Percolation on isoradial graphs

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Homogeneous percolation on $\mathbb{Z}^2$: all edges have intensity $p \in [0, 1]$.

Question: is there an infinite connected component?

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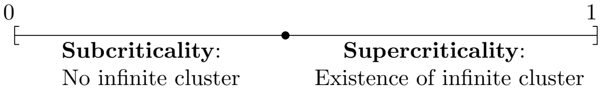

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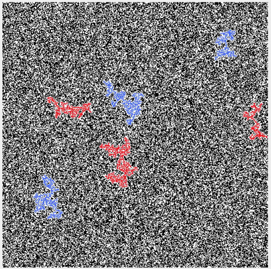
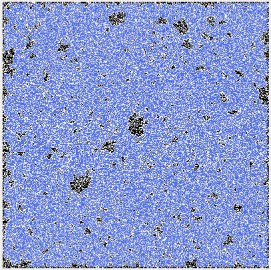
Question: is there an infinite connected component?

0 —————•————— 1

Subcriticality:
No infinite cluster
Exponential tail for cluster size
Trivial large scale behaviour!

Supercriticality:
Existence of infinite cluster
Unique infinite cluster
Exponential tail for distance to infinite cluster.
Trivial large scale behaviour!





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0 — p_c — 1

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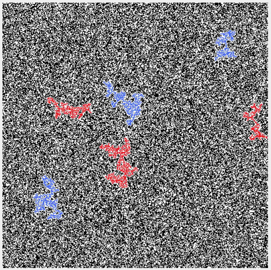
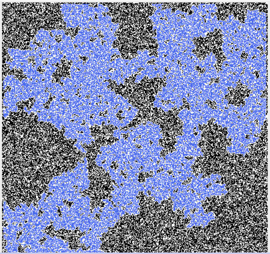
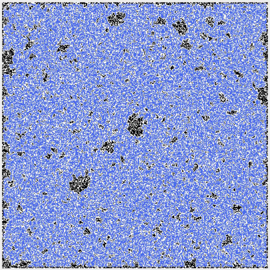
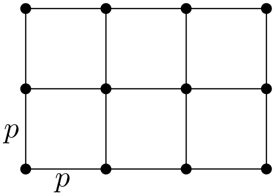
Criticality
Scale invariance
Large scale limit.
Universality.







Homogeneous bond percolation on $\mathbb{Z}^2$



Theorem (Kesten 80)

$p \leq \frac{1}{2}$, *a.s. no infinite cluster;*

$p > \frac{1}{2}$, *a.s. existence of infinite cluster.*

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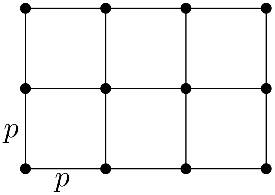

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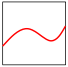
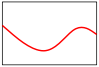
Method:

self-duality + RSW + sharp-threshold

$\mathbb{P}_{\frac{1}{2}}\Big($  $\Big) = \frac{1}{2} \Rightarrow \mathbb{P}_{\frac{1}{2}}\Big($  $\Big) \geq c \Rightarrow \mathbb{P}_{\frac{1}{2}+\epsilon}(0 \leftrightarrow \infty) > 0$

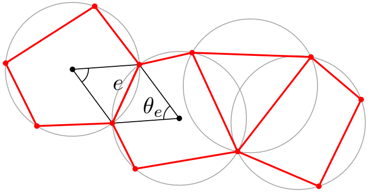
Also implies:

$p < 1/2 \Rightarrow$ exponential decay.

$p > 1/2 \Rightarrow$ exponential decay of holes in infinite cluster.

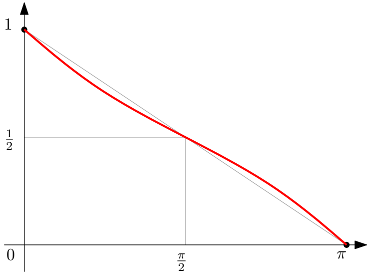
$p = 1/2 \Rightarrow$ power-law bounds.

Isoradial percolation

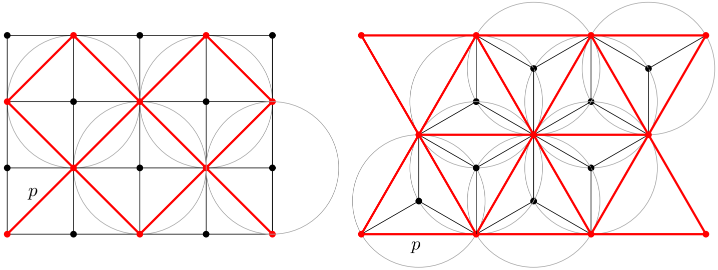


Each face of G is inscribed in a circle of radius 1.
$\mathbb{P}_G$ percolation with p_e:

$$\frac{p_e}{1-p_e} = \frac{\sin\left(\frac{\pi-\theta(e)}{3}\right)}{\sin\left(\frac{\theta(e)}{3}\right)}.$$



Inhomogeneous models on lattices



$p = \frac{1}{2}$, $p = 2\sin\frac{\pi}{18}$

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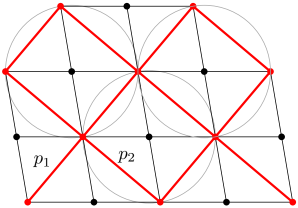

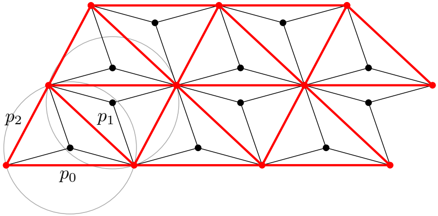

$p_1 + p_2 = 1,$

$\kappa_\triangle(\mathbf{p}) = p_0 + p_1 + p_2 - p_0 p_1 p_2 = 1$

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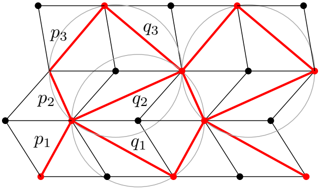
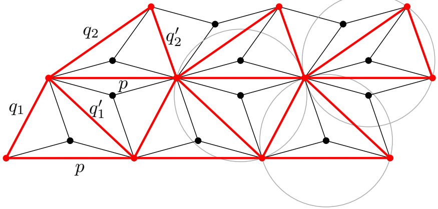


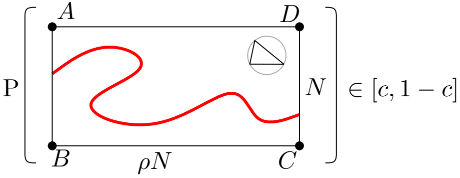


$p_i + q_i = 1,$

$\kappa_\triangle(p, q_i, q'_i) = p + q_i + q'_i - pq_iq'_i = 1$

The box-crossing property (RSW)

A model satisfies the box-crossing property if for all rectangles $ABCD$ there exists $c(BC/AB) = c(\rho) > 0$ s. t. for all N large enough:



Equivalent for the primal and dual model.

Theorem

If $\mathbb{P}_{\mathbf{p}}$ satisfies the box-crossing property, then it is critical.

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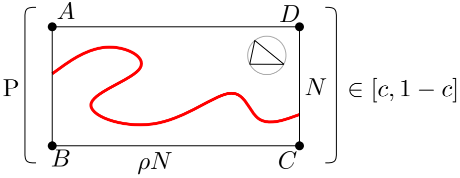

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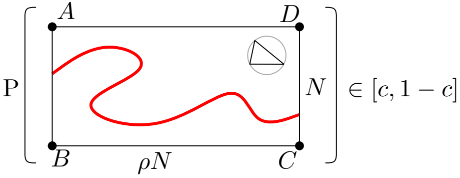

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Results I: the box-crossing property

For a periodic isoradial graph G with the percolation measure $\mathbb{P}_G$

Theorem (G.Grimmet, I.M.)

$\mathbb{P}_G$ satisfies the box-crossing property.

Corollary

$\mathbb{P}_G$ is critical.

- $\mathbb{P}_{\mathbf{p}}(\textit{infinite cluster}) = 0$,
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Arm exponents

For a critical percolation measure $\mathbb{P}$,
as $n \to \infty$, we expect:

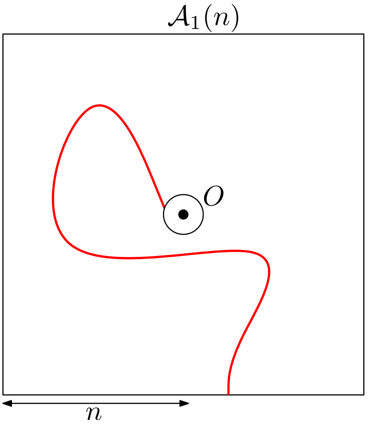
- one-arm exponent $\frac{5}{48}$:

$$\mathbb{P}(\mathrm{rad}(C_0) \geq n) = \mathbb{P}(A_1(n)) \approx n^{-\rho_1},$$

- $2j$-alternating-arms exponents $\frac{4j^2-1}{12}$:

$$\mathbb{P}[A_{2j}(n)] \approx n^{-\rho_{2j}}.$$

Moreover ρ_i **does not depend on the underlying model**.



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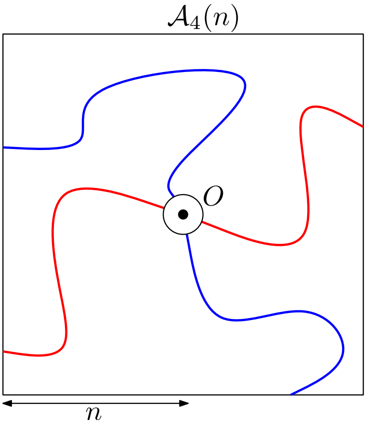

Power-law bounds are given by the box-crossing property.

For $\mathbb{P}_{\mathbf{p}}$ critical we expect:

Exponents at criticality.

Volume exponent $\delta = \frac{91}{5}$:
$\mathbb{P}_{\mathbf{p}}(|C_0| = n) \approx n^{-1-1/\delta}$.

Connectivity exponent $\eta = \frac{5}{24}$:
$\mathbb{P}_{\mathbf{p}}(0 \leftrightarrow x) \approx |x|^{-\eta}$.

Radius exponent $\rho = \frac{48}{5}$:
$\mathbb{P}_{\mathbf{p}}(\mathrm{rad}(C_0) = n) \approx n^{-1-1/\rho}$.

$(\rho = \frac{1}{\rho_1})$

Exponents near criticality.

Percolation probability $\beta = \frac{5}{36}$:
$\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0| = \infty) \approx \epsilon^{\beta}$ as $\epsilon \downarrow 0$.

Correlation length $\nu = \frac{4}{3}$:
$\xi(\mathbf{p} - \epsilon) \approx \epsilon^{-\nu}$ as $\epsilon \downarrow 0$, were
$-\frac{1}{n} \log \mathbb{P}_{\mathbf{p}-\epsilon}(\mathrm{rad}(C_0) \geq n) \to_{n\to\infty} \frac{1}{\xi(\mathbf{p}-\epsilon)}$.

Mean cluster-size $\gamma = \frac{43}{18}$:
$\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0|; |C_0| < \infty) \approx |\epsilon|^{-\gamma}$ as $\epsilon \to 0$.

Gap exponent $\Delta = \frac{91}{36}$:
$\frac{\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0|^{k+1}; |C_0| < \infty)}{\mathbb{P}_{\mathbf{p}+\epsilon}(|C_0|^{k}; |C_0| < \infty)} \approx |\epsilon|^{-\Delta}$. for $k \geq 1$, as $\epsilon \to 0$.

Kesten scaling relations: these exponents are functions of 1 and 4 arm exponents.
(some symmetry conditions are necessary)

Results II: arm exponents

For a periodic isoradial graph G with the percolation measure $\mathbb{P}_G$

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For $k \in \{1, 2, 4, \dots\}$ there exist constants $c_1, c_2 > 0$ such that:

$$c_1 \mathbb{P}_{\mathbb{Z}^2}[A_k(n)] \le \mathbb{P}_G[A_k(n)] \le c_2 \mathbb{P}_{\mathbb{Z}^2}[A_k(n)],$$

for $n \in \mathbb{N}$.

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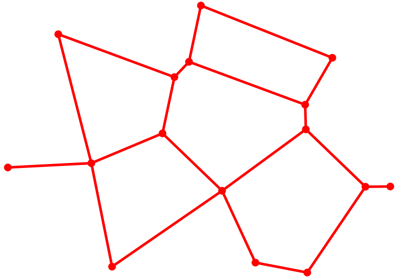
Corollary

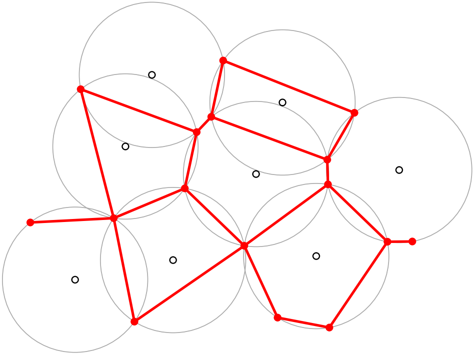
The one arm exponent and the $2j$ alternating arm exponents are universal for percolation on isoradial graphs.

Isoradial Graphs



G isoradial graph

Isoradial Graphs



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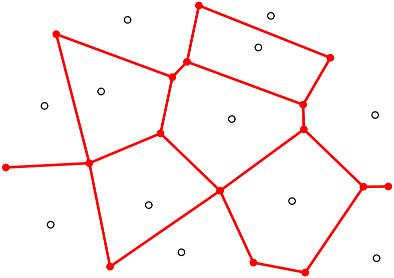

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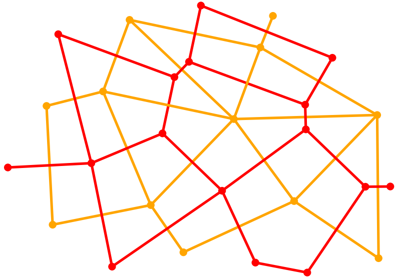

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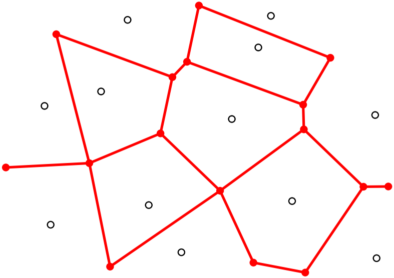

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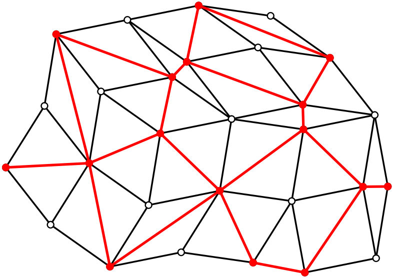

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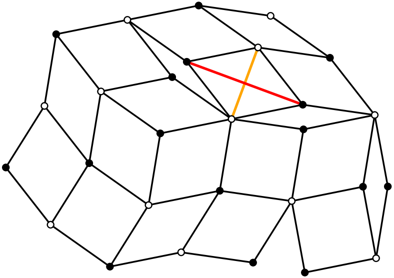

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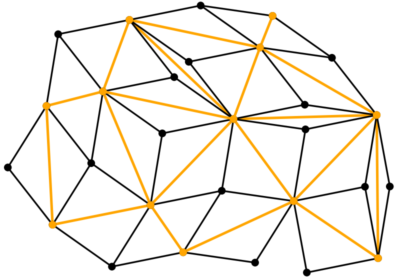

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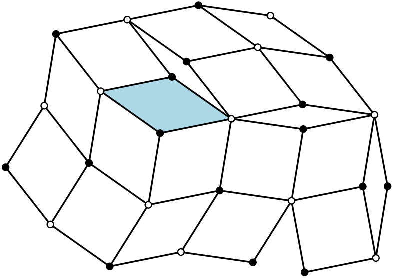

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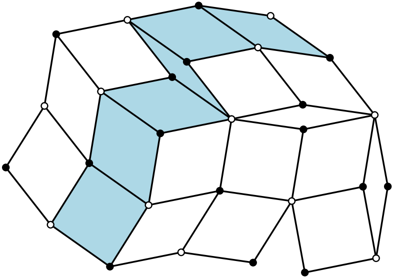

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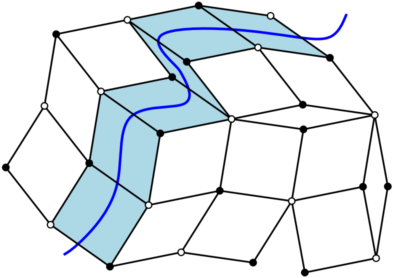

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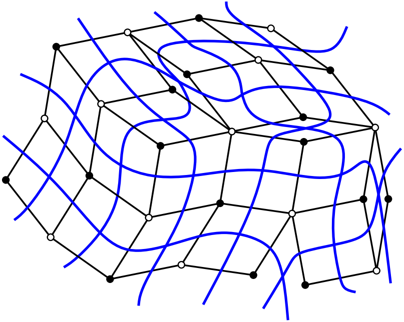

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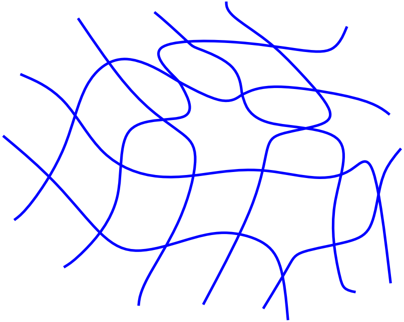
Track system

Isoradial Graphs



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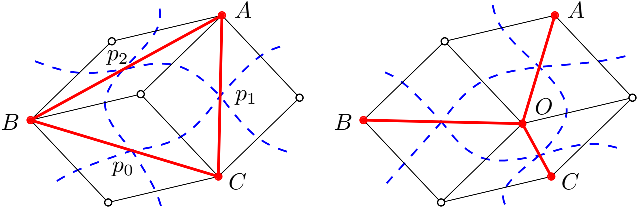
Star–triangle transformation



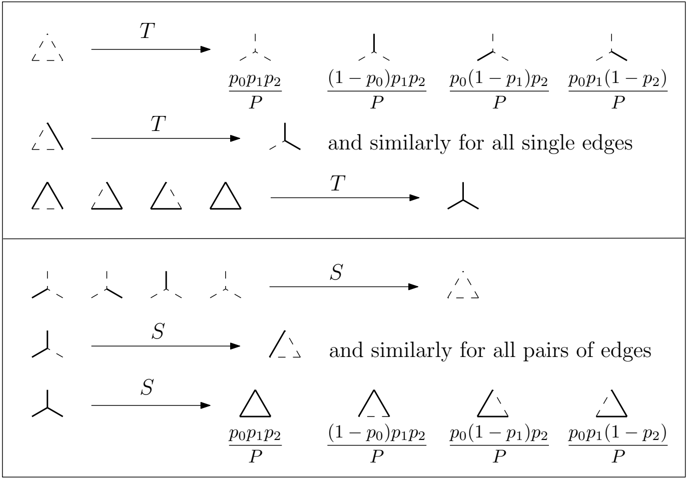
$$\kappa_{\triangle}(\mathbf{p}) = p_0 + p_1 + p_2 - p_0 p_1 p_2 = 1.$$

Take ω, respectively ω', according to the measure on the left, respectively right. The families of random variables

$$\left(x \overset{\omega}{\longleftrightarrow} y : x, y = A, B, C\right), \quad \left(x \overset{\omega'}{\longleftrightarrow} y : x, y = A, B, C\right),$$

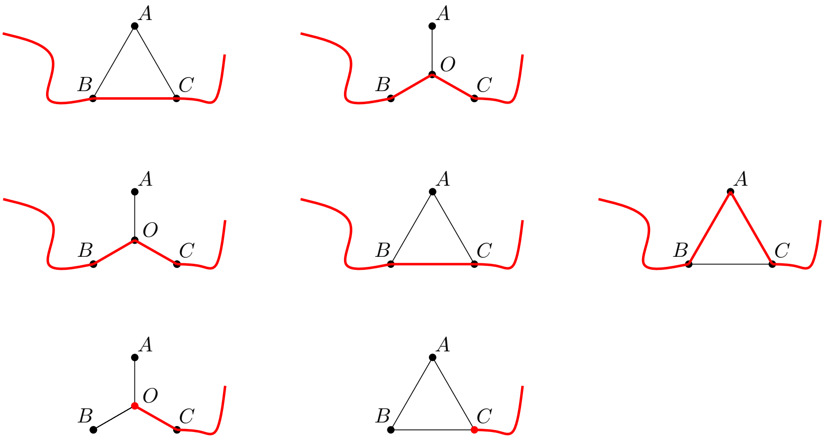
have the same joint law.

Coupling



where $P = (1 - p_0)(1 - p_1)(1 - p_2)$.

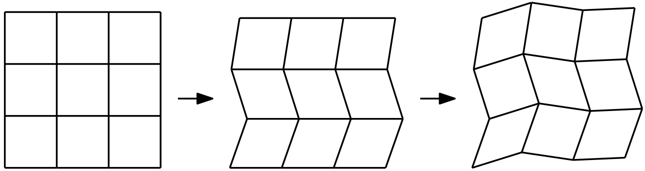
Path transformation



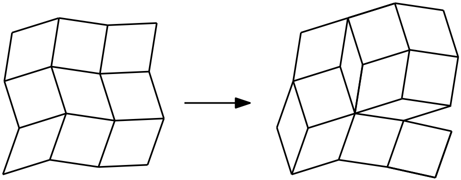
Strategy of proofs

Transform a regular square lattice into any isoradial graph; preserve properties (such as box-crossing property and arm exponents)

Step 1: From regular square lattice to any embedding of the square lattice.



Step 2: From square lattices to all periodic isoradial graphs



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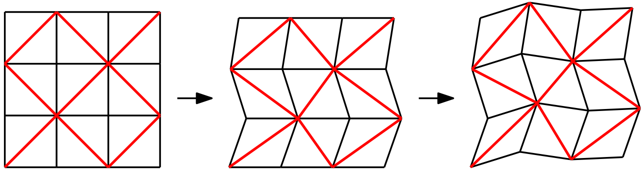

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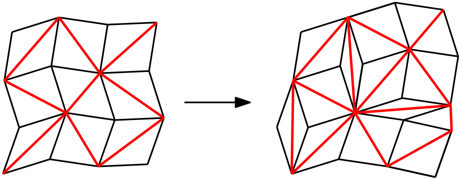
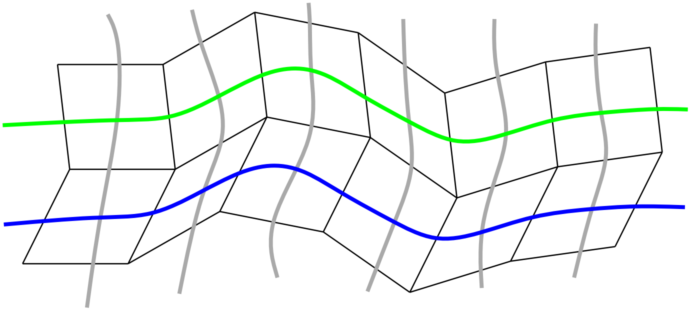


Track exchange

Two parallel tracks s_1 and s_2 with no intersection between them.
We may exchange s_1 and s_2 using star–triangle transformations.



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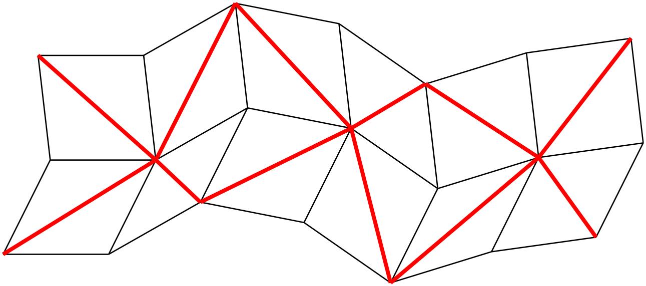

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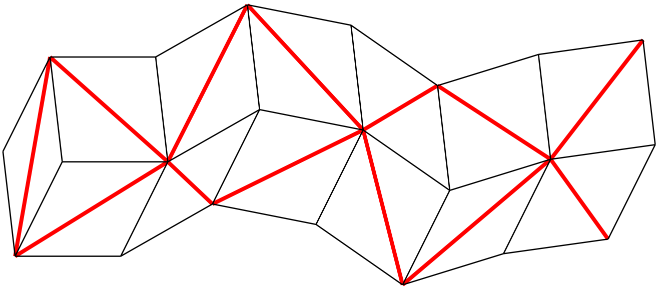

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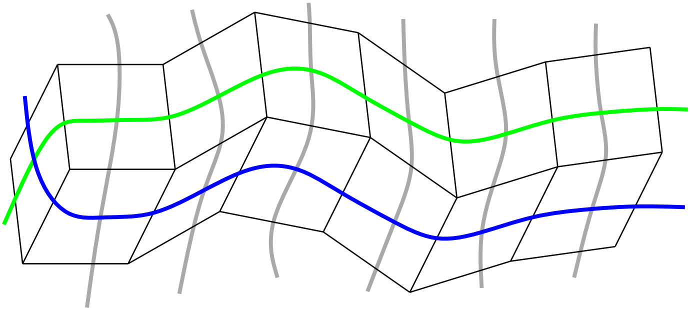

Track exchange

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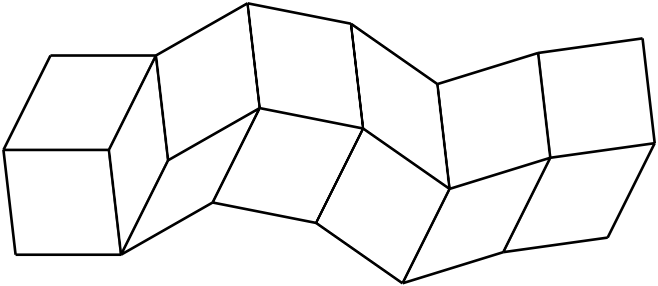

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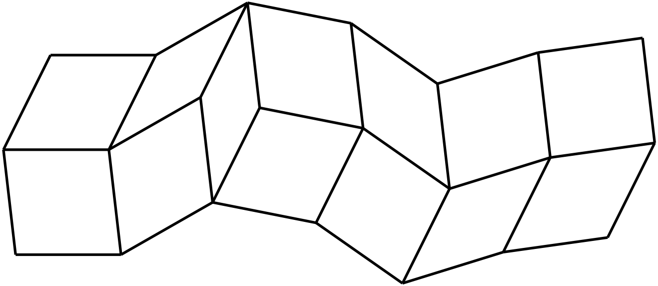

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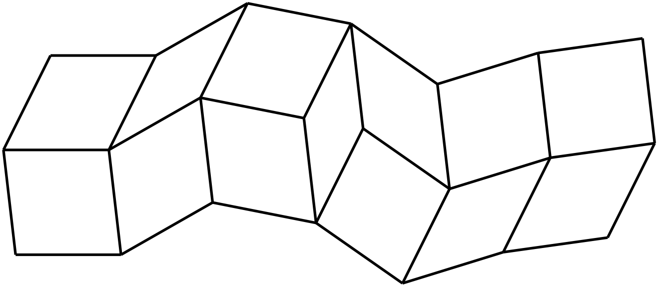

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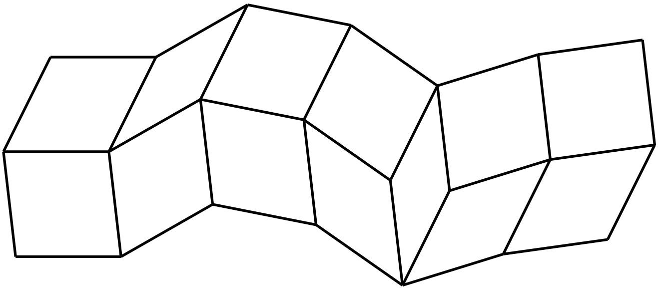

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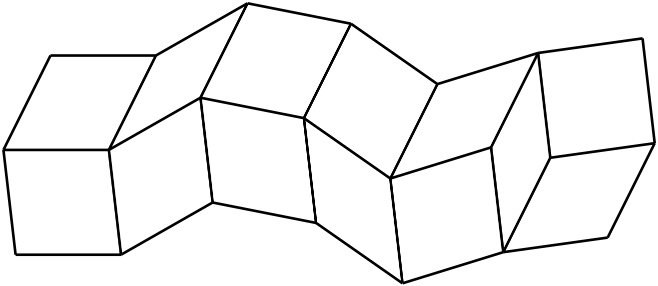

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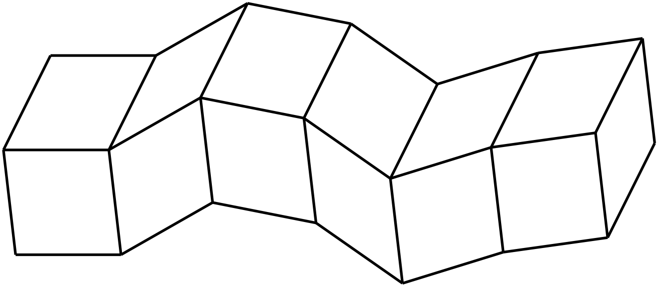

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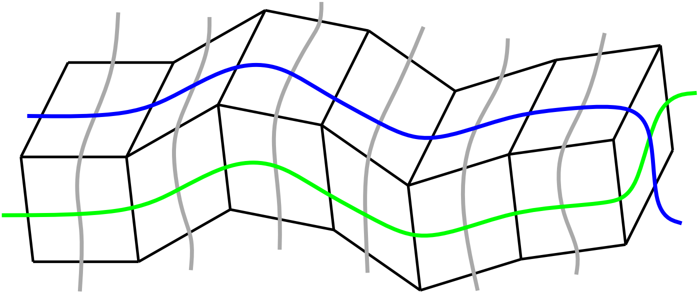

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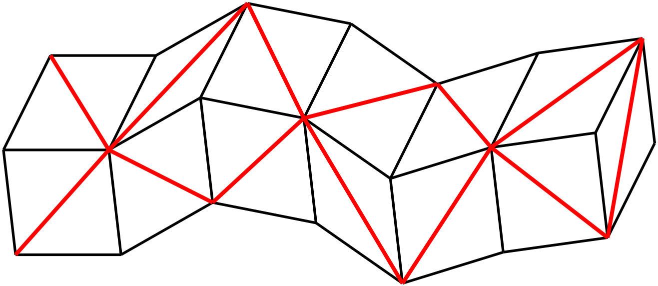

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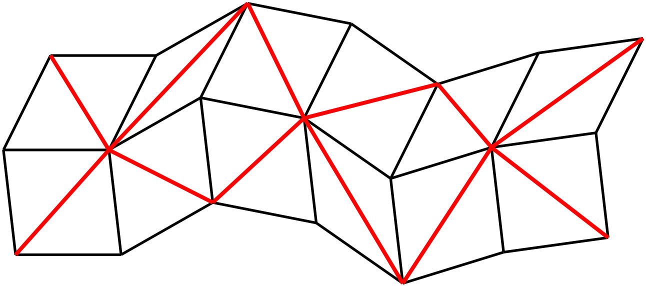

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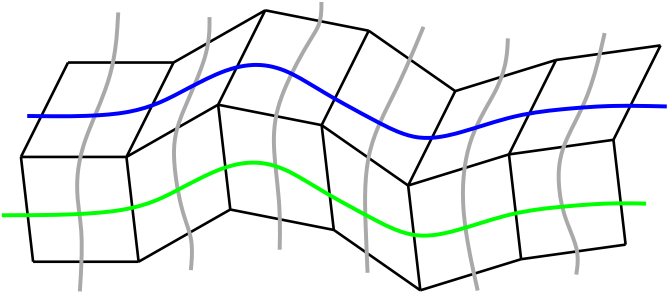
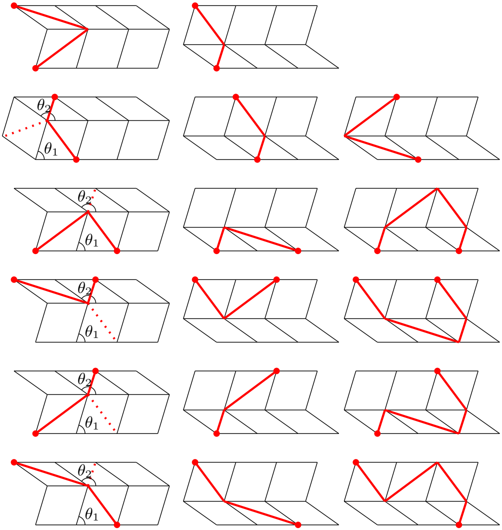


Initial configuration	Principal outcome	Secondary outcome	Probability of secondary outcome



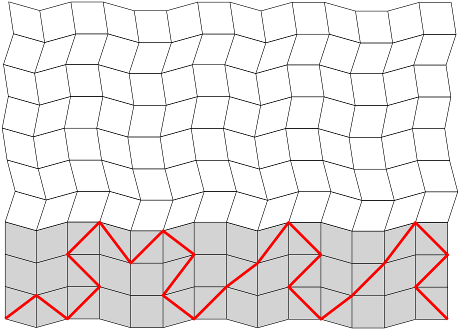
Probability of secondary outcome, by row:

1. —
2. $\frac{p_{\pi-\theta_1} p_{\theta_2}}{p_{\theta_1} p_{\pi-\theta_2}}$
3. $\frac{p_{\pi-\theta_1} p_{\pi-\theta_2+\theta_1}}{p_{\theta_1} p_{\theta_2-\theta_1}}$
4. $\frac{p_{\theta_2} p_{\pi-\theta_2+\theta_1}}{p_{\pi-\theta_2} p_{\theta_2-\theta_1}}$
5. $\frac{p_{\theta_2} p_{\pi-\theta_2+\theta_1}}{p_{\pi-\theta_2} p_{\theta_2-\theta_1}}$
6. $\frac{p_{\pi-\theta_1} p_{\pi-\theta_2+\theta_1}}{p_{\theta_1} p_{\theta_2-\theta_1}}$

Open paths are preserved (unless the deleted edge was part of the path).

Transport of horizontal crossings

Construct a mixed isoradial square lattice:
"regular" in the gray part, "irregular" in the rest.



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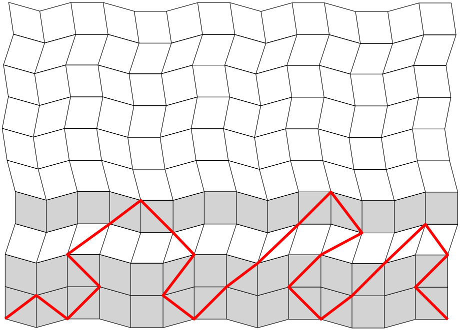

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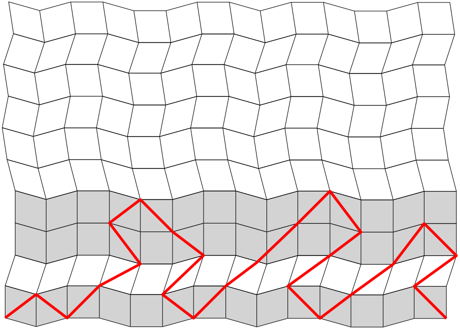

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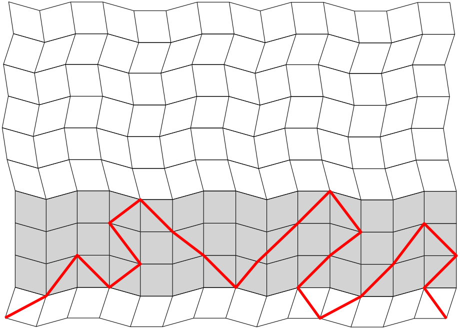

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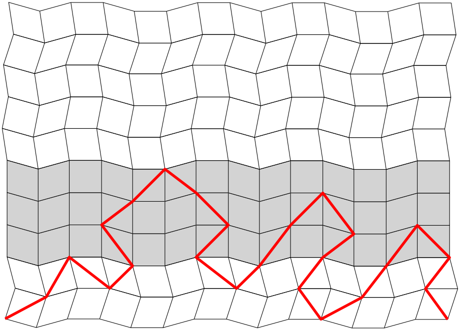

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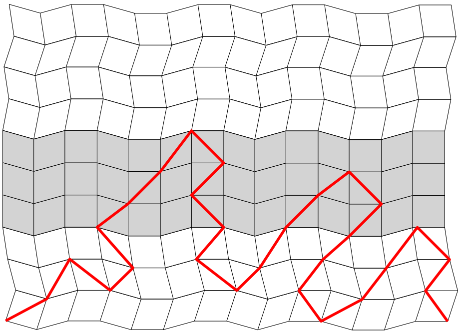

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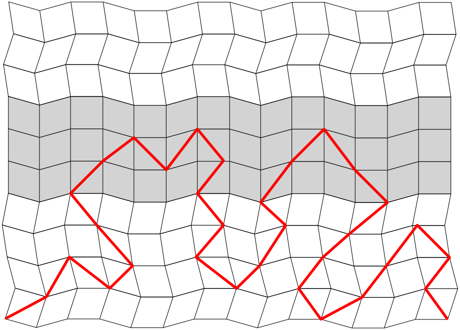

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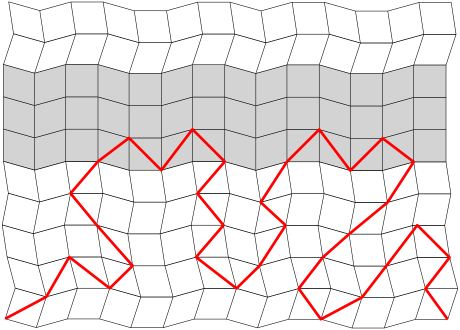

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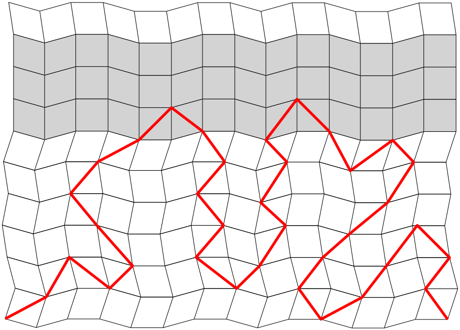

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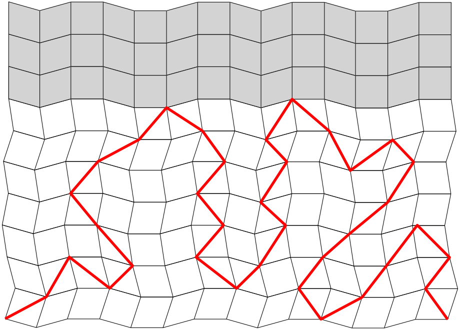

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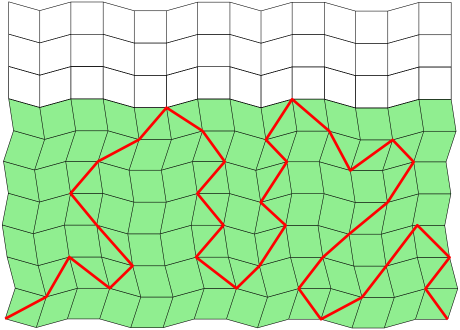

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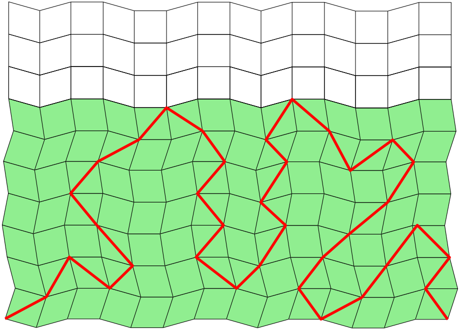
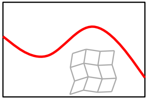
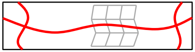
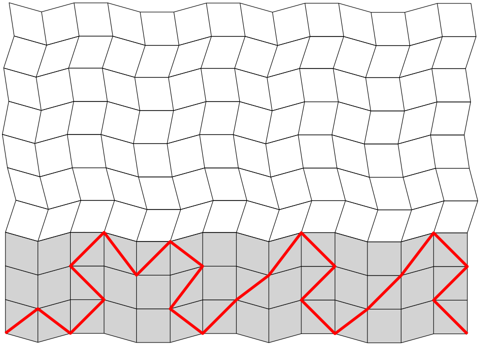


We obtain lower bounds on horizontal crossings in the irregular part:

$\mathbb{P}_{irreg}\Bigg($  $\Bigg) \geq c \cdot \mathbb{P}_{reg}\Bigg($  $\Bigg)$

Transport of horizontal crossings



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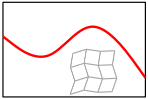
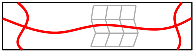
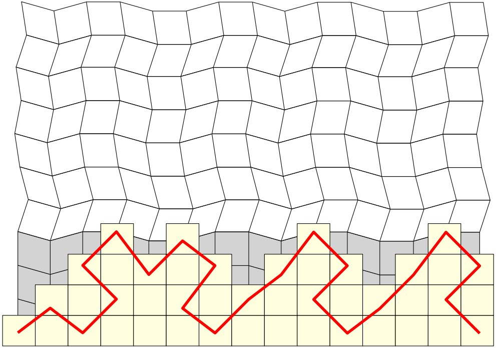
$\mathbb{P}_{irreg}$ () $\geq c \cdot \mathbb{P}_{reg}$ ()

Transport of horizontal crossings



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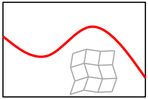
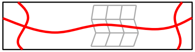
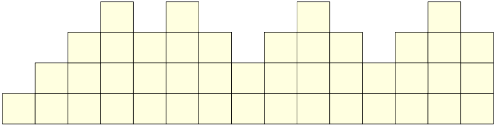
$\mathbb{P}_{irreg}\Bigg($  $\Bigg) \geq c \cdot \mathbb{P}_{reg}\Bigg($  $\Bigg)$

Transport of horizontal crossings



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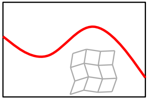
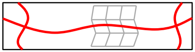
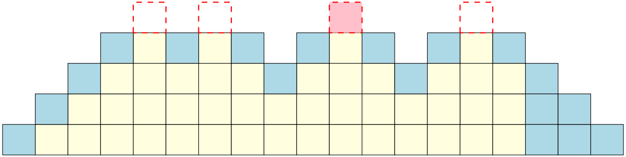
$\mathbb{P}_{irreg}$ () $\geq c \cdot \mathbb{P}_{reg}$ ()

Transport of horizontal crossings



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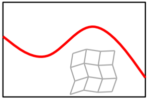
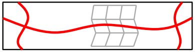
$$\mathbb{P}_{irreg}\Big(\ \Big) \geq c \cdot \mathbb{P}_{reg}\Big(\ \Big)$$





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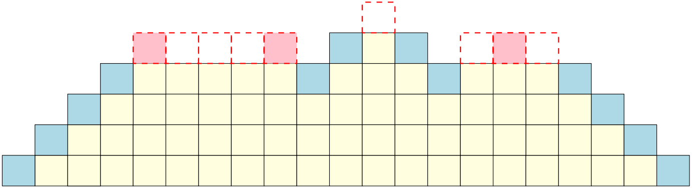

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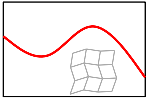
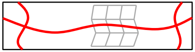
$\mathbb{P}_{irreg}$ () $\geq c \cdot \mathbb{P}_{reg}$ ()

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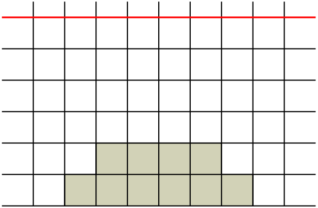

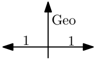

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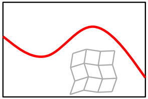
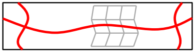
$\mathbb{P}_{irreg}$ () $\geq c \cdot \mathbb{P}_{reg}$ ()

Transport of vertical crossings

Construct a mixed isoradial square lattice:
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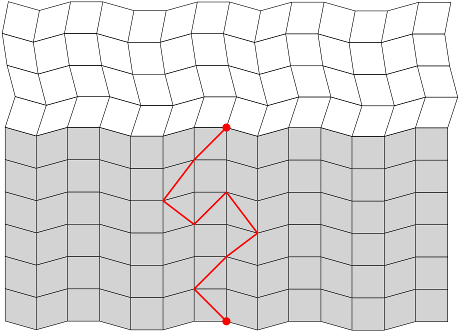

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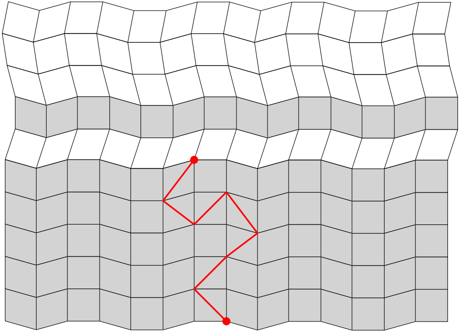

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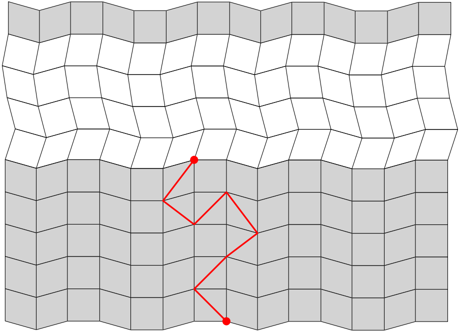

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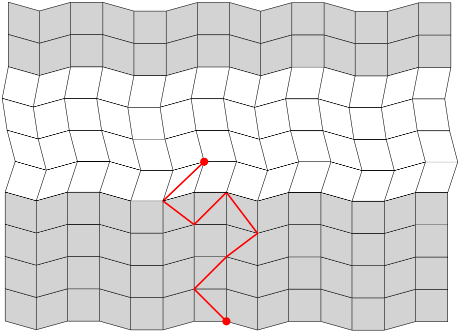

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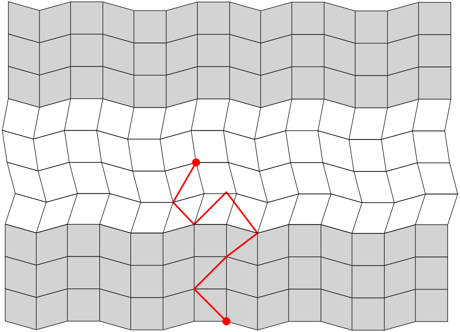

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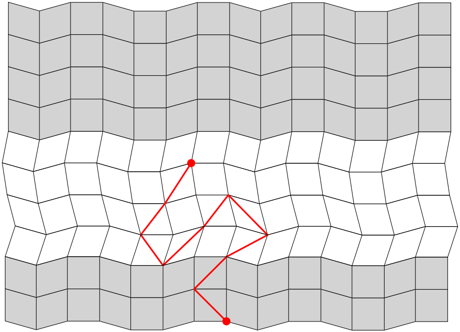

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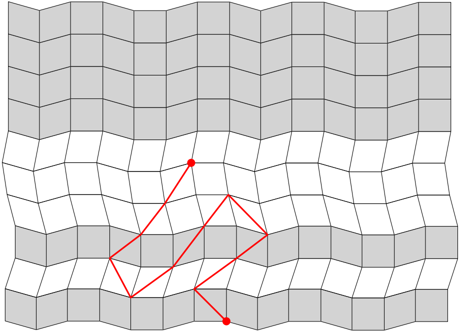

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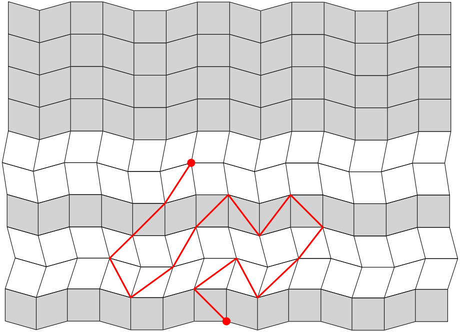

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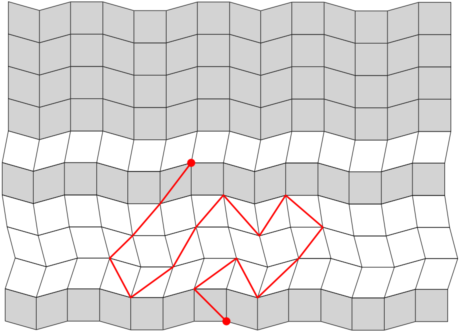

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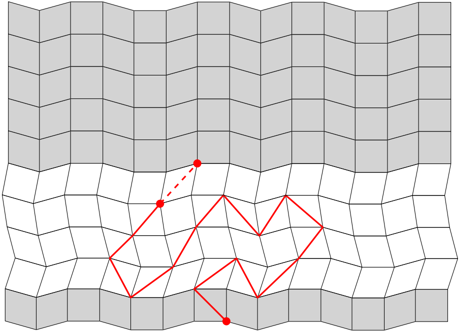

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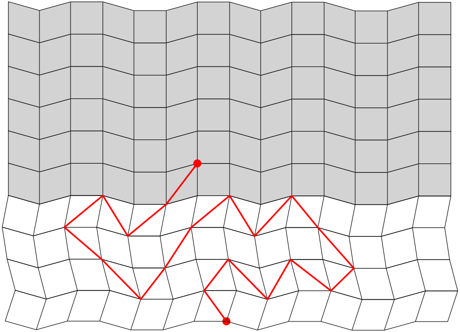
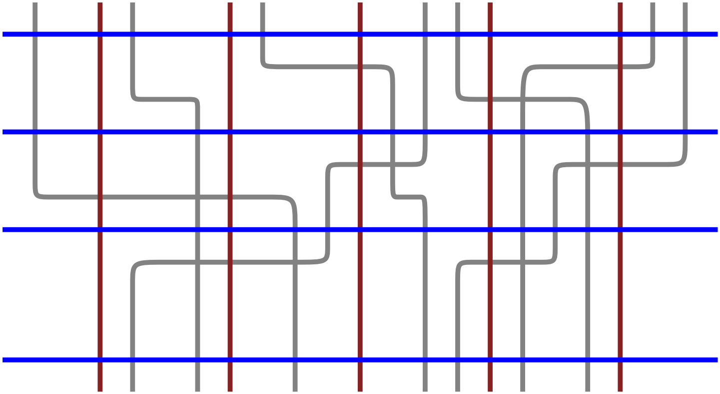
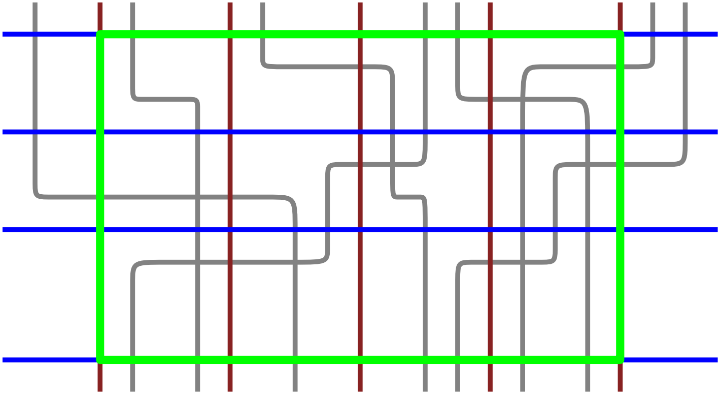


Track stacking



Track stacking



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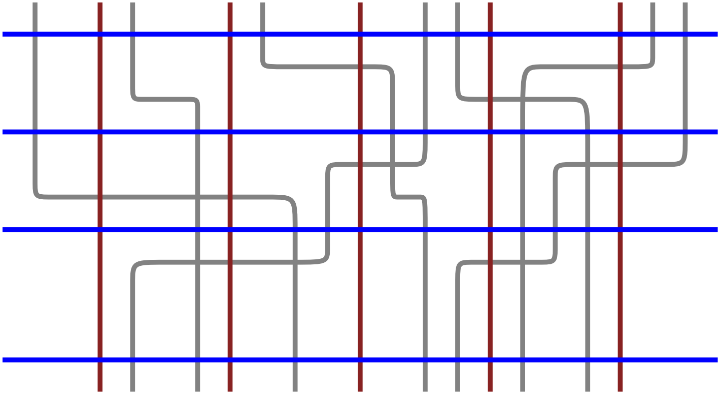

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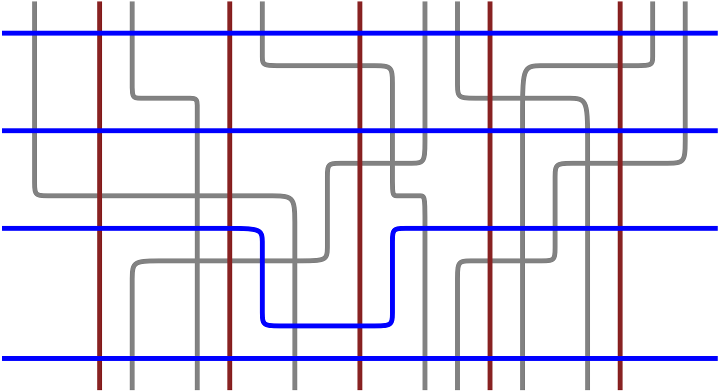
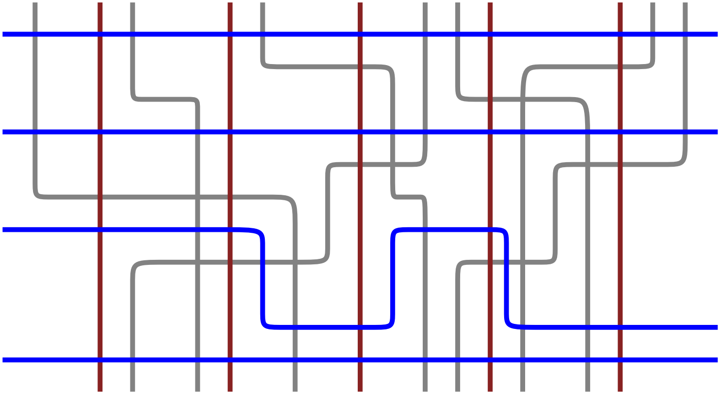
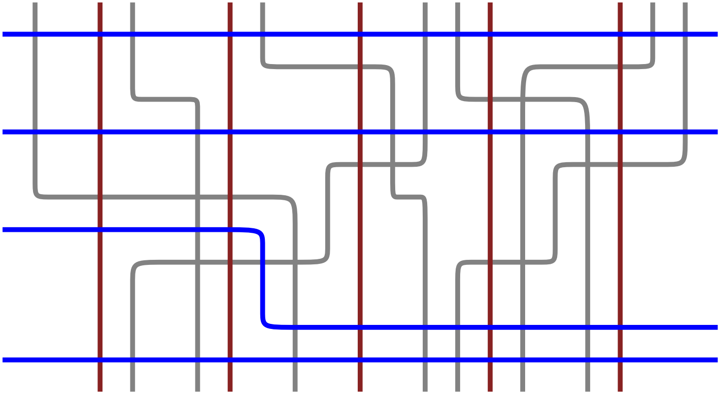


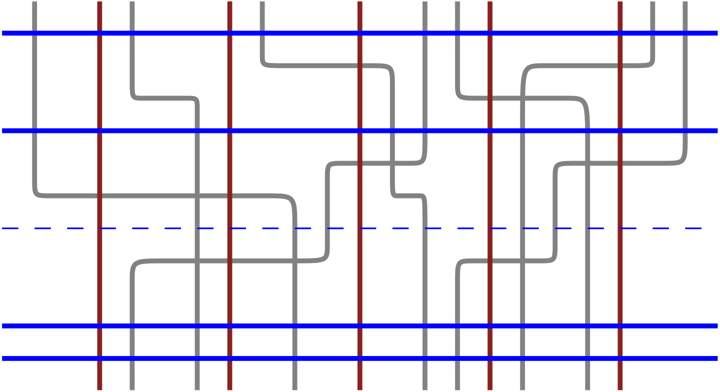
Track stacking



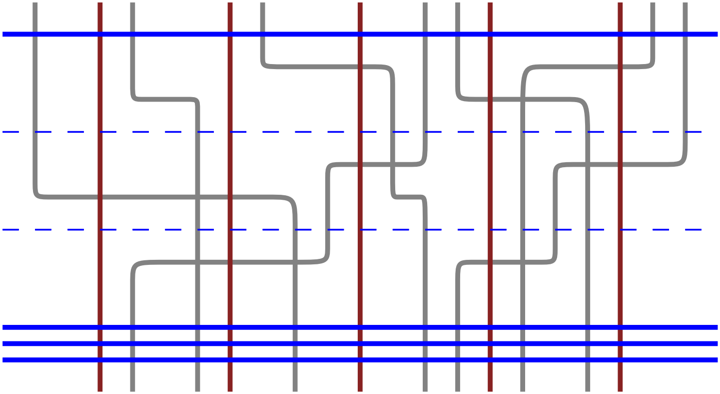
Track stacking



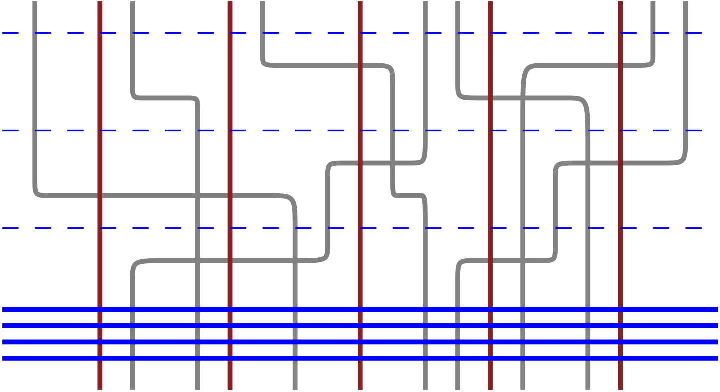
Track stacking



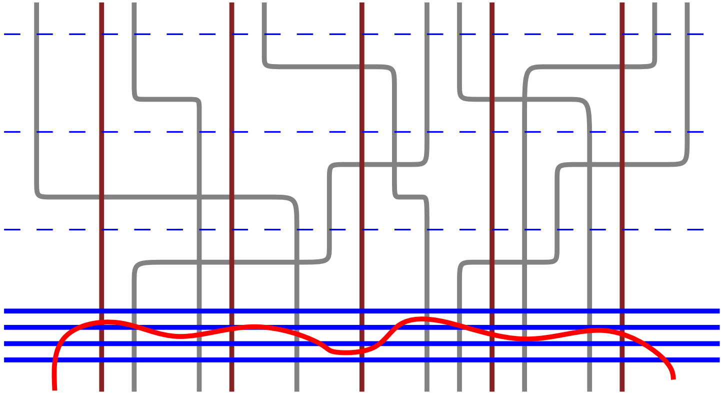
Track stacking



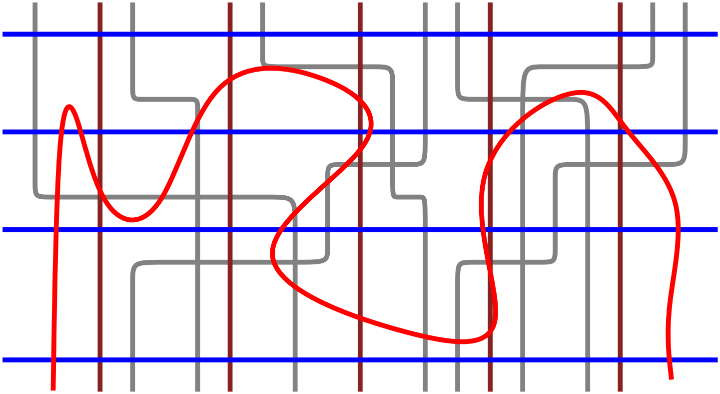
Track stacking



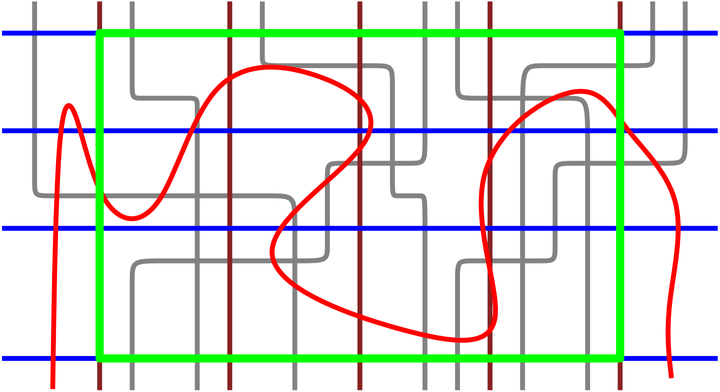
Track stacking



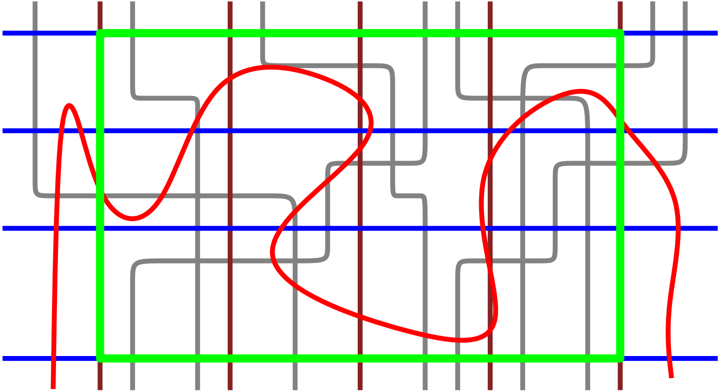
Track stacking

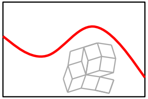
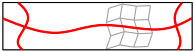


Track stacking



Track stacking

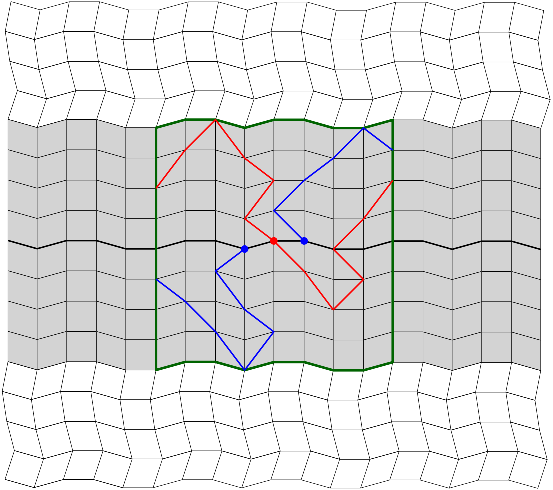


$\mathbb{P}_{irreg}\Big($  $\Big) \geq c \cdot \mathbb{P}_{reg}\Big($  $\Big)$

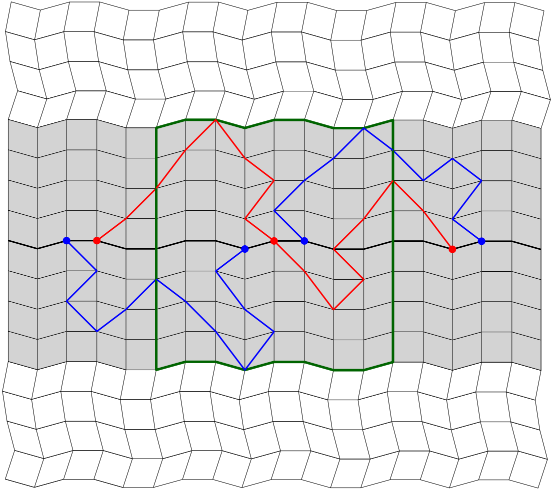
Transport of the arm exponents . . .

. . . using the same strategy as for the box-crossing property.

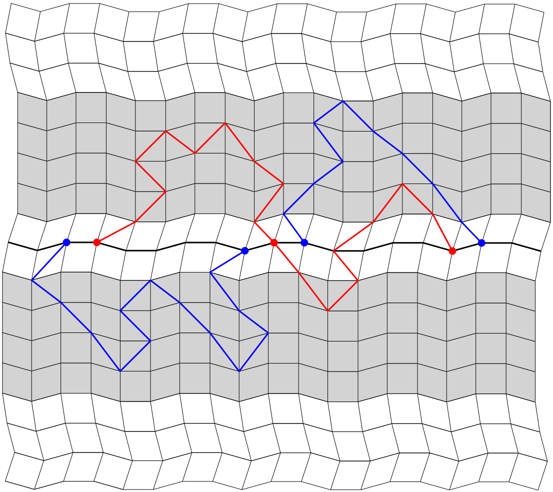
Square lattices



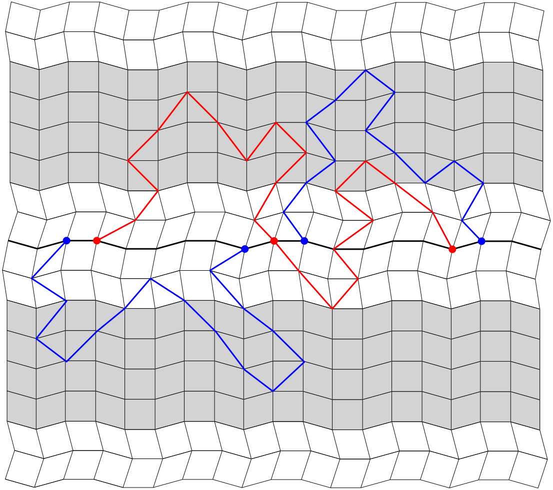
Square lattices



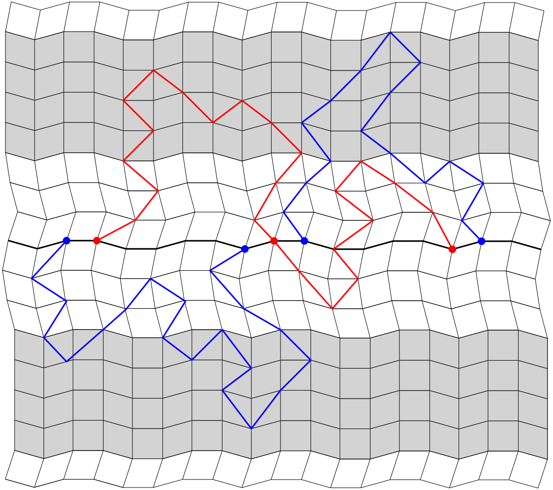
Square lattices



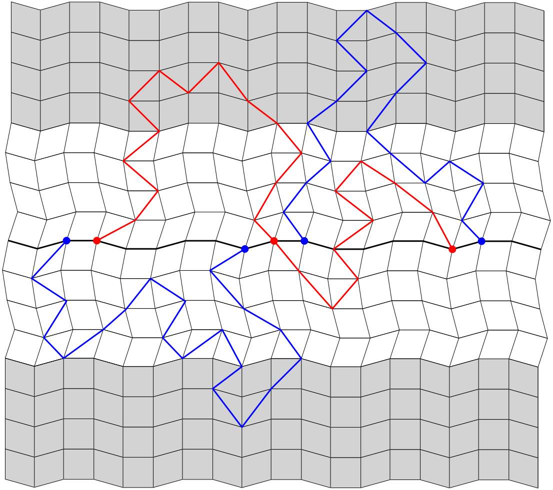
Square lattices



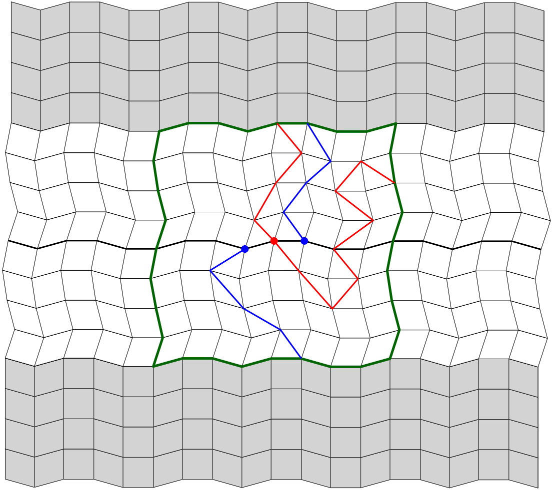
Square lattices



Square lattices

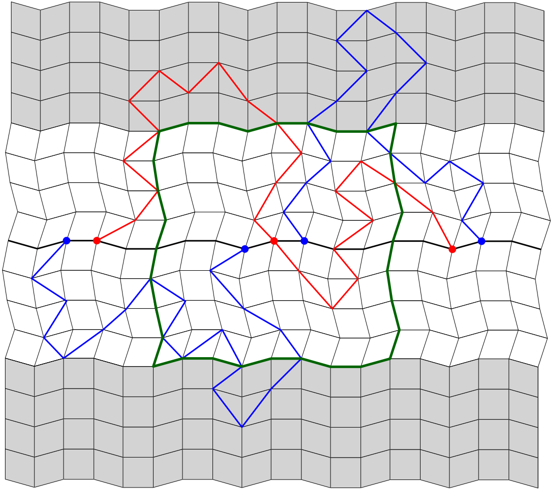


Square lattices



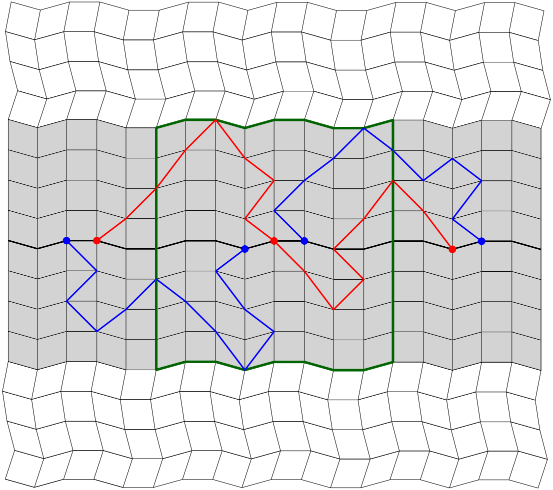
$$c_1 \mathbb{P}_{reg}(A_k(n)) \leq \mathbb{P}_{irreg}(A_k(n))$$

Square lattices



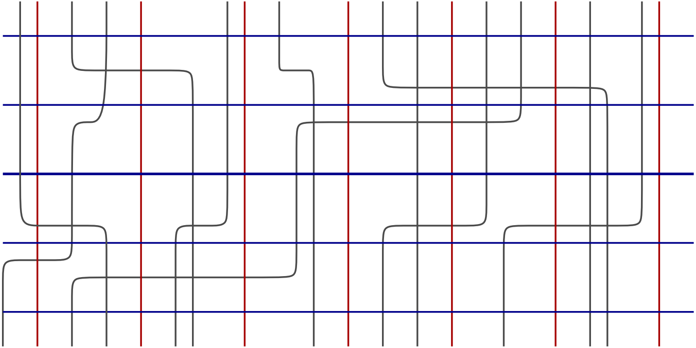
$$c_1 \mathbb{P}_{reg}(A_k(n)) \leq \mathbb{P}_{irreg}(A_k(n))$$

Square lattices

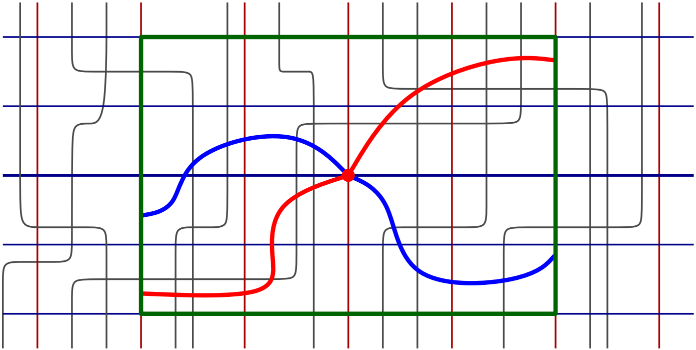


$$c_1 \mathbb{P}_{reg}(A_k(n)) \leq \mathbb{P}_{irreg}(A_k(n)) \leq c_2 \mathbb{P}_{reg}(A_k(n)).$$

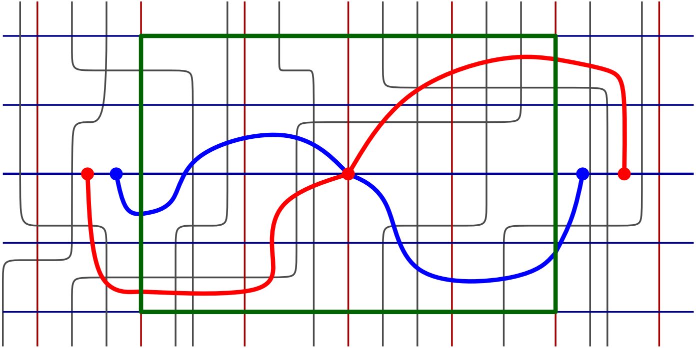
From square lattices to general graphs



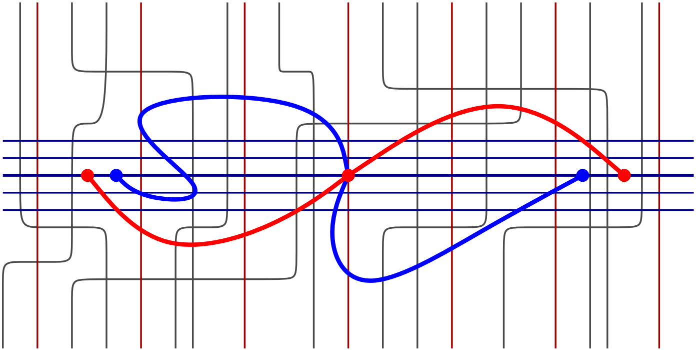
From square lattices to general graphs



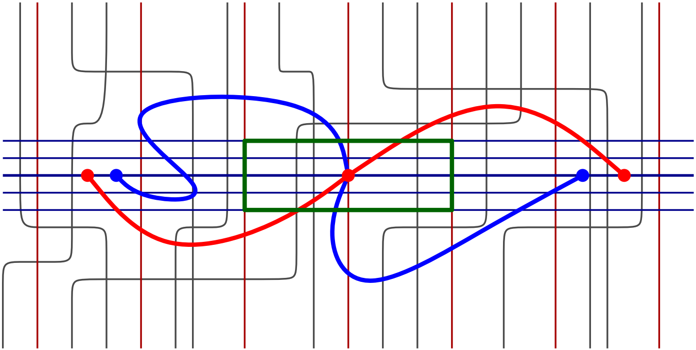
From square lattices to general graphs



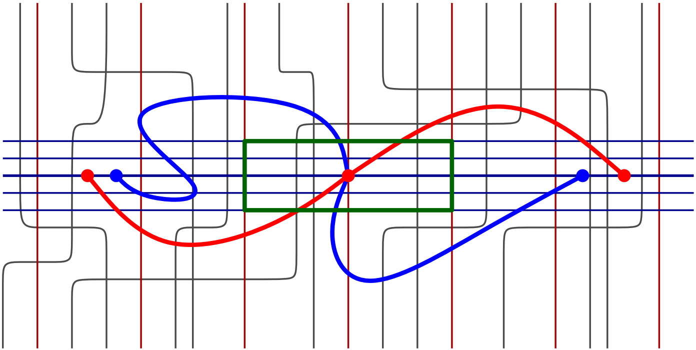
From square lattices to general graphs



From square lattices to general graphs



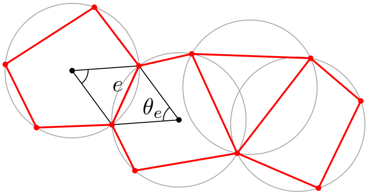
From square lattices to general graphs



$$c_1 \mathbb{P}_{sq}(A_k(n)) \leq \mathbb{P}_{gen}(A_k(n)) \leq c_2 \mathbb{P}_{sq}(A_k(n)).$$

Isoradial Random Cluster

Let G be a finite isoradial graph



$\phi_{G,q}$ random cluster with parameters $q \geq 1$ and p_e:

$$\frac{p_e}{1-p_e} = \sqrt{q}\frac{\sin(\frac{r}{\pi}(\pi-\theta))}{\sin(\frac{r}{\pi}\theta)}, \qquad \text{where } r = \arccos\left(\frac{\sqrt{q}}{2}\right)$$

with the measure given by

$$\phi_{G,q}(\omega) = \frac{1}{Z_G}\prod_{e\in E:\omega_e=1} p_e \prod_{e\in E:\omega_e=0} (1-p_e)\cdot q^{\#\text{clusters}},$$

for $\omega \in \{0,1\}^E$.

Boundary conditions

$$\phi_{G,q}(\omega) = \frac{1}{Z_G} \prod_{e \in E:\omega_e=1} p_e \prod_{e \in E:\omega_e=0} (1-p_e) \cdot q^{\#\text{clusters}},$$

for $\omega \in \{0,1\}^E$.

Different boundary conditions lead to different measures:

- **Wired boundary conditions** $\Rightarrow \phi^1_{G,q}$:
  all clusters touching the boundary are counted as the same one.
- **Free boundary conditions** $\Rightarrow \phi^0_{G,q}$:
  clusters touching the boundary are counted separately.

$$\phi^0_{G,q} \leq_{\mathrm{st}} \phi^1_{G,q}$$

May define infinite volume measures (on infinite graphs G) by taking limits. These may depend on boundary conditions.

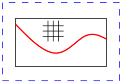
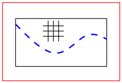
RSW on square lattice

On the regular square lattice:

Theorem (Duminil-Copin, Sidoravicius, Tassion '15)

Depending on q two behaviour are possible:

- *continuous phase transition* ($\phi^0_{\mathbb{Z}^2,q} = \phi^1_{\mathbb{Z}^2,q}$), *then*

  $\phi\Bigg($  $\Bigg) \geq c > 0$ and $\phi\Bigg($  $\Bigg) \geq c > 0$

- *discontinuous phase transition* ($\phi^0_{\mathbb{Z}^2,q} < \phi^1_{\mathbb{Z}^2,q}$), *then*
  $\phi^0_{\mathbb{Z}^2,q}$ *has exponential decay;* $\phi^1_{\mathbb{Z}^2,q}$ *has infinite cluster*

Moreover, for $q \leq 4$*, the phase transition is continuous.*

Expected: for $q > 4$, the phase transition is discontinuous.

Work in progress

With H. Duminil-Copin and J.H. Li:

- For all periodic isoradial graphs G, ϕ_G is critical
- The phase transition is of the same type for all periodic isoradial graphs
- For $q \leq 4$, the arm exponents are the same for all periodic isoradial graphs

Idea of proof:

The star-triangle transformation applies to isoradial random cluster.
The randomness in the star-triangle transformations is independent.
The same estimates apply.

Differences:

Adding/removing edges changes the measure.
Scaling relations do not apply.

Thank you!