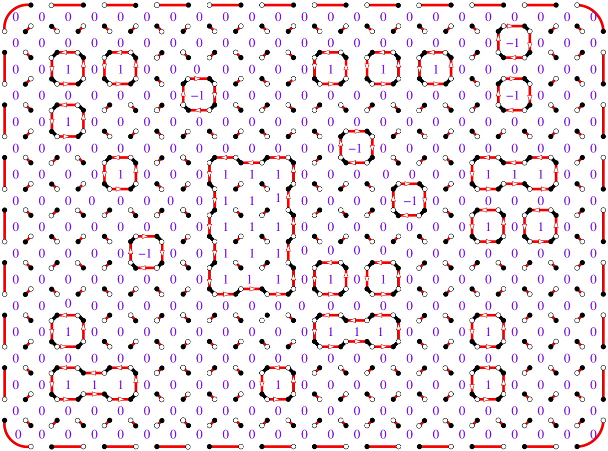

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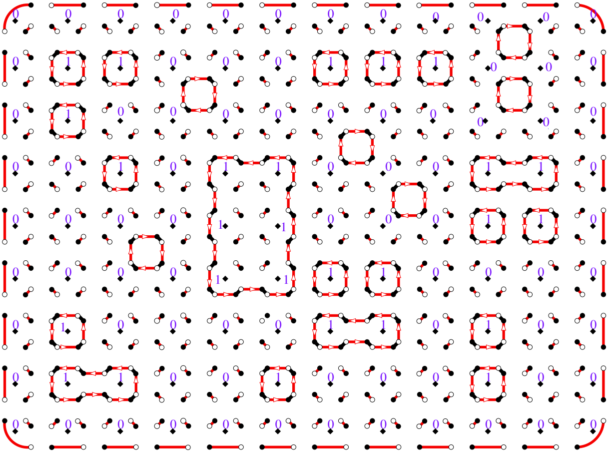

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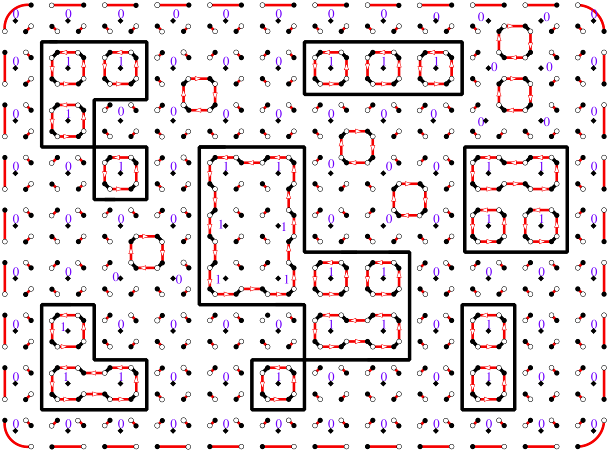

Pictorial proof of "Polygon configurations of the graph G^* are level lines of the restriction of the height function."

Isoradial graphs

- A graph G is isoradial if it is planar and can be embedded in the plane in such a way that all faces are inscribed in a circle of radius 1, and that the circumcenters are in the interior of the faces (Duffin-Mercat-Kenyon).

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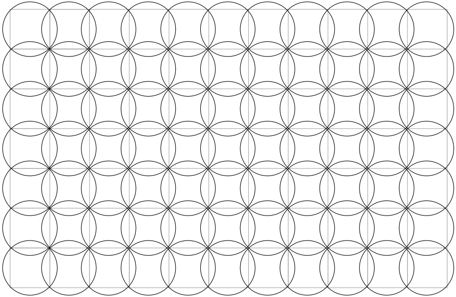

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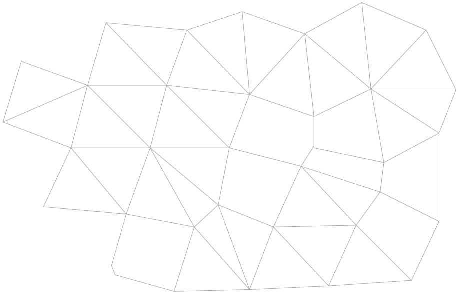

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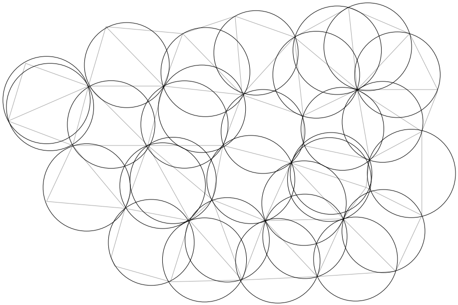
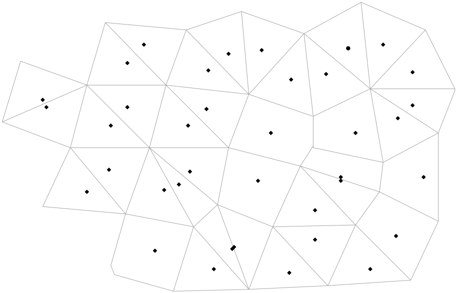


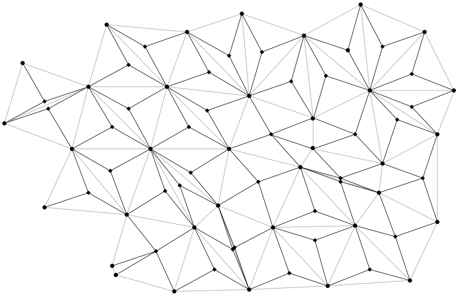
Associated rhombus graph, angles

- Take the circumcenters.



Associated rhombus graph, angles

- Join the circumcenters to the vertices of the graph G.
  $\Rightarrow$ Associated rhombus graph $\mathsf{G}^\diamond$.



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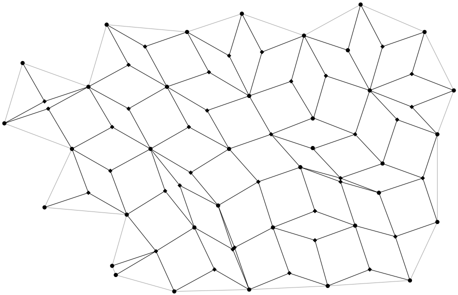
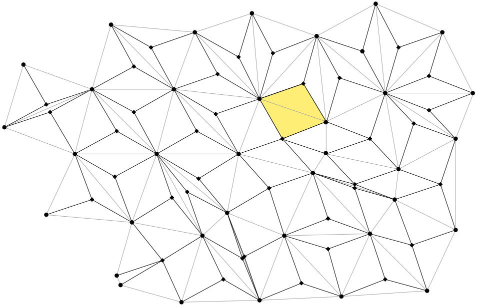
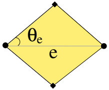


Associated rhombus graph, angles

- To every edge e is assigned the half-angle θ_e of the corresponding rhombus.





Critical 2-dimensional Ising model on isoradial graphs

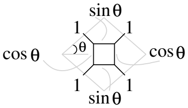
- The Ising model defined on an isoradial graph G is critical if the coupling constants are given by, for every edge e:

$$J_{\mathsf{e}} = \frac{1}{2}\log\left(\frac{1+\sin\theta_{\mathsf{e}}}{\cos\theta_{\mathsf{e}}}\right).$$

  (Z-invariance + duality [Baxter], proof in period. case [Li-Dum.& Cim.])

  Example: $\mathsf{G}=\mathbb{Z}^2$: $\forall$ edge e, $\theta_{\mathsf{e}}=\frac{\pi}{4}$, $J_{\mathsf{e}}=\frac{1}{2}\log(1+\sqrt{2})$.
  $\Rightarrow$ critical temperature computed by Kramers & Wannier.

- The corresponding bipartite graph G^{Q} is also isoradial, and the weights are the critical dimer weights:



Back to Wilson's conjecture

Conjecture (Wilson)

The scaling limit of polygon configurations separating ±1 clusters of the critical XOR-Ising model are contour lines of the Gaussian free field, with the heights of the contours spaced $\sqrt{2}$ times as far apart as they are for [...] the double dimer model on the square lattice.

Theorem (Boutillier, dT)

XOR-polygon configurations of the double Ising model on G *have the same law as level lines of a restriction of the height function of the bipartite dimer model on* G^{Q}*, with an explicit coupling.*

Theorem (dT)

The height function (as a random distribution) of the critical dimer model defined on a bipartite graph converges weakly in law to $\frac{1}{\sqrt{\pi}}$ *a Gaussian free field of the plane.*

Back to Wilson's conjecture

Suppose we had strong form of convergence, allowing for convergence of level lines. Then:

level lines of h^ε	$\rightarrow$	level lines of GFF	
$(k, k \in \mathbb{Z})$		$(\sqrt{\pi}k, k \in \mathbb{Z})$	
$(k+\frac{1}{2}, k \in \mathbb{Z})$		$(\frac{\sqrt{\pi}}{2}(2k+1), k \in \mathbb{Z})$	XOR loops

For the critical double dimer model. The height function is $h_1^\varepsilon - h_2^\varepsilon$, where h_1 and h_2 are independent, and each converges weakly in distribution to $\frac{1}{\sqrt{\pi}}$ a Gaussian free field. Thus, $h_1 - h_2$ converges weakly in distribution to $\frac{\sqrt{2}}{\sqrt{\pi}}$ a Gaussian free field.

level lines of $h_1^\varepsilon - h_2^\varepsilon$	$\rightarrow$	level lines of GFF	
$(k, k \in \mathbb{Z})$		$(\frac{\sqrt{\pi}}{\sqrt{2}}k, k \in \mathbb{Z})$	
$(k+\frac{1}{2}, k \in \mathbb{Z})$		$(\frac{\sqrt{\pi}}{2\sqrt{2}}(2k+1), k \in \mathbb{Z})$	d-dimer loops