Variational Formulas and Disorder Regimes of Random Walks in Random Potentials

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The model

Random walk in random potential (RWRP) on $\mathbb{Z}^d$, with $d \geq 1$, has three ingredients.

(i) **The underlying walk:** Simple RW with a finite set $\mathcal{R} \in \mathbb{Z}^d$ of allowed steps (covers directed and undirected). Induces a probability measure P_x on paths starting at $x \in \mathbb{Z}^d$. Expectations denoted by E_x.

(ii) **The environment:** Take a probability space $(\Omega, \mathfrak{S}, \mathbb{P})$ with a group $\{T_x : x \in \mathbb{Z}^d\}$ of measurable transformations. Assume $\mathbb{P}$ is invariant and ergodic w.r.t. this group. Expectations denoted by $\mathbb{E}$. Each $\omega \in \Omega$ is an environment.

(iii) **The potential:** Take a measurable function $V : \Omega \times \mathcal{R} \to \mathbb{R}$. For every $\omega \in \Omega$, $x \in \mathbb{Z}^d$ and $z \in \mathcal{R}$, the quantity $V(T_x\omega, z)$ is the potential at the ordered pair $(x, x+z)$ in the environment ω.

Given $n \geq 1$ and $\omega \in \Omega$, define the quenched RWRP probability measure

$$Q^{\omega}_{n,x}((X_i)_{i\geq 0} \in \cdot) = \frac{1}{\mathcal{Z}^{\omega}_{n,x}} E_x \left[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1})} \mathbb{1}_{\{(X_i)_{i\geq 0} \in \cdot\}} \right]$$

on paths starting at any $x \in \mathbb{Z}^d$. Here, $(X_i)_{i\geq 0}$ denotes the random path with increments $Z_{i+1} = X_{i+1} - X_i$, and

$$\mathcal{Z}^{\omega}_{n,x} = E_x \left[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1})} \right]$$

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We can take the underlying walk to be an RWRE with kernel given by some $p : \Omega \times \mathcal{R} \to [0, 1]$. This is equivalent to adding $\log p(\omega, z) + \log |\mathcal{R}|$ to the potential $V(\omega, z)$.

The canonical environment space: Assume without much LOG that $\Omega = \Gamma^{\mathbb{Z}^d}$ for some Borel $\Gamma \subset \mathbb{R}$ and $\mathfrak{S}$ is the Borel σ-algebra. Environments are of the form $\omega = (\omega_y)_{y \in \mathbb{Z}^d}$, and $\{T_x : x \in \mathbb{Z}^d\}$ are translations defined by $(T_x\omega)_y = \omega_{x+y}$.

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Some extra assumptions (for later parts of the talk):

(Dir) *Directed nearest-neighbor walk:* $\mathcal{R} = \{e_1, \dots, e_d\}$, the standard basis for $\mathbb{R}^d$, with $d \geq 2$.

(Ind) *Independent environment:* The components of $\omega = (\omega_x)_{x\in\mathbb{Z}^d}$ are i.i.d. under $\mathbb{P}$.

(Loc) *Local potential:* There exists a $V_o : \Gamma \times \mathcal{R} \to \mathbb{R}$ such that $V(\omega, z) = V_o(\omega_0, z)$ for every $\omega \in \Omega$ and $z \in \mathcal{R}$.

These assumptions enable us to use martingale techniques.
If V_o does not depend on z, then RWRP is a directed polymer.
We keep the z dependence to cover RWRE as well as for certain large deviations applications.

Quenched free energy and large deviations

In a previous work [RasSepYil2013], we proved the $\mathbb{P}$-a.s. existence of the quenched free energy

$$\Lambda_q(V) := \lim_{n\to\infty} \frac{1}{n} \log \mathcal{Z}^{\omega}_{n,0}.$$

The statement of this result requires the following definition.

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Definition

A measurable function $F : \Omega \times \mathcal{R} \to \mathbb{R}$ is said to be a centered cocycle if it satisfies the following conditions.

(i) Centered: $\mathbb{E}[|F(\cdot, z)|] < \infty$ and $\mathbb{E}[F(\cdot, z)] = 0$ for every $z \in \mathcal{R}$.

(ii) Cocycle: $\sum_{i=0}^{n-1} F(T_{x_i}\omega, z_{i+1})$ depends only on $\omega \in \Omega$, x_0 and x_n.

The class of centered cocycles is denoted by $\mathcal{K}_0$.

Theorem (RasSepYil2013)

The limiting quenched free energy exists $\mathbb{P}$-a.s., is deterministic, and satisfies

$$\begin{aligned}\Lambda_q(V) = & \inf_{F \in \mathcal{K}_0} \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \left\{ \log \left(\sum_{z \in \mathcal{R}} \frac{1}{|\mathcal{R}|} e^{V(\omega,z)+F(\omega,z)} \right) \right\} \\ =: & \inf_{F \in \mathcal{K}_0} K(V,F) \in (-\infty, \infty].\end{aligned}$$

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$$=: \inf_{F \in \mathcal{K}_0} K(V,F) \in (-\infty, \infty].$$

- The result is for $V : \Omega \times \mathcal{R}^\ell \to \mathbb{R}$ with arbitrary $\ell \geq 1$.
- Gives two variational formulas for $\Lambda_q(V)$, but we omit the second one here.
- **Technical condition:** $V \in \mathcal{L}$. It holds if $\mathbb{P}$ is stationary & ergodic and V is bounded, or if the environment is i.i.d. and $V \in L^p(\mathbb{P})$ for some $p > d$. In general, tradeoff between the degree of mixing in $\mathbb{P}$ and the moment of V required.
- This assumption does not rule out $\Lambda_q(V) = \infty$.

- The proof is based on applications of ergodic and minimax theorems, developed in [KosRezVar2006, KosVar2008] for stochastic homogenization of viscous HJ equations.
- The existence of the a.s. limit of the quenched free energy can be shown more easily (without giving any formulas for $\Lambda_q(V)$) by subadditivity arguments and additional estimates, e.g., concentration inequalities or lattice animal bounds. Done in full generality in [RasSep2014].

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- Gives a quenched large deviation principle (LDP) under $Q^{\omega}_{n,x}$ for the empirical measure

$$R_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T_{X_i}\omega, Z_{i+1}}$$

  as well as for the average velocity X_n/n (via contraction). Rate functions have natural variational formulas.
- Covers/strengthens various previous results for RWRE/RWRP.

Directed i.i.d. case: disorder regimes

Assume (Dir), (Ind) and (Loc). Define the annealed free energy

$$\Lambda_a(V) := \log\left(\sum_{z\in\mathcal{R}} \frac{1}{|\mathcal{R}|}\mathbb{E}\left[e^{V(\cdot,z)}\right]\right) \in (-\infty,\infty].$$

Then,

$$W_n(\omega) := \frac{\mathcal{Z}^{\omega}_{n,0}}{\mathbb{E}[\mathcal{Z}^{\omega}_{n,0}]} = E_0\left[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1}) - n\Lambda_a(V)}\right]$$

is a nonnegative martingale w.r.t. $(\mathfrak{S}_n)_{n\geq 1}$, and

$$W_\infty := \lim_{n\to\infty} W_n \quad \mathbb{P}\text{-a.s.}$$

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is a nonnegative martingale w.r.t. $(\mathfrak{S}_n)_{n\geq 1}$, and

$$W_\infty := \lim_{n\to\infty} W_n \quad \mathbb{P}\text{-a.s.}$$

exists. Moreover, $\{W_\infty = 0\}$ is a tail event, and the Kolmogorov zero-one law implies the following dichotomy:

$$\begin{aligned} &\text{either} \quad \mathbb{P}(W_\infty = 0) = 0 \quad \text{(weak disorder);}\\ &\text{or} \quad \mathbb{P}(W_\infty = 0) = 1 \quad \text{(strong disorder).}\end{aligned}$$

The so-called annealing bound (Jensen) gives $\Lambda_q(V) \leq \Lambda_a(V)$.
In the case of weak disorder, we have

$$0 = \lim_{n\to\infty} \frac{1}{n} \log W_n(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \mathcal{Z}^{\omega}_{n,0} - \Lambda_a(V)$$
$$= \Lambda_q(V) - \Lambda_a(V)$$

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for $\mathbb{P}$-a.e. ω.
Therefore,

$$\Lambda_q(V) < \Lambda_a(V) \quad \text{(very strong disorder)}$$

is a sufficient condition for strong disorder.
However, it is not known whether it is necessary for strong disorder, more about this later.

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Theorem (ImbSpe1988, Bol1989, SonZho1996, ComYos2006, ComVar2006, Lac2010)

Assume (Dir), (Ind) and (Loc) are satisfied, and that V does not depend on z. (Not generalized to RWRP.)

(a) *There exist $0 \leq \beta_c = \beta_c(V, d) \leq \beta_c' = \beta_c'(V, d) \leq \infty$ such that the directed polymer with potential βV is in*
  - (i) *weak disorder if $\beta \in \{0\} \cup (0, \beta_c)$,*
  - (ii) *strong disorder if $\beta \in (\beta_c, \infty)$, and*
  - (iii) *very strong disorder if $\beta \in (\beta_c', \infty)$.*

(b) *$\beta_c = \beta_c(V, d)$ and $\beta_c' = \beta_c'(V, d)$ satisfy*
  - (i) *$\beta_c > 0$ if $d \geq 4$, and*
  - (ii) *$\beta_c' = 0$ if $d = 2, 3$.*

Results: Quenched free energy in the general case

Observe that

$$K(V,F) := \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \left\{ \log \left(\sum_{z \in \mathcal{R}} \frac{1}{|\mathcal{R}|} e^{V(\omega,z)+F(\omega,z)} \right) \right\}$$
$$= \mathbb{P}\text{-}\operatorname*{ess\,sup}_{\omega} \left\{ \log \left(\sum_{z \in \mathcal{R}} \frac{e^{V(\omega,z)}}{|\mathcal{R}|} \frac{g(T_z\omega)}{g(\omega)} \right) \right\}$$

when F is of the form

$$F(\omega,z) = (\nabla^* g)(\omega,z) := \log \left(\frac{g(T_z\omega)}{g(\omega)} \right)$$

for some $g \in L^+(\Omega, \mathfrak{S}, \mathbb{P})$.

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for some $g \in L^+(\Omega,\mathfrak{S},\mathbb{P})$. Here and throughout,

$$L^+(\Omega,\mathfrak{S}',\mathbb{P}) := \{g : \Omega \to \mathbb{R} : g \text{ is } \mathfrak{S}'\text{-meas. and } 0 < g(\omega) < \infty \text{ for } \mathbb{P}\text{-a.e. } \omega\} \quad \text{and}$$
$$L^{++}(\Omega,\mathfrak{S}',\mathbb{P}) := \{g : \Omega \to \mathbb{R} : g \text{ is } \mathfrak{S}'\text{-meas. and } \exists\, c > 0 \text{ s.t. } c < g(\omega) < \infty \text{ for } \mathbb{P}\text{-a.e. } \omega\}$$

for every σ-algebra $\mathfrak{S}' \subset \mathfrak{S}$ on Ω.

Theorem (RasSepYil2015)

Assume $V \in \mathcal{L}$. Then,

$$\Lambda_q(V) = \inf_{F \in \mathcal{K}_0} K(V, F) \tag{qVar0}$$

$$= \inf_{g \in L^+} K(V, \nabla^* g) \tag{qVar1}$$

$$= \inf_{g \in L^{++}} K(V, \nabla^* g). \tag{qVar2}$$

Here, L^+, L^{++} stand for (i) $L^+(\Omega, \mathfrak{S}, \mathbb{P})$, $L^{++}(\Omega, \mathfrak{S}, \mathbb{P})$ in general and (ii) $L^+(\Omega, \mathfrak{S}_0^\infty, \mathbb{P})$, $L^{++}(\Omega, \mathfrak{S}_0^\infty, \mathbb{P})$ in the directed case.

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Remark

$\mathcal{K}_0$ is the $L^1(\Omega, \mathfrak{S}, \mathbb{P})$-closure of

$$\{\nabla^* g : \ \exists\, C > 0 \text{ s.t. } C^{-1} < g(\omega) < C \text{ for } \mathbb{P}\text{-a.e. } \omega\}.$$

Unfortunately, our understanding of $\mathcal{K}_0$ does not go much beyond this characterization. Thus, for applications, (qVar1) and (qVar2) are perhaps more useful than (qVar0).

A few words about the proof of the theorem

Define

$$\bar{\Lambda}_q(V) := \limsup_{n\to\infty} \frac{1}{n} \log \mathcal{Z}^{\omega}_{n,0}.$$

Then,

$$\bar{\Lambda}_q(V) \leq \inf_{F\in\mathcal{K}_0} K(V,F) \leq \inf_{g\in L^+} K(V,\nabla^* g) \leq \inf_{g\in L^{++}} K(V,\nabla^* g) \leq \bar{\Lambda}_q(V).$$

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The first inequality hinges on a certain control on the minima of path integrals of centered cocycles on large sets which is implied by an ergodic theorem. The second and third inequalities are trivial. The last inequality follows from an elementary spectral argument. Finally, $\bar{\Lambda}_q(V) = \Lambda_q(V)$ by subadditivity [RasSep2014]. Hence, there is no need for minimax here, and this proof is independent of the previous one.

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(qVar0) always has a minimizer (nontrivial fact). What about (qVar1) and (qVar2)?

Results: Annealed free energy in the directed i.i.d. case

Recall

$$\Lambda_a(V) := \log\left(\sum_{z\in\mathcal{R}} \frac{1}{|\mathcal{R}|}\mathbb{E}\left[e^{V(\cdot,z)}\right]\right) \in (-\infty,\infty].$$

Theorem (RasSepYil2015)

Assume (Dir), (Ind), and (Loc). Then,

$$\Lambda_a(V) = \inf_{g\in L^+\cap L^1} K(V,\nabla^* g) \tag{aVar1}$$

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Here, L^+, L^{++} and L^1 stand for $L^+(\Omega,\mathfrak{S}_0^\infty,\mathbb{P})$, $L^{++}(\Omega,\mathfrak{S}_0^\infty,\mathbb{P})$ and $L^1(\Omega,\mathfrak{S}_0^\infty,\mathbb{P})$, respectively.

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Remark

We know that $\Lambda_q(V) < \Lambda_a(V)$ in the case of very strong disorder. This is particularly interesting as all of these sets are dense in $\mathcal{K}_0$.

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Do (aVar1) and (aVar2) have minimizers?

Theorem (RasSepYil2015)

Assume (Dir), (Ind), (Loc), and $\Lambda_a(V) < \infty$.

(a) *(aVar1) has a minimizer if and only if there is weak disorder. In this case, the minimizer is unique (up to a multiplicative constant), and equal to* W_∞.

(b) *(aVar2) has no minimizers unless* $\mathcal{Z}^{\omega}_{1,0}$ *is* $\mathbb{P}$*-essentially constant, i.e., the RWRP is nothing but an RWRE (with zero potential).*

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Other characterizations of weak disorder have been previously given: delocalization, $L^1(\mathbb{P})$-convergence or uniform integrability of the martingale $(W_n)_{n\geq 1}$, etc. As far as we know, ours is the first variational characterization of weak disorder for RWRP. Its proof builds on an earlier characterization [ComYos2006] for directed polymers.

Analysis of (qVar1) and (qVar2) in the directed i.i.d. case

Assume (Dir), (Ind) and (Loc). In the case of weak disorder, $\Lambda_q(V) = \Lambda_a(V) < \infty$. Therefore, the unique minimizer W_∞ of (aVar1) in $L^+ \cap L^1$ is also a minimizer of (qVar1) in the larger space L^+. However, it is not a-priori clear whether W_∞ is the unique minimizer of (qVar1).

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What about in strong disorder? Do (qVar1) and (qVar2) have minimizers then? This is more difficult.

Define

$$h_n^\lambda(\omega) := E_0\left[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1}) - n\lambda}\right]$$

for every $n \geq 1$, $\lambda \in \mathbb{R}$ and $\omega \in \Omega$, and consider the future measurable functions

$$\underline{h}_\infty^\lambda(\omega) := \liminf_{n\to\infty} h_n^\lambda(\omega) \qquad \text{and} \qquad \overline{h}_\infty^\lambda(\omega) := \limsup_{n\to\infty} h_n^\lambda(\omega).$$

With this notation, $W_n = h_n^\lambda$ and $W_\infty = \underline{h}_\infty^\lambda = \overline{h}_\infty^\lambda$ when $\lambda = \Lambda_a(V) < \infty$. For general $\lambda \in \mathbb{R}$, we know that

$$\lim_{n\to\infty} \frac{1}{n} \log h_n^\lambda(\omega) = \Lambda_q(V) - \lambda$$

holds for $\mathbb{P}$-a.e. ω. Therefore,

$$\mathbb{P}(\underline{h}_\infty^\lambda = \overline{h}_\infty^\lambda = 0) = 1 \text{ if } \lambda > \Lambda_q(V), \text{ and}$$
$$\mathbb{P}(\underline{h}_\infty^\lambda = \overline{h}_\infty^\lambda = \infty) = 1 \text{ if } \lambda < \Lambda_q(V).$$

Hence, the only nontrivial choice of parameter is $\lambda = \Lambda_q(V)$.

When $\lambda = \Lambda_q(V)$, drop λ from the notation. Each of the events

$$\{\underline{h}_\infty = 0\},\ \{0 < \underline{h}_\infty < \infty\},\ \{\underline{h}_\infty = \infty\},$$
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has $\mathbb{P}$-probability zero or one. The idea is to use these sets to characterize the disorder regimes.

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Regarding nonexistence of minimizers, we have the following result.

Theorem (RasSepYil2015)

Assume (Dir), (Ind), (Loc) and $V \in \mathcal{L}$. If there is strong disorder and $\mathbb{P}(\overline{h}_\infty = 0) = 0$, then (qVar1) and (qVar2) have no minimizers.

A sufficient condition for $\mathbb{P}(\bar{h}_\infty = \infty) = 1$

Let

$$H_n(\omega) := E_0\left[e^{\sum_{i=0}^{n-1} V(T_{X_i}\omega, Z_{i+1}) - n\Lambda_q(V)}\mathbb{1}_{\{X_n = (n/d,\ldots,n/d)\}}\right]$$

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Proposition

Assume (Dir), (Ind), (Loc) and $V \in \mathcal{L}$. If there exists an increasing sequence $(a(n))_{n \geq 1}$ such that

$$\lim_{n\to\infty} a(n) = \infty, \;\; \lim_{n\to\infty} \frac{a(n-1)}{a(n)} = 1, \;\; \textit{and} \;\; \limsup_{n\to\infty} \mathbb{P}\left(\log H_n \geq a(n)\right) > 0,$$

then

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{\log H_n}{a(n)} \geq 1 \right) = 1.$$

In particular, $\mathbb{P}(\bar{h}_\infty = \infty) = 1$.

[BorCorRem2013] shows that $n^{-1/3} \log H_n$ has an F_{GUE} distributional limit for the log-gamma polymer model on $\mathbb{Z}^2$ with parameter $\gamma \in (0, \gamma^*)$ for some $\gamma^* > 0$.
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Corollary

Assume (Dir), (Ind), (Loc) and $V \in \mathcal{L}$. Then, (qVar1) and (qVar2) do not always have any minimizers in the case of very strong disorder.

Remarks and open problems

- The critical inverse temperatures $\beta_c = \beta_c(V, d)$ and $\beta'_c = \beta'_c(V, d)$ satisfy $\beta_c = \beta'_c = 0$ for $d = 2, 3$, and it is natural to expect that $\beta_c = \beta'_c$ for every $d \geq 2$. However, this is an open problem.
- Furthermore, it is generally believed that there is strong disorder at β_c for $d \geq 4$. The latter claim is supported by the analogous result in the context of directed polymers on trees [KahPey1976].

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- Furthermore, it is generally believed that there is strong disorder at β_c for $d \geq 4$. The latter claim is supported by the analogous result in the context of directed polymers on trees [KahPey1976].
- With this background, here is our conjecture regarding the very strong disorder regime.

Conjecture

Assume (Dir), (Ind), (Loc) and $V \in \mathcal{L}$. Then,

$$\mathbb{P}(0 = \underline{h}_\infty < \overline{h}_\infty = \infty) = 1$$

whenever there is very strong disorder.

If this conjecture is indeed true, it would readily give the following quenched characterization of the disorder regimes:

(a) If there is weak disorder, then

$$\mathbb{P}(0 < \underline{h}_\infty = \overline{h}_\infty < \infty) = 1.$$

(b) If there is critically strong disorder, then

$$\mathbb{P}(\underline{h}_\infty = \overline{h}_\infty = 0) = 1.$$

(c) If there is very strong disorder, then

$$\mathbb{P}(0 = \underline{h}_\infty < \overline{h}_\infty = \infty) = 1.$$

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- Parts (a) and (b) are trivial since $\Lambda_q(V) = \Lambda_a(V)$ and $\underline{h}_\infty = \overline{h}_\infty = W_\infty$.
- Part (c) would imply that (qVar1) and (qVar2) never have any minimizers in the case of very strong disorder.

The result of Borodin et al. that we have used is a form of KPZ universality and is expected to hold for a large class of models. However, our proposition is much more modest since it does not require any sharp estimates such as the $n^{1/3}$ scaling in KPZ universality. Indeed, slowly growing sequences, e.g., $a(n) = \log \log \log n$, would suffice.

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Finally, observe that our theorem is not applicable in the (hypothetical) case of critically strong disorder since, then, $\mathbb{P}(\bar{h}_\infty = 0) = 1$. Therefore, we refrain from making any claims regarding the existence of any minimizers of (qVar1) and (qVar2) in that case.

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Thank you for your attention.