Arm Exponents for Critical Ising and FK-Ising Model IRS 2017 Random Geometry

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Outline

1. Percolation
   - What are the arm exponents ?
   - Why we are interested in the arm exponents ?
   - How to derive these exponents ?
2. SLE and Arm Exponents
3. Ising and FK-Ising
4. Proof
5. Further questions

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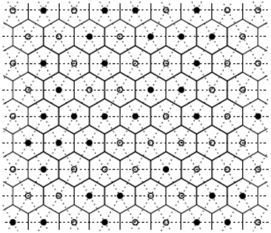
Percolation



Site percolation on triangular lattice : each site is chosen independently to be black or white with probability p or $1 - p$.

- When $p < 1/2$, white sites dominate.
- When $p > 1/2$, black sites dominate.
- When $p = 1/2$, critical, the system converges to something nontrivial.

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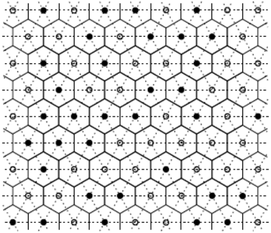

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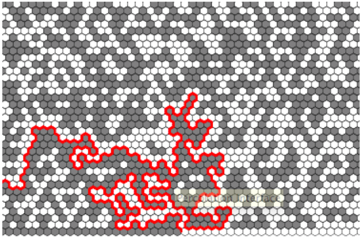
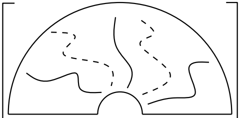


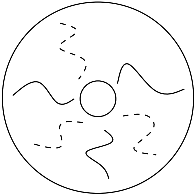
- Describe the critical percolation via interfaces between black and white.
- The interface converges to SLE(6) as mesh-size goes to zero. (Smirnov)

What are the arm exponents ?

Boundary arm exponents

$p_n^+(r,R) = P\Big[$  $\Big] \approx R^{-\alpha_n^+}, \quad R \to \infty$

Interior arm exponents

$p_n(r,R) = P\Big[$  $\Big] \approx R^{-\alpha_n}, \quad R \to \infty$

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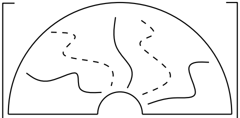

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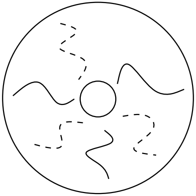

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Near critical percolation, Kesten

Correlation length : for $p > 1/2$, let $L(p)$ be the smallest n s.t.

$$\mathbb{P}_p[\text{crossing of } \Lambda_n] \geq 1 - \delta$$

- For n below $L(p)$, we have RSW and thus the situation is almost the same as the critical case. $L(p) \to \infty$ as $p \to 1/2$.

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- For n below $L(p)$, we have RSW and thus the situation is almost the same as the critical case. $L(p) \to \infty$ as $p \to 1/2$.
- By Russo's formula, we have

$$(p - 1/2)L(p)^2 p_4(L(p)) \asymp 1.$$

- Combining with 4-arm exponent $p_4(n) \approx n^{-5/4}$,
- we obtain

$$L(p) \approx (p - 1/2)^{-4/3}.$$

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Near critical percolation, Kesten

The density of the infinite cluster : for $p > 1/2$,

$$\theta(p) := \mathbb{P}_p[0 \leftrightarrow \infty], \quad \theta(p) \to 0 \text{ as } p \to 1/2.$$

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The density of the infinite cluster : for $p > 1/2$,

$$\theta(p) := \mathbb{P}_p[0 \leftrightarrow \infty], \quad \theta(p) \to 0 \text{ as } p \to 1/2.$$

- Once we arrive at $L(p)$, we are not far from ∞ :

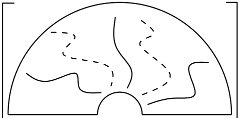
$$\theta(p) \asymp \mathbb{P}_p[0 \leftrightarrow L(p)] = p_1(L(p)).$$

- Combining with 1-arm exponent $p_1(n) \approx n^{-5/48}$,
- we obtain

$$\theta(p) \approx (p - 1/2)^{5/36}.$$

How to derive these exponents ?

Boundary arm exponents

$p_n^+(r,R) = P\Big[$  $\Big] \approx R^{-\alpha_n^+}, \quad R \to \infty$

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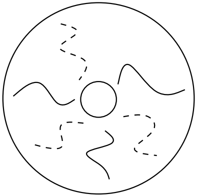
$p_n(r,R) = P\Big[$  $\Big] \approx R^{-\alpha_n}, \quad R \to \infty$

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Boundary arm exponents

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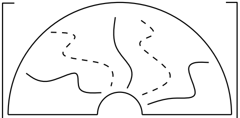

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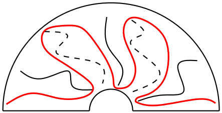

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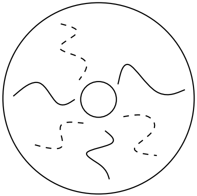

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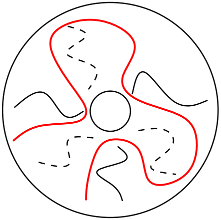

Question : $\alpha_n^+ =?$, $\alpha_n =?$

How to derive these exponents ?

Critical site percolation on triangular lattice. The interface converges to SLE(6). (Smirnov)

Arm exponents for SLE(6). (Lawler, Schramm, Werner)

Quasi-multiplicativity :

$$p_n(r, R') \asymp p_n(r, R) p_n(R, R')$$

Conclusion

Arm exponents for critical percolation :

$$\alpha_n^+ = n(n+1)/6, \quad \alpha_n = (n^2 - 1)/12.$$

Questions

Question 1

Understand the relation between other critical lattice models and SLE

Question 2

Calculate the arm exponents for SLE

Question 3

Derive the arm exponents for the critical lattice models.

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Question 2

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Today's topic

Question 3

Derive the arm exponents for the critical lattice models.

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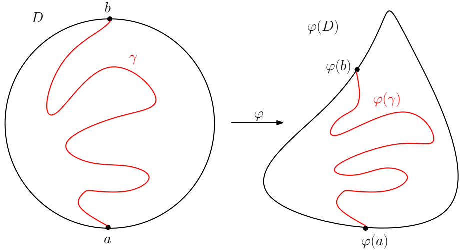
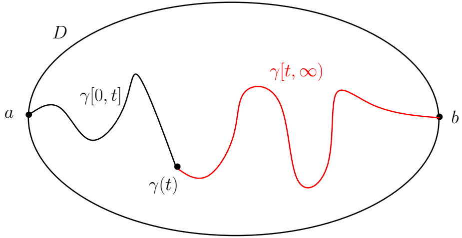
SLE (Schramm Loewner Evolution)

Random fractal curves in $D \subset \mathbb{C}$ from a to b. Candidates for the scaling limit of discrete Statistical Physics models.



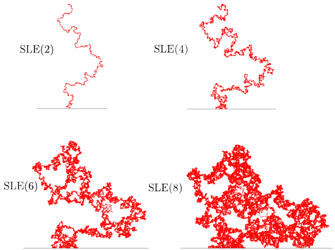
Conformal invariance :
If γ is in D from a to b,
and $\varphi : D \to \varphi(D)$ conformal map,
then $\varphi(\gamma) \overset{d}{\sim}$ the one in $\varphi(D)$ from $\varphi(a)$ to $\varphi(b)$.



Domain Markov property :
the conditional law of
$\gamma[t,\infty)$ given $\gamma[0,t] \overset{d}{\sim}$
the one in $D \setminus \gamma[0,t]$ from $\gamma(t)$ to b.

Examples of SLE

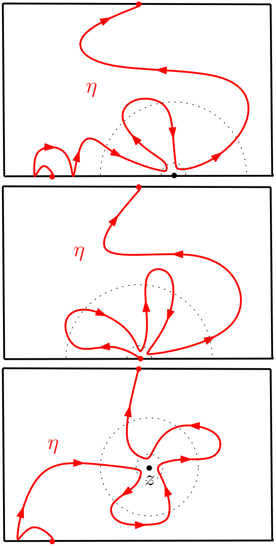
One parameter family of growing processes SLE_κ for $\kappa \geq 0$.
Simple, $\kappa \in [0,4]$; Self-touching, $\kappa \in (4,8)$; Space-filling, $\kappa \geq 8$.



Courtesy to Tom Kennedy.

- $\kappa = 2$: LERW
- $\kappa = 8$: UST
  (Lawler, Schramm, Werner)
- $\kappa = 3$: Critical Ising
- $\kappa = 16/3$: FK-Ising
  (Chelkak, Duminil-Copin, Hongler, Kempainen, Smirnov)
- $\kappa = 6$: Percolation
  (Camia, Newman, Smirnov)

Arm Exponents of SLE

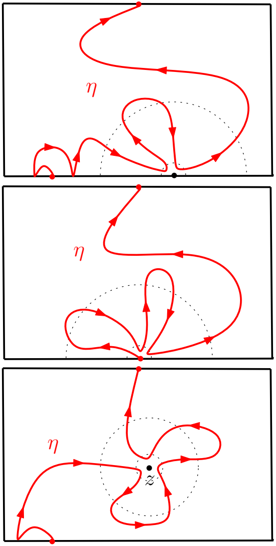


- For SLE_κ with $\kappa \in (0, 8)$,

$$\alpha^+_{2j-1} = j(4j + 4 - \kappa)/\kappa,$$
$$\alpha^+_{2j} = j(4j + 8 - \kappa)/\kappa,$$
$$\alpha_{2j} = \left(16j^2 - (4 - \kappa)^2\right)/(8\kappa).$$

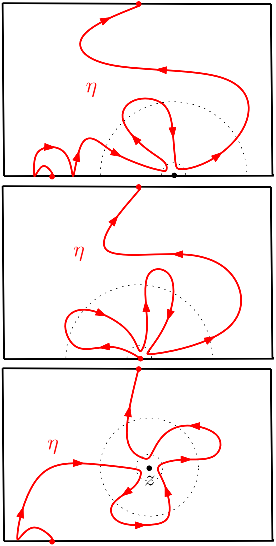
- SLE_κ with $\kappa \geq 8$.
- Some variants of SLE_κ with $\kappa \in (4, 8)$.

Relation to the fractal dimensions of SLE



- $1-\alpha_1^+$: dimension of the intersection with the boundary. (Alberts, Sheffield)

Relation to the fractal dimensions of SLE



- $1-\alpha_1^+$: dimension of the intersection with the boundary. (Alberts, Sheffield)
- $2-\alpha_2$: dimension of the trace. (Beffara)
- $2-\alpha_3$: dimension of the frontier. (Duality)

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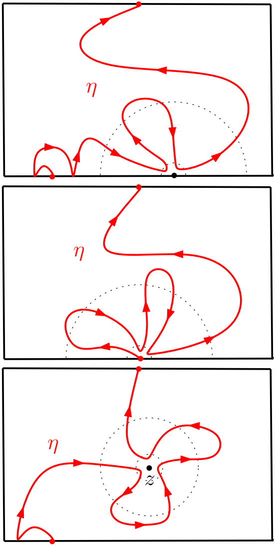

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- $2-\alpha_4$: dimension of the double point. (Miller, Wu)

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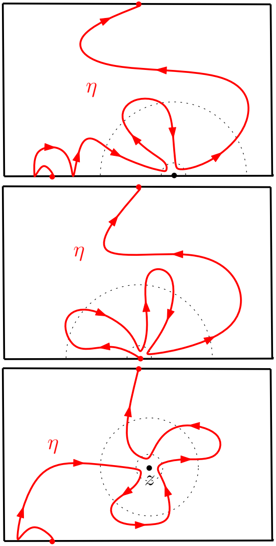

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- $2-\alpha_3$: dimension of the frontier. (Duality)
- $2-\alpha_4$: dimension of the double point. (Miller, Wu)
- $\alpha_6 > 2$ for $\kappa \in (4,8)$: no triple point.
- $\alpha_6 = 2$ for $\kappa \geq 8$: countably many triple points.

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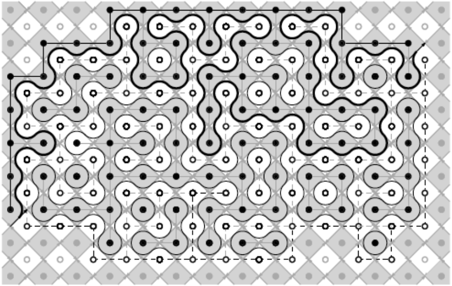
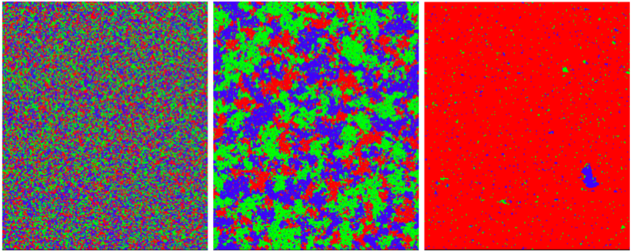
Random cluster model



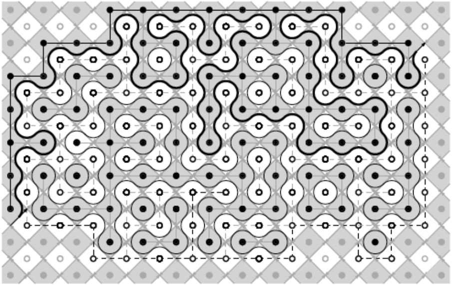
Random cluster on $\mathbb{Z}^2$ with edge-weight $p \in [0,1]$ and cluster-weight $q > 0$ is the probability measure given by

$$\phi_{p,q}(\omega) \propto p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega)}$$

At critical $p = p_c(q)$, the system converges to something nontrivial.



FK-Ising model, RCM with $q = 2$



Critical FK-Ising on $\mathbb{Z}^2$ with Dobrushin boundary condition. The interface converges to $\mathrm{SLE}_{16/3}$ (Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov).

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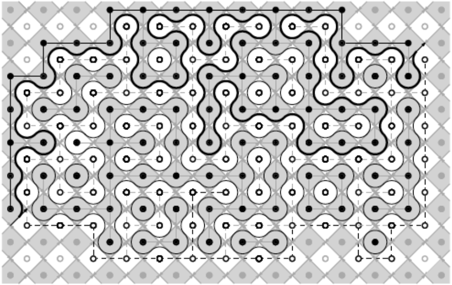

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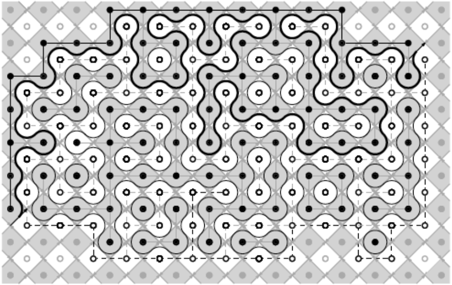

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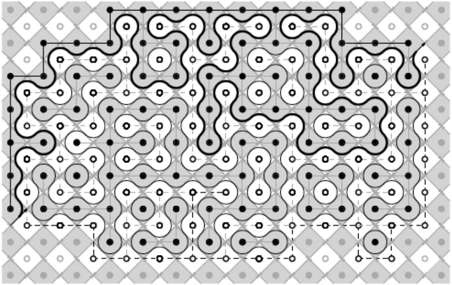

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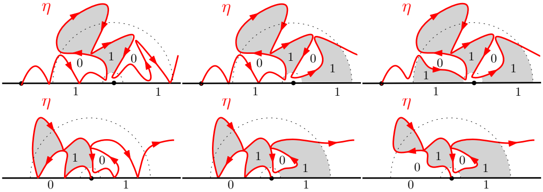
FK-Ising model

Boundary arm exponents : 6 patterns



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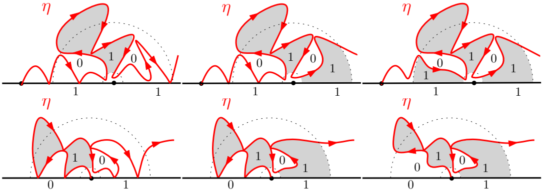

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Interior arm exponents : 3 patterns

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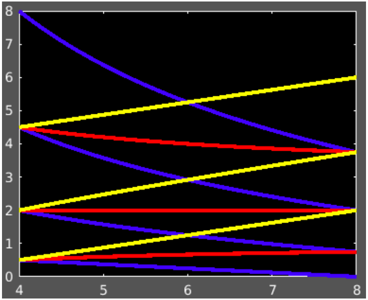
red : (101), (10101), (1010101)

yellow : (1100), (110100), (11010100)



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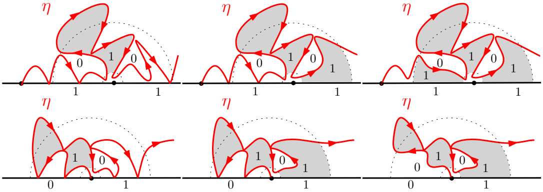

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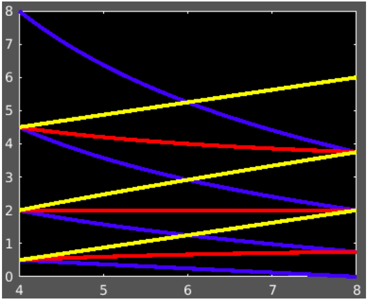

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Universal arm exponents for RCM

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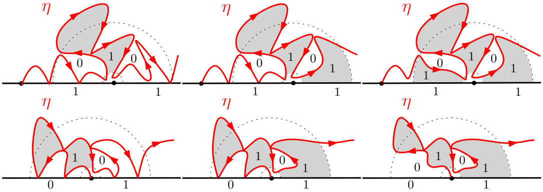

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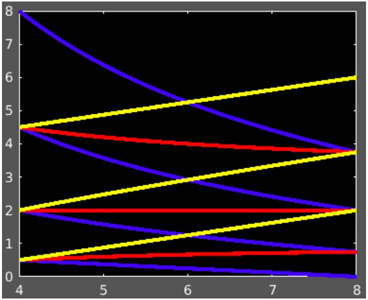

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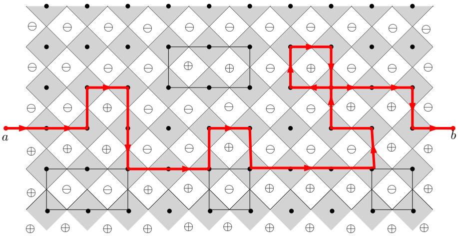
Question : Why they are monotone in κ ?

Ising model

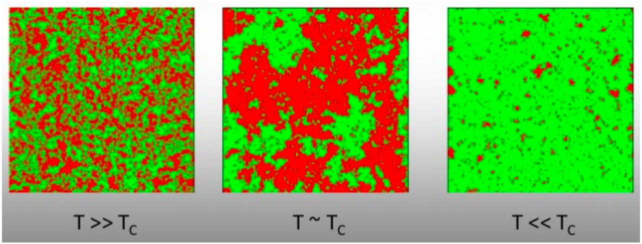


Spin Ising model on $\mathbb{Z}^2$: Each vertex x has a spin $\sigma_x \in \{-1, +1\}$, inverse temperature $\beta > 0$, the probability measure given by

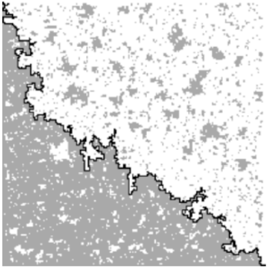
$$\mu_\beta(\sigma) \propto \exp(\beta \sum_{x \sim y} \sigma_x \sigma_y)$$

$$\propto \exp(-2\beta \# \text{disagree})$$

At critical $\beta = \beta_c$, the system converges to something nontrivial.



Critical Ising model, Dobrushin boundary condition



courtesy to Smirnov.

Critical Ising model on $\mathbb{Z}^2$ with Dobrushin boundary condition. The interface converges to SLE_3 (Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov).

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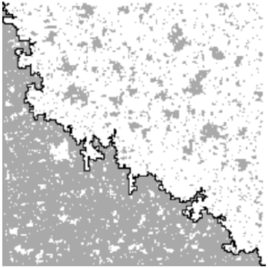

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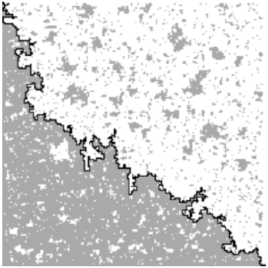

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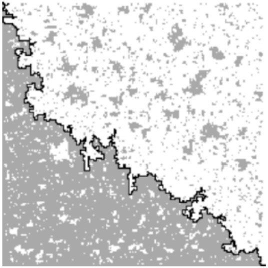

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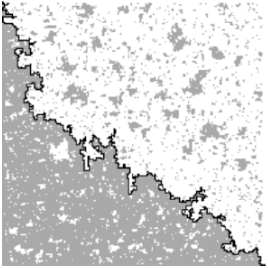
Conclusion

Arm exponents for critical Ising with Dobrushin boundary condition.

Critical Ising model, free boundary condition



courtesy to Smirnov.

Critical Ising model on $\mathbb{Z}^2$ with free boundary condition. The interface converges to $\mathrm{SLE}_3(-3/2; -3/2)$ (Hongler, Kytölä, Izyurov).

Arm exponents for $\mathrm{SLE}_3(-3/2; -3/2)$.

Quasi-multiplicativity (Chelkak, Duminil-Copin, Hongler).

Conclusion

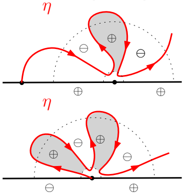
Arm exponents for critical Ising with free boundary condition

Critical Ising model, Arm exponents

Interior arm exponents : alternating $\alpha_{2j} = (16j^2 - 1)/24$.

Boundary arm exponents : 6 patterns

b.c. $(\ominus\oplus)$



$\alpha_j^+ \approx j^2/3$.

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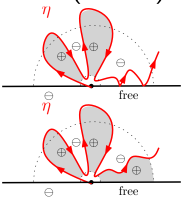
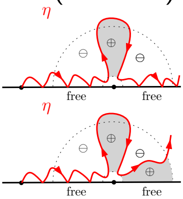


$\beta_j^+ = j(j+1)/3$.

b.c.(freefree)



$\gamma_j^+ = j(2j-1)/6$.

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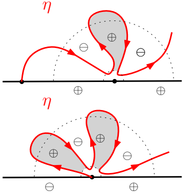

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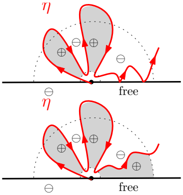

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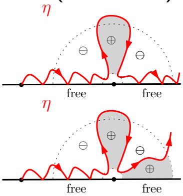

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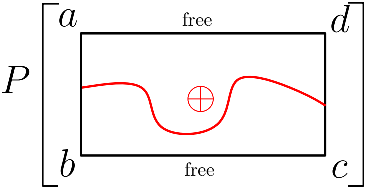
The asymptotic of the arm exponents is uniform over b.c. :

$$\alpha_j^+, \beta_j^+, \gamma_j^+ \approx j^2/\kappa, \quad \forall \kappa.$$

Critical Ising model, Cardy's formula



(Benoist, Duminil-Copin, Hongler)

- It converges to $f(\Omega, a, b, c, d)$.
- It is conformal invariant.
- Thus, it only depends on the extremal length L.
- But $f(L) = ?$

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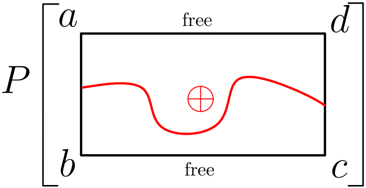

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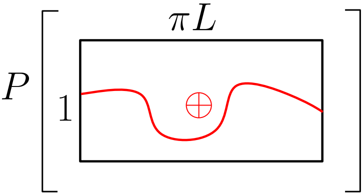

$$\approx \exp(-L/6)$$

Relation to KPZ formula

$\mathrm{SLE}_\kappa \leftrightarrow \gamma$-Liouville Quantum Gravity with $\kappa = \gamma^2$. KPZ formula

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta.$$

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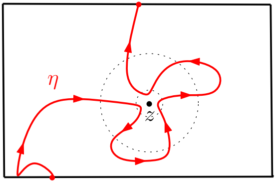
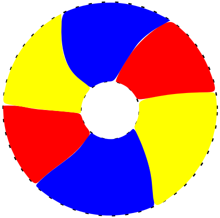


Euclidean Exponents :

$$x_{2j}^b = \frac{j(4j+4-\kappa)}{\kappa}, \quad x_{2j}^i = \frac{16j^2 - (\kappa-4)^2}{16\kappa}.$$



Quantum Exponents :

$$\Delta_{2j}^b = \frac{4j}{\kappa}, \qquad \Delta_{2j}^i = \frac{1}{2}\left(\Delta_{2j}^b + \frac{\kappa-4}{\kappa}\right).$$

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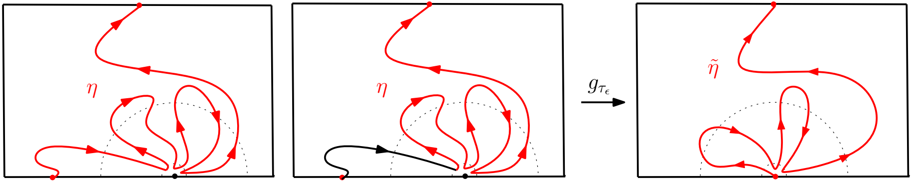
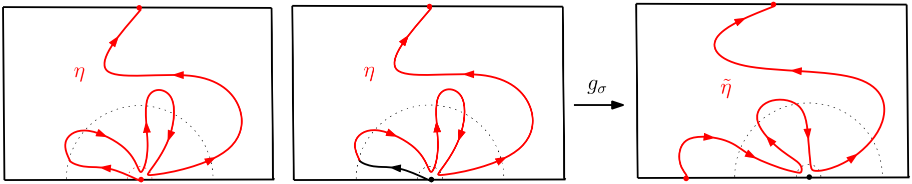
Reduce from $2n+1$ to $2n$



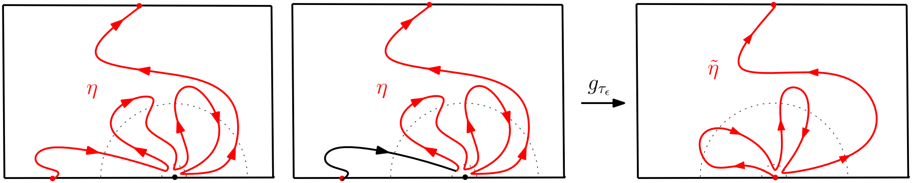
$$\mathbb{P}[(2n+1)\text{ arms}] \approx \mathbb{E}[(g'_{\tau_\epsilon}(1)\epsilon)^{\alpha^+_{2n}}], \quad \alpha^+_{2n+1} = u_1(\alpha^+_{2n}) + \alpha^+_{2n}.$$

Reduce from $2n$ to $2n-1$



$$\mathbb{P}[2n\text{ arms}] \approx \mathbb{E}[(g'_{\sigma}(\epsilon)\epsilon)^{\alpha^+_{2n-1}}], \quad \alpha^+_{2n} = u_2(\alpha^+_{2n-1}) + \alpha^+_{2n-1}.$$

Reduce from $2n+1$ to $2n$



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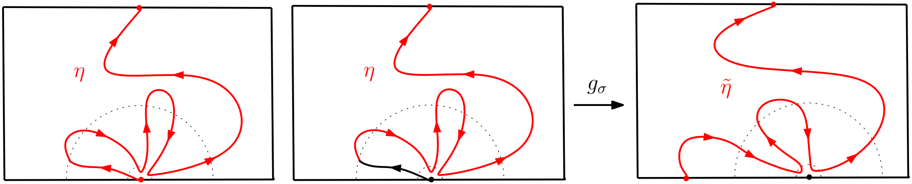

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Difficulty 1 : Only for well-oriented crossings. Solved by RSW.
Difficulty 2 : Need a strong one-point estimate.

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Further questions—Monochromatic ?

Question

The arm exponents we obtained are “alternating" arm exponents.
What are the other patterns of arm exponents ?

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One arm exponent

For general $\kappa \in (4,8)$, we know that

$$\tilde{\alpha}_1 = (8-\kappa)(3\kappa-8)/(32\kappa).$$

Percolation : $\kappa = 6$, $\tilde{\alpha}_1 = 5/48$.
FK-Ising : $\kappa = 16/3$, $\tilde{\alpha}_1 = 1/8$.

Thanks !

References

Critical percolation

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- *Critical exponents for 2D percolation,* Smirnov, Werner
- *One-arm exponent for critical 2D percolation,* Lawler, Schramm, Werner

Ising and FK-Ising

- *Convergence of Ising interfaces to SLE,* Chelkak, Duminil-Copin, Hongler, Kemppainen, Smirnov
- *Ising interfaces and free boundary conditions,* Hongler, Kytola
- *Crossing probabilities in topological rectangles for the critical planar FK-Ising model,* Chelkak, Duminil-Copin, Hongler

Arm exponents

- *Polychromatic arm exponents for the critical planar FK-Ising model*, Wu
- *Alternating arm exponents for the critical planar Ising model*, Wu
- *Boundary arm exponents for SLE*, Wu, Zhan