Extrema of log-correlated fields: duality and freezing, value and position of the maximum, and applications

P. Le Doussal *LPTENS, Paris*

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- log correlated Gaussian fields (LCF), freezing duality conjecture

  PDF of the value of the maximum

- moments of the position of the maximum of LCF on an interval

  Moments of Jacobi ensemble GUE-CP, fBm0

  Y. V. Fyodorov, PLD, arXiv 1511.04258, J. Stat. Phys. (2016)

- joint PDF of value max and min on the circle

  Edwards-Anderson order parameter

  X. Cao, PLD, arXiv 1604.02282, EPL, 114 (2016) 40003

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  X. Cao, PLD, arXiv 1604.02282, EPL, 114 (2016) 40003

- PDF value second minimum

  Also: 1stepRSB <-> Freezing, k-th order statistics

  X. Cao, Y. Fyodorov, PLD, arXiv 1610.02226, SciPost Phys. 1, 011 (2016)

- 2D ? PDF of position of max and Liouville field theory

  X. Cao, A. Rosso, R. Santachiara, PLD, arXiv 1611.02193

David Carpentier
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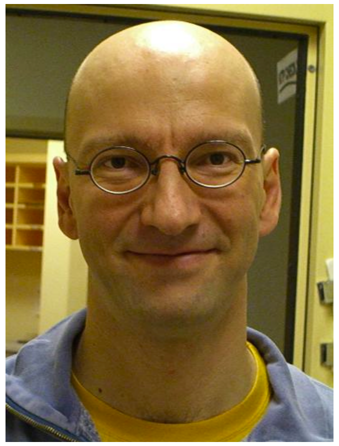

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Derrida, Spohn (1988) — prehistory: DPCT hierarchical LCF

D. Carpentier, PLD, PRE (2001) — prehistory: trans. inv. log-correlated gaussian field
Coulomb gas RG predicts freezing of generating function
2D freezing in Liouville and Sinh-Gordon

Fyodorov, Bouchaud J Phys A (2008) — freezing, PDF of minimum on the circle

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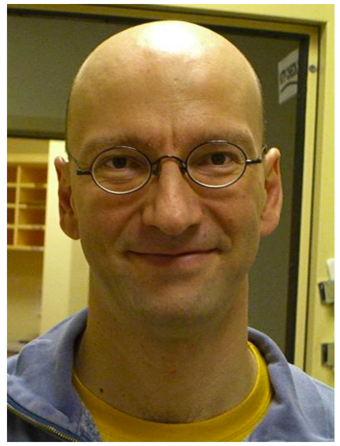

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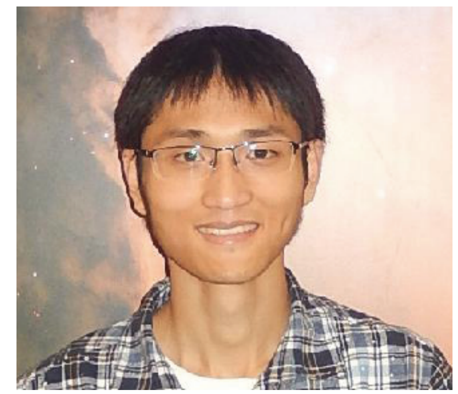
PhD thesis
March 30 !!



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Stat. Mech. approach to extreme value stat

$$Z = \sum_r e^{-V(r)/T} \qquad \beta = 1/T$$

particle in a random potential

$$F = -T \ln Z \overset{T=0}{\rightarrow} V_{min} = min_r V(r)$$

freezing transition $T < T_c$ Z dominated by one (or a few) minima

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$$\overline{(\dots)} \equiv \mathbb{E}\{(\dots)\}$$

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Log-correlated Gaussian field

UV info IR info

$$a \ll |r - r'| \ll L$$

$$r \in \mathbb{R}^d$$

$$\overline{(V(r) - V(r'))^2} \simeq 4 \ln \frac{|r - r'|}{a}$$

$d = 2$ GFF $P[V] \sim e^{\frac{1}{8\pi} \int d^2 r (\nabla V)^2}$

$d = 1$ along a curve in 2d line r=(x,0)

$d = +\infty$ solvable RSB circle, interval

Coulomb gas RG D. Carpentier, PLD, PRE (2001)

integrate small scale fluctuations

$$V(\mathbf{r}) = V^{>}(\mathbf{r}) + v(\mathbf{r}) \qquad z(\mathbf{r}) = e^{-\beta v(\mathbf{r})}$$

Gaussian
Log-correlated

non-Gaussian
short-range correlated

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introduce generating function

$$G_{\ell;\beta}(y) = \langle e^{-e^{\beta(y-v)}} \rangle_{P_\ell(v)}$$

$$\frac{1}{d}\partial_l G(x) = \frac{\sigma}{d}\partial_x^2 G - G(1-G)$$

KPP equation
use results

Bramson

Derrida-Spohn

traveling wave $G_\ell(x) \to g_c(x + m(\ell))$ $\partial_\ell m(\ell) = c(T) = \partial_\ell \overline{F}(\ell)$

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DPCT

traveling wave $G_\ell(x) \to g_c(x + m(\ell))$ $\quad \partial_\ell m(\ell) = c(T) = \partial_\ell \overline{F}(\ell)$

velocity c freezes at Tc $\quad F \simeq f \ln M \quad M = (L/a)^d$

$$f = -\left(\frac{T}{T_c} + \frac{T_c}{T}\right)T_c \qquad T > T_c$$

$$f = -2T_c \qquad T < T_c$$

freezing transition

$$T_c = \frac{1}{\sqrt{d}}$$

similarity to REM

Universal predictions
for log-REM's

$$G_{\beta=+\infty}(y) = Prob(y < v)$$

O(1) random variable

$$V_{min} \simeq T_c\left(-2\ln M + \frac{3}{2}\ln\ln M + v\right)$$

universal tail $\quad P(v) \simeq_{v \to -\infty} -v e^{v}$

exact form of the CDF=front g(x) ?

$$V_{min} \simeq T_c(-2\ln M + \frac{3}{2}\ln\ln M + v)$$

Conjectures for P(v)

from "continuation" from high temperature

Circle: Fyodorov, Bouchaud J Phys A 41 372001 (2008)

Interval: Fyodorov, PLD, Rosso J. Stat. Mech.P10005 (2009)

Exact solutions from high-temperature

discrete model $Z_M = \sum_{j=1}^{M} e^{-\beta V_i}$

$\beta = 1/T$

$C_{jk} = \overline{V_j V_k}$

$C_{jj} = 2\ln M + W$ $\quad W \geq 0$

- CIRCLE (circular log-REM) $C_{j\neq k} = -2\ln|e^{i\theta_j} - e^{i\theta_k}|$ $\quad \theta_j = \frac{2\pi j}{M}$

$M \to \infty$ periodic 1/f noise in $[0, 2\pi[$

- INTERVAL $C_{j\neq k} = -2\ln|x_j - x_k|$ $\quad x_j = \frac{j}{M} \in [0,1]$

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$$\overline{Z_M^n} \simeq M^{n(1+\beta^2)} I_n(\beta) \qquad n\beta^2 < 1$$

$$I_n(\beta) = \prod_{i=1}^{n} \int_0^{2\pi} \frac{d\theta_i}{2\pi} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^{-2\beta^2} = \frac{\Gamma(1 - n\beta^2)}{\Gamma(1 - \beta^2)^n}$$ Dyson

$$I_n(\beta) = \prod_{i=1}^{n} \int_0^{1} dx_i \prod_{j<k} \frac{1}{|x_j - x_k|^{2\beta^2}} = \prod_{j=1}^{n} \frac{\Gamma(1 - (j-1)\beta^2)^2 \Gamma(1 - j\beta^2)}{\Gamma(2 - (n+j-2)\beta^2)\Gamma(1 - \beta^2)}$$ Selberg

convergent for $n\beta^2 < 1$

- circle: exact solution in high temperature phase

positive integer moments

$$z = \Gamma(1-\beta^2) Z = e^{-\beta f} \qquad \overline{z^n} = \overline{e^{-\beta n f}} = \Gamma(1 - n\beta^2)$$

$$g_\beta(y) := \overline{e^{-z e^{\beta y}}} = \overline{e^{-e^{\beta(y-f)}}} \qquad \beta < \beta_c \qquad g_\beta(y) = \int_0^\infty dt \, \exp\{-t - e^{\beta y} \, t^{-\beta^2}\}$$

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conjecture: the whole function g(y) freezes at $\beta_c = 1$

$$g_{\beta \geq \beta_c}(y) = g_{\beta_c}(y) \longrightarrow$$

predicts the PDF of the minimum

$$V_{min} \simeq -2 \ln M + \frac{3}{2} \ln \ln M + v$$

Fyodorov, Bouchaud

$$\mathrm{Prob}(v > y) = g_{\beta=+\infty}(y) = g_{\beta=1^-}(y) = 2 e^{y/2} K_1(e^{y/2})$$

Fyodorov, PLD, Rosso J. Stat. Mech.(2009) exact solution for interval

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- duality $\beta < \beta_c$ $$g_\beta(y) = g_{1/\beta}(y)$$

continued

$$\sum_{n=1}^\infty \frac{s^n}{n!} \overline{y^n}^c = \ln \Gamma(1 + s\beta) + \ln \Gamma(1 + \frac{s}{\beta})$$

inside high-T phase only!

found to be exact for interval and circle

=> $$\partial_\beta g_\beta(y)|_{\beta=\beta_c^-} = 0 \quad , \quad \text{for all } \; y$$

- numerical tests

Freezing-duality conjecture

Thermodynamic quantities which for $\beta < 1$ are **duality-invariant** *functions of the inverse temperature β, that is remain invariant under the transformation $\beta \to \beta^{-1}$,* "**freeze**" *in the low temperature phase, that is retain for all $\beta > 1$ the value they acquired at the point of self-duality $\beta = 1$.*

not everything freezes..

e.g. PDF of free energy does not freezes

Freezing-duality conjecture

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PDF of value of maximum of LCF on interval

$$Q(V_m) = LT^{-1}_{n \to V_m} \Gamma(1-n) S(n) \qquad \overline{e^{-nV_m}} = \Gamma(1-n) S(n)$$

$$S(n) = \frac{G(1)G(2+a)G(2+b)G(4+a+b-2n)}{G(1-n)G(2+a-n)G(2+b-n)G(4+a+b-n)}$$

G(z) is Barnes function

Fyodorov, PLD, Rosso J. Stat. Mech.(2009)

summary in Appendix G of
Y. V. Fyodorov, PLD, arXiv 1511.04258

Next question: PDF of position of maximum on interval ?

Non exhaustive!

Freezing conjecture of g(y)

T. Madaule, R. Rhodes, V. Vargas. Glassy phase and freezing of log-correlated Gaussian potentials. *Ann. Appl. Probab.* **26** Number 2, 643-690 (2016)

E. Subag and O. Zeitouni. Freezing and decorated Poisson point processes. *Commun. Math. Phys.* **337**, Issue 1, pp 55-92 (2015)

Conjecture Vm = -2 log N + 3/2 log log N + v/universal tail of v (bound?)

M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field *Comm. Pure Appl. Math.* **65** 1-20 (2012)

J. Ding, R. Roy, O. Zeitouni. Convergence of the centered maximum of log-correlated Gaussian fields. e-preprint arXiv:1503.04588 (2015)

GUE-CP

Y. V. Fyodorov and N. J. Simm. On the distribution of maximum value of the characteristic polynomial of GUE random matrices. e-preprint arXiv:1503.07110 (2015)

Selberg moment problem-Barnes distributions

D. Ostrovsky. Mellin Transform of the Limit Lognormal Distribution, *Comm. Math. Phys.* **288**, 287-310 (2009).

D. Ostrovsky. Selberg Integral as a Meromorphic Function. *Int. Math. Res. Not.* **2012** 41 pp (2012).

D. Ostrovsky. Theory of Barnes Beta Distributions. *Electron. Commun. Prob.* **18**, no. 59, 116, (2012).

D. Ostrovsky. On Barnes Beta Distributions, Selberg Integral and Riemann Xi. *Forum Mathematicum.* DOI: 10.1515/forum-2013-0149, September 2014.

Position of the maximum

on an interval

Three examples of log-correlated fields

value of the minimum (max) — moments, PDF

position of the minimum (max) — moments

- log-correlated Gaussian random potential with background potential (LCGP)

- maxima of GUE characteristic polynomial (GUE-CP)

- Fractional Brownian motion with Hurst index H=0 (fBm0)

on an interval ⟷ related to RMT Jacobi ensemble

$x \in D$ — $y \in [0, 1]$

Log-correlated Gaussian random potential with a background potential (edge charges) (LCGP)

- random part $\mathbb{E}\{V(x)V(x')\} = C_\epsilon(x-x')$

$|x|>0 \qquad \lim_{\epsilon\to 0} C_\epsilon(x) = -2\ln|x|$

$\overline{(\ldots)} \equiv \mathbb{E}\{(\ldots)\}$

$C_\epsilon(0) = 2\ln(1/\epsilon)$

- deterministic part $V_0(x) = -\bar{a}\ln x - \bar{b}\ln(1-x)$

$D = [0,1]$

$V_m = \min_{x\in D}(V(x)+V_0(x)) \longrightarrow$

Fyodorov, PLD, Rosso J. Stat. Mech.(2009)

summary in Appendix G of
Y. V. Fyodorov, PLD, arXiv 1511.04258

$x_m = \mathrm{Argmin}_{x\in D}\big(V(x)+V_0(x)\big)$

repelling charges $\bar{a},\bar{b}>0$

in absence of random potential $x_m^0 = \frac{\bar{a}}{\bar{a}+\bar{b}}$

GUE characteristic polynomial (GUE-CP)

$$p_N(x) = \det(xI - H)$$

with H a GUE random matrix H

$$P(H) \propto \exp(-2N\mathrm{Tr}(H^2))$$

consider for large N

$$\phi_N(x) = 2\log|p_N(x)| - 2\mathbb{E}(\log|p_N(x)|)$$

value of maximum ?

position of maximum ?

it is a Gaussian log-correlated field !

Forrester Frankel (2004) Garoni (2005) Krasovsky (2007)

$$D = [-1, 1]$$

$$\overline{e^{\beta \sum_{a=1}^{n} \phi_N(x_a)}} \simeq A_n \prod_{a=1}^{n} (1 - x_a^2)^{\beta^2/2} \prod_{a<b} |x_a - x_b|^{-2\beta^2}$$

$$A_n = [C(\beta)(N/2)^{\beta^2} e^{-\beta C'(0)} 2^{-\beta^2(n-1)}]^n$$

$$V(x) = -\phi_N(x)$$

study minimum of V(x)

Fractional Brownian motion with Hurst index H=0 (fBm0)

$$\mathbb{E}\left\{B_H^{(\eta)}(x_1)B_H^{(\eta)}(x_2)\right\} = \phi_H^{(\eta)}(x_1) + \phi_H^{(\eta)}(x_2) - \phi_H^{(\eta)}(x_1 - x_2)$$

- Gaussian, self-similar index H

$$\phi_H^{(\eta)}(x) = \frac{1}{2}\int_0^\infty \frac{e^{-2\eta s}}{s^{1+2H}}\left(1 - \cos(xs)\right) ds$$

- stationary increments

$$\lim_{H\to 0}\phi_H^{(\eta)}(x) = \frac{1}{4}\log\frac{x^2+4\eta^2}{4\eta^2}$$

Question: min and argmin of fBm0 in interval D=[0,1]

$$V(x) = 2\, B_0^{(\eta)}(x)$$

note: $V(0) = 0$ at $x = 0$

$$\overline{e^{-2\beta\ \sum_{i=1}^n B_0^{(\eta)}(x_i)}} \approx (2\eta)^{n(\gamma(n-1)-a)}\prod_{i=1}^n x_i^a \prod_{i<j}^n |x_i - x_j|^{-2\beta^2}$$

$$a = 2n\beta^2$$

Statistical mechanics approach:

1) introduce partition sum

$$Z_\beta = \int_D e^{-\beta V(x)} \mu(x)\, dx \qquad \beta = 1/T$$

- GUE-CP $\quad \mu(x) = \rho(x)^q \qquad \rho(x) = \frac{2}{\pi}\sqrt{1-x^2}$

- LCGP $\quad \mu(x) = 1 \qquad V(x) \to V(x) + V_0(x)$

- fBm0 $\quad \mu(x) = 1 \qquad V(x) = 2B_0(x)$

Statistical mechanics approach:

1) introduce partition sum

$$Z_\beta = \int_D e^{-\beta V(x)} \mu(x)\, dx$$

Gibbs measure

$$p_\beta(x) = \frac{1}{Z_\beta} \mu(x) e^{-\beta V(x)}$$

PDF of position of minimum $\mathcal{P}(x) = \overline{\delta(x - x_m)}$ $\overline{(\ldots)} \equiv \mathbb{E}\{(\ldots)\}$

$$\mathcal{P}(x) = \lim_{\beta \to \infty} \overline{p_\beta(x)}$$

study average Gibbs measure (its moments) as a function of $\beta = 1/T$

Statistical mechanics approach: replica

1) **introduce partition sum**

$$Z_\beta = \int_D e^{-\beta V(x)} \mu(x)\, dx$$

Gibbs measure

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$$\mathcal{P}(x) = \lim_{\beta\to\infty} \overline{p_\beta(x)}$$

study average Gibbs measure (its moments) as a function of $\beta = 1/T$

2) **introduce replica**

define

$$p_{\beta,n}(x) = \mu(x)\overline{e^{-\beta V(x)} Z_\beta^{n-1}}$$

$$= \int_{x_1\in D} \dots \int_{x_n \in D} \overline{e^{-\beta \sum_{=1}^n V(x_i)}}\, \delta(x - x_1) \prod_{i=1}^n \mu(x_i)\, dx_i$$

so that

$$\overline{p_\beta(x)} = \lim_{n\to 0} p_{\beta,n}(x)$$

Statistical mechanics approach: Jacobi ensemble

$$p_{\beta,a,b,n}(y) = \int_0^1 .. \int_0^1 \prod_{i=1}^{n} dy_i y_i^a (1-y_i)^b \prod_{1\leq i<j\leq j\leq n} \frac{1}{|y_i - y_j|^{2\beta^2}} \delta(y - y_1)$$

$$< y^k >_{\beta,a,b,n} := \frac{1}{\mathcal{Z}_n} \int_0^1 dy y^k p_{\beta,a,b,n}(y)$$

moments in a given sample

$$< y^k >_{\beta,a,b} = \frac{\int_0^1 dy y^k y^a (1-y)^b e^{-\beta V(y)}}{\int_0^1 dy y^a (1-y)^b e^{-\beta V(y)}}$$

disorder averaged moments

$$\overline{< y^k >_{\beta,a,b}} = \lim_{n\to 0} < y^k >_{\beta,a,b,n}$$

- GUE-CP

$$x = 1 - 2y \qquad a = b = \frac{q+\beta^2}{2} \qquad \mu(x) = \rho(x)^q$$

$$x \in [-1,1] \qquad y \in [0,1] \qquad \rho(x) = \tfrac{2}{\pi}\sqrt{1-x^2}$$

- LCGP

$$x = y \in [0,1] \qquad a = \beta\bar{a} \qquad b = \beta\bar{b}$$

- fBm0

$$a = 2n\beta^2 \qquad b = 0$$

Jacobi ensemble

JPDF

$$\mathcal{P}_J(\mathbf{y})d\mathbf{y} = \frac{1}{\mathcal{Z}_n}\prod_{i=1}^{n} dy_i y_i^a (1-y_i)^b |\Delta(\mathbf{y})|^{2\kappa} \qquad \Delta(\mathbf{y}) = \prod_{1\le i<j\le n}(y_i - y_j)$$

$$\mathcal{Z}_n = Sl_n(\kappa, a, b) := \int_{[0,1]^n} |\Delta(\mathbf{y})|^{2\kappa} \prod_{i=1}^{n} y_i^a (1-y_i)^b dy_i$$

normalization is

Selberg integral

$$= \prod_{j=0}^{n-1} \frac{\Gamma\left(a+1+\kappa j\right)\Gamma\left(b+1+\kappa j\right)\Gamma\left(1+\kappa(j+1)\right)}{\Gamma\left(a+b+2+\kappa(n+j-1)\right)\Gamma\left(1+\kappa\right)}$$

$$< f(\mathbf{y}) >_J := [Sl_n(\kappa, a, b)]^{-1} \int_{[0,1]^n} f(\mathbf{y})\, |\Delta(\mathbf{y})|^{2\kappa} \prod_{i=1}^{n} y_i^a (1-y_i)^b dy_i$$

$$< y^k >_{\beta,a,b,n} := \left\langle \frac{1}{n}\sum_{r=1}^{n} y_r^k \right\rangle_J \Bigg|_{\kappa=-\beta^2}$$

disordered model needs
analytic continuations !

moments of Jacobi ensemble

recursions   Mehta's book

Savin, Sommers (2006) chaotic transport cavities

contour integral method (Borodin and Gorin)

Francesco Mezzadri, Alexi Reynolds, arXiv 1510.02390

Forrester's book   Y. V. Fyodorov, PLD, arXiv 1511.04258, J. Stat. Phys. (2016)

freezing

we will find that all moments are duality-invariant

conjecture: they freeze at $\beta = 1$

i.e. the whole disorder averaged Gibbs measure freezes

$$\mathcal{P}(x) = \lim_{\beta \to 1} \overline{p_\beta(x)}$$

$$\overline{(\dots)} \equiv \mathbb{E}\left\{(\dots)\right\}$$

The Kadell integral

K.W.J. Kadell Adv. Math. 130 33 (1997)

$$\int_{[0,1]^n} P_\lambda^{(1/\kappa)}(\mathbf{y})|\Delta(\mathbf{y})|^{2\kappa}\prod_{i=1}^{n} y_i^a(1-y_i)^b dy_i \;= n! \qquad\qquad \alpha = 1/\kappa$$

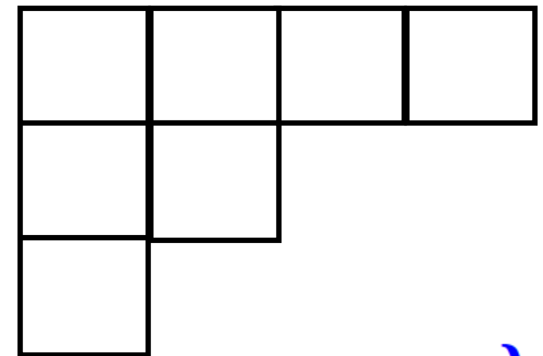
$$\prod_{i=1}^{n}\frac{\Gamma\left(\lambda_i+a+1+\kappa(n-i)\right)\Gamma\left(b+1+\kappa(n-i)\right)}{\Gamma\left(\lambda_i+a+b+2+\kappa(2n-i-1)\right)}\prod_{1\le i<j\le n}\frac{\Gamma\left(\lambda_i-\lambda_j+\kappa(j-i+1)\right)}{\Gamma\left(\lambda_i-\lambda_j+\kappa(j-i)\right)}$$

averages of Jack polynomials
symmetric, homogeneous, indexed by partitions

for empty partition gives Selberg integral

Partitions

$$\lambda = (4,2,1) \qquad |\lambda| = 7$$

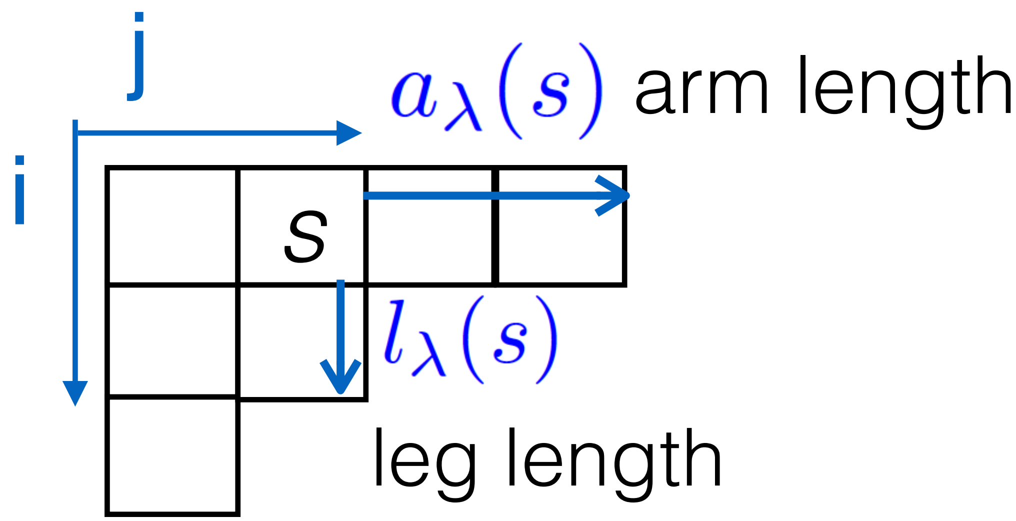


$$\ell(\lambda) = 3$$

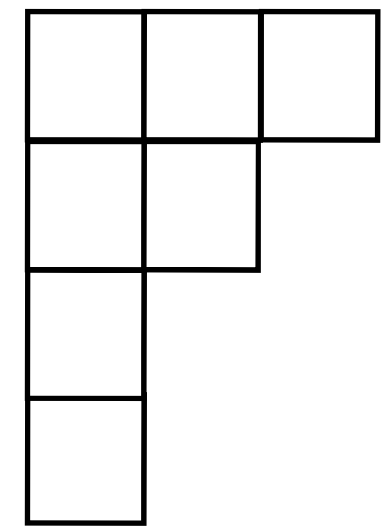
$$\lambda \equiv \lambda_1 \geq \lambda_2 .. \geq \lambda_{\ell(\lambda)} \geq 1$$

$$\lambda = \{(i,j) \in \mathbb{Z}^2; 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$$

$$s = (i,j)$$



λ^*



$$c(\lambda,\alpha,t) := \prod_{s\in\lambda} (\alpha a_\lambda(s) + l_\lambda(s) + t)$$

$a'_\lambda(s) = j - 1 = x$ co-arm

$l'_\lambda(s) = i - 1 = y$ co-leg

power sums $p_\lambda(\mathbf{y}) = \prod_{i=1}^{\ell(\lambda)} \sum_{r=1}^{n} y_r^{\lambda_i}$ $\alpha = 1/\kappa$

define scalar product as $< p_\lambda, p_\mu >= \delta_{\lambda\mu} z_\lambda \alpha^{\ell(\lambda)}$

$$z_\lambda = 1^{q_1} 2^{q_2} .. q_1! q_2! ..$$

q_p = number of rows of length p

Jack functions satisfy

$$< J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} >= c(\lambda, \alpha, 1) c(\lambda, \alpha, \alpha) \delta_{\lambda\mu}$$

$$J_\lambda^{(\alpha)}(\mathbf{y}) = c(\lambda, \alpha, 1) P_\lambda^{(\alpha)}(\mathbf{y})$$

$$J_\lambda^{(\alpha)} = c(\lambda, \alpha, 1) m_\lambda + \sum_{\nu < \lambda} u_{\lambda\nu} m_\nu$$

monomial symmetric functions

$$m_{(211)}(\mathbf{y}) = y_1^2 y_2 y_3 + y_1 y_2^2 y_3 + y_1 y_2 y_3^2$$

G. Macdonald book

Symmetric functions and Hall polynomials

R.P. Stanley, Adv. Math. 77 76 (1989).

$$J_\lambda^{(\alpha)} = \sum_\nu \theta_\nu^\lambda(\alpha) p_\nu \qquad p_\nu = \sum_\lambda \gamma_\nu^\lambda(\alpha) J_\lambda^{(\alpha)}$$

$$\longrightarrow \qquad \gamma_\mu^\lambda(\alpha) = \frac{\theta_\mu^\lambda(\alpha) z_\mu \alpha^{\ell(\mu)}}{c(\lambda,\alpha,1)c(\lambda,\alpha,\alpha)}$$

$$c(\lambda,\alpha,t) := \prod_{s\in\lambda}(\alpha a_\lambda(s) + l_\lambda(s) + t)$$

G. Macdonald book

$$\theta_{(k)}^\lambda(\alpha) = \prod_{s-\{1,1\}} (\alpha a'_\lambda(s) - l'_\lambda(s))$$

$$\sum_{r=1}^n y_r^k = p_{(k)}(\mathbf{y})$$

$$= \sum_{\lambda,|\lambda|=k} \frac{k\alpha}{c(\lambda,\alpha,1)c(\lambda,\alpha,\alpha)} \prod_{s-\{1,1\}} (\alpha a'_\lambda(s) - l'_\lambda(s)) \quad J_\lambda^{(\alpha)}(\mathbf{y})$$

express moments in terms of Jacks

The Kadell integral

K.W.J. Kadell Adv. Math. 130 33 (1997)

$$\int_{[0,1]^n} P_\lambda^{(1/\kappa)}(\mathbf{y})|\Delta(\mathbf{y})|^{2\kappa} \prod_{i=1}^{n} y_i^a(1-y_i)^b dy_i \quad = n!$$

$$\prod_{i=1}^{n} \frac{\Gamma\left(\lambda_i+a+1+\kappa(n-i)\right)\Gamma\left(b+1+\kappa(n-i)\right)}{\Gamma\left(\lambda_i+a+b+2+\kappa(2n-i-1)\right)} \prod_{1\le i<j\le n} \frac{\Gamma\left(\lambda_i-\lambda_j+\kappa(j-i+1)\right)}{\Gamma\left(\lambda_i-\lambda_j+\kappa(j-i)\right)}$$

$$\downarrow$$

$$\left\langle J_\lambda^{1/\kappa}(\mathbf{y})\right\rangle_J = \kappa^{-|\lambda|} \prod_{i=1}^{\ell(\lambda)} \frac{(a+1+\kappa(n-i))_{\lambda_i}}{(a+b+2+\kappa(2n-i-1))_{\lambda_i}} (\kappa(n-i+1))_{\lambda_i}$$

Moments of Jacobi ensemble: explicit expression

k positive integer

$$\left\langle \frac{1}{n}\sum_{j=1}^{n} y_j^k \right\rangle_J = \sum_{\lambda, |\lambda|=k} A_\lambda a_\lambda^+$$

positive moments

$$\left\langle \frac{1}{n}\sum_{j=1}^{n} y_j^{-k} \right\rangle_J = \sum_{\lambda, |\lambda|=k} A_\lambda a_\lambda^-$$

negative moments

$$A_\lambda = \frac{k(\lambda_1-1)!}{(\kappa(\ell(\lambda)-1)+1)_{\lambda_1}} \prod_{i=2}^{\ell(\lambda)} \frac{(\kappa(1-i))_{\lambda_i}}{(\kappa(\ell(\lambda)-i)+1)_{\lambda_i}} \prod_{1\le i<j\le \ell(\lambda)} \frac{\kappa(j-i)+\lambda_i-\lambda_j}{\kappa(j-i)}$$

$$\times \frac{1}{n} \prod_{i=1}^{\ell(\lambda)} \frac{(\kappa(n-i+1))_{\lambda_i}}{(\kappa(\ell(\lambda)-i+1))_{\lambda_i}} \prod_{1\le i<j\le \ell(\lambda)} \frac{(\kappa(j-i+1))_{\lambda_i-\lambda_j}}{(\kappa(j-i-1)+1)_{\lambda_i-\lambda_j}}$$

$$a_\lambda^+ = \prod_{i=1}^{\ell(\lambda)} \frac{(a+1+\kappa(n-i))_{\lambda_i}}{(a+b+2+\kappa(2n-i-1))_{\lambda_i}}$$

$$(x)_n = x(x+1)..(x+n-1)$$

$$a_\lambda^- = \prod_{i=1}^{\ell(\lambda)} \frac{(a+1+\kappa(i-1))_{-\lambda_i}}{(a+b+2+\kappa(n+i-2))_{-\lambda_i}}$$

$$\left\langle \frac{1}{n}\sum_{j=1}^{n} y_j^k \right\rangle_{J,\kappa=0} = \frac{(a+1)_k}{(a+b+2)_k}$$

$\sim \ \kappa^{\ell(\lambda)-1}$ from only $\lambda = (k)$

Duality invariance of the moments

$$< y^k >_{\beta,a,b,n} = < y^k >_{\beta',a',b',n'} \qquad \kappa = -\beta^2$$

$$\beta' = 1/\beta \quad n' = \beta^2 n \quad a' = a/\beta^2 \quad b' = b/\beta^2$$

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$$< y^k >_{\beta,a,b,n} := \left\langle \frac{1}{n} \sum_{r=1}^{n} y_r^k \right\rangle_J \Bigg|_{\kappa=-\beta^2} = \sum_{\lambda,|\lambda|=k} \lim_{\epsilon\to 0} \frac{k}{n\epsilon} \prod_{s=(i,j)\in\lambda} B_\lambda^\epsilon(s) \Bigg|_{\kappa=-\beta^2}$$

$$B_\lambda^\epsilon(s)\big|_{\kappa=-\beta^2}\Big|_{\beta,n,a,b} = B_{\lambda'}^\epsilon(s')\big|_{\kappa=-\beta'^2}\Big|_{\beta',n',a',b'}$$

duality invariance of the Jacobi moments

$\longleftrightarrow$ exchange partition with its dual partition

Duality invariance of the moments

$$< y^k >_{\beta,a,b,n} = < y^k >_{\beta',a',b',n'}$$

<=> invariant combinations

$$t = -\beta n$$
$$\bar{a} = a/\beta$$
$$\bar{b} = b/\beta$$

$$\beta' = 1/\beta \quad n' = \beta^2 n \quad a' = a/\beta^2 \quad b' = b/\beta^2$$

Consequences (under duality-freezing conjecture)

all 3 models: freezing transition at $\beta = 1$

- GUE-CP $\quad a = b = \frac{q+\beta^2}{2} \quad\quad q' = 1 + \frac{q-1}{\beta^2}$ $\quad$ choose $q = 1$ duality-invariant
- LCGRP $\quad a = \beta\bar{a} \quad b = \beta\bar{b}$ $\quad$ duality-invariant
- fBm0 $\quad a = 2n\beta^2$, $b = 0$ $\quad$ duality-invariant

First moment (k=1)

only partition $\lambda = (1)$ contributes

$$< y \ >_{\beta,a,b,n} := \left\langle \frac{1}{n} \sum_{r=1}^{n} y_r \right\rangle_J \Big|_{\kappa=-\beta^2}$$

all are rational functions

$$< y >_{\beta,a,b,n} = \frac{1 + a - \beta^2(n-1)}{2 + a + b - 2\beta^2(n-1)}$$

dependence in n is simple

$$\overline{< y \ >_{\beta,a,b}} = \lim_{n\to 0} < y \ >_{\beta,a,b,n} \longrightarrow \overline{< y >_{\beta,a,b}} = \frac{1 + a + \beta^2}{2 + a + b + 2\beta^2}$$

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- LCGP $\quad \overline{< y >_\beta} = \dfrac{1 + \bar{a}\beta + \beta^2}{2 + (\bar{a} + \bar{b})\beta + 2\beta^2} \quad \beta < 1$

duality-invariant

freezes - remains constant for $\beta > 1 \longrightarrow$

$$\overline{y_m} = \frac{2 + \bar{a}}{4 + \bar{a} + \bar{b}}$$

$$\overline{y_m} - \frac{1}{2} = \frac{\bar{a} - \bar{b}}{2(\bar{a} + \bar{b} + 4)}$$

- GUE-CP $\quad a = b = \frac{1+\beta^2}{2}$

$\longrightarrow \overline{< y >_\beta} = \dfrac{1}{2}$

- fBm0 $\quad a = b = 0$

without random potential:

$$< y >_{P^0} - \frac{1}{2} = \frac{\beta(\bar{a} - \bar{b})}{2(2 + \beta(\bar{a} + \bar{b}))}$$

$$y_m^0 = \bar{a}/(\bar{a} + \bar{b})$$

More results

- higher moments:

  - GUE-CP $\overline{x_m^2} = \frac{13}{49} = 0.265306..$ $\quad \mathrm{Ku} = -\frac{541}{507} = -1.06706.. = \frac{\overline{y_m^4}^c}{\overline{y_m^2}^2}$

compare with moments of semi-circle law $< x^2 >_\rho = \frac{1}{4}$ $\quad \mathrm{Ku} = -1$

$$\langle x^k \rangle_\rho = \int_{-1}^{+1} dx x^k \rho(x) \qquad \rho(x) = \frac{2}{\pi}\sqrt{1-x^2}$$

  - fBm0 $\overline{y_m^2} = \frac{17}{50}$ $\quad \mathrm{Ku} = -1.26363..$

compare with uniform on [0,1] $< y^2 >_{P^0} = \frac{1}{3}$ $\quad \mathrm{Ku} = -1.2$

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  compare with uniform on [0,1] $\quad < y^2 >_{P^0} = \frac{1}{3} \quad \mathrm{Ku} = \ -1.2$

- negative moments:

  - LCGP $\quad \overline{< y^{-1} >_\beta} = \frac{1+a+b+\beta^2}{a} \qquad \overline{y_m^{-1}} = \frac{2+\bar{a}+\bar{b}}{\bar{a}}$

$$\mathcal{P}(y_m) \sim y_m^a ? \qquad b_c = -1 - \beta^2$$

  - GUE-CP

$$\bar{a} = \bar{b} \to 1 \qquad \overline{y_m^{-1}} = 4 \qquad \overline{(1-x_m)^{-1}} = 2$$

$$< \frac{1}{1-x} >_\rho = 2$$

Numerics for position of maximum of GUE-CP by Nick Simm

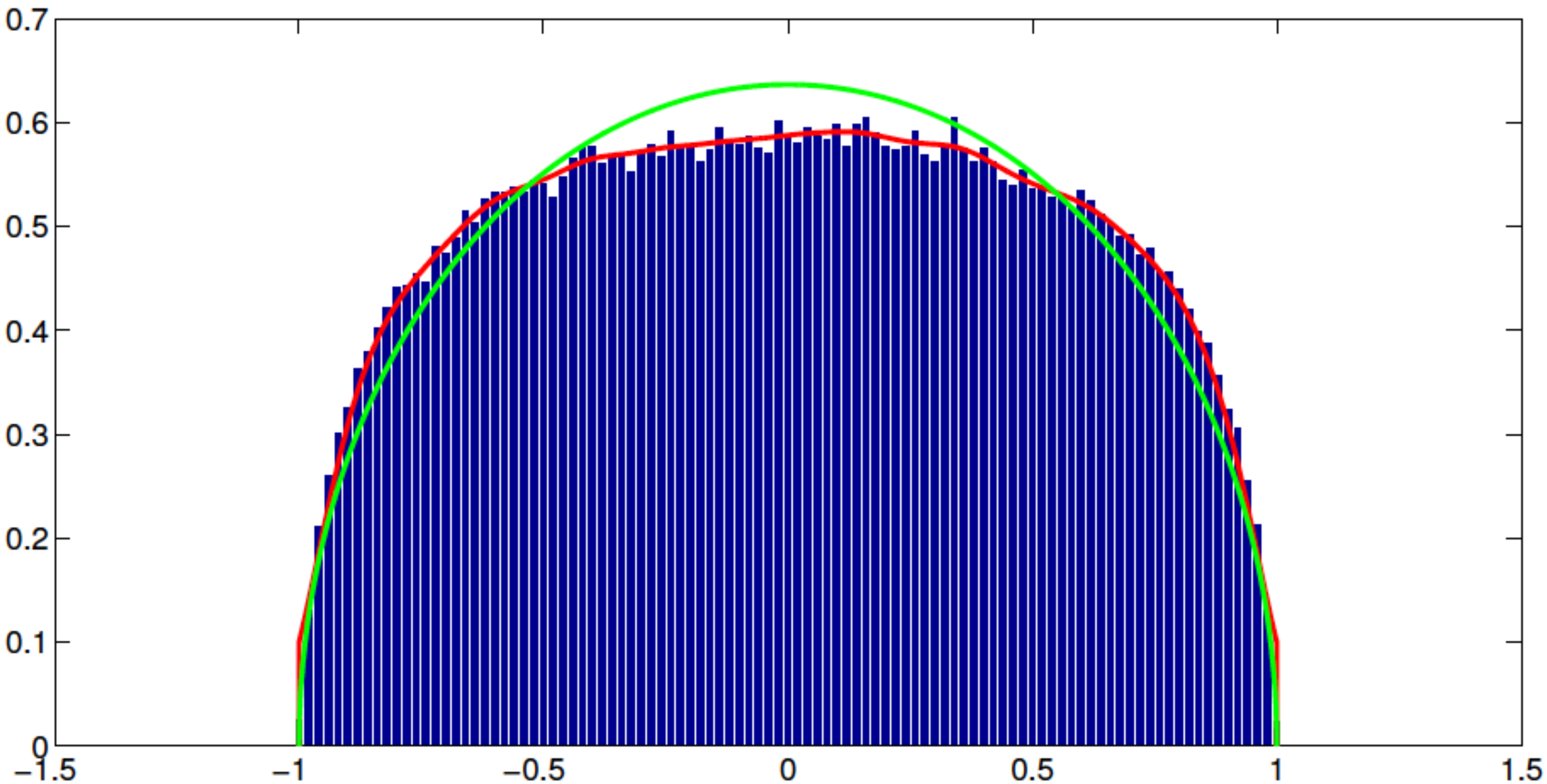
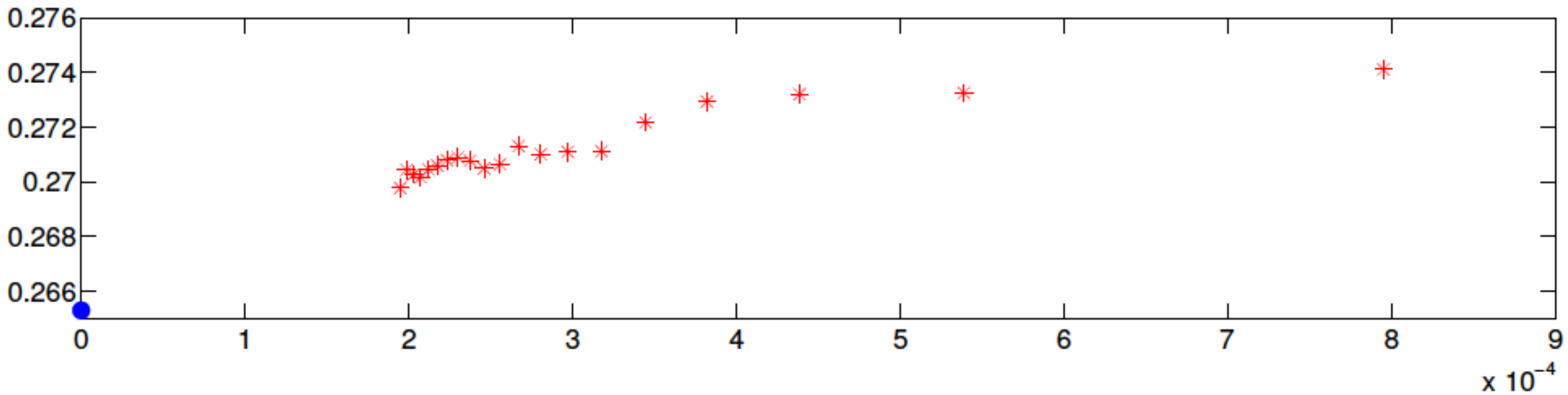


Figure 1. Histogram of values x_m for the position of the maximum of the characteristic polynomial for size $N = 3000$ GUE matrices with $250,000$ realizations. We use the numerical method described in Section 3 of [21]. The curve fitting the histogram (red) differs from the semi-circular density (green) at most by 0.099.

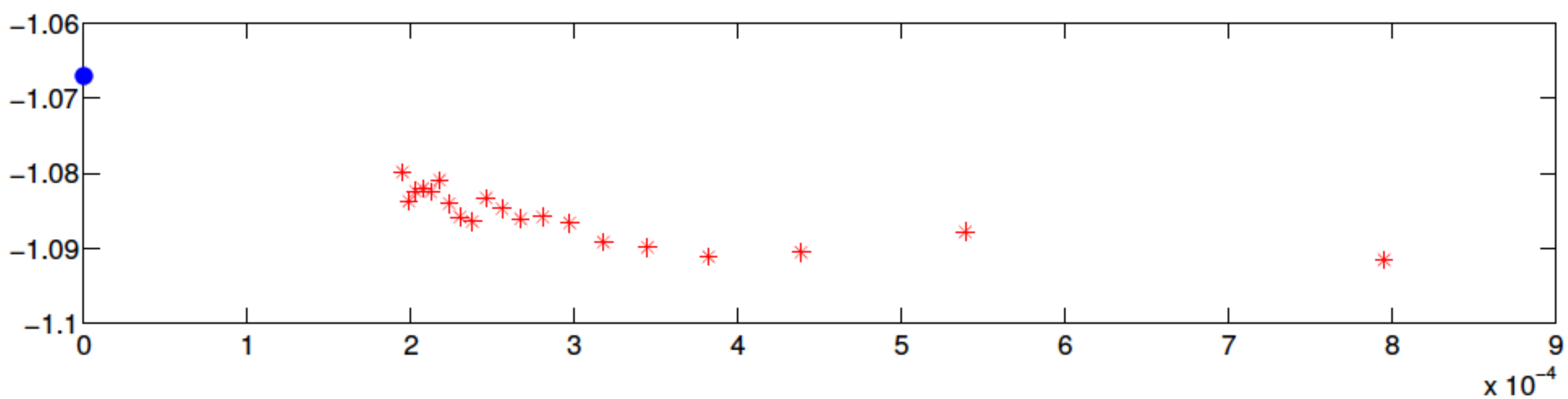


Variance

prediction 0.265..

semi-circle 0.25

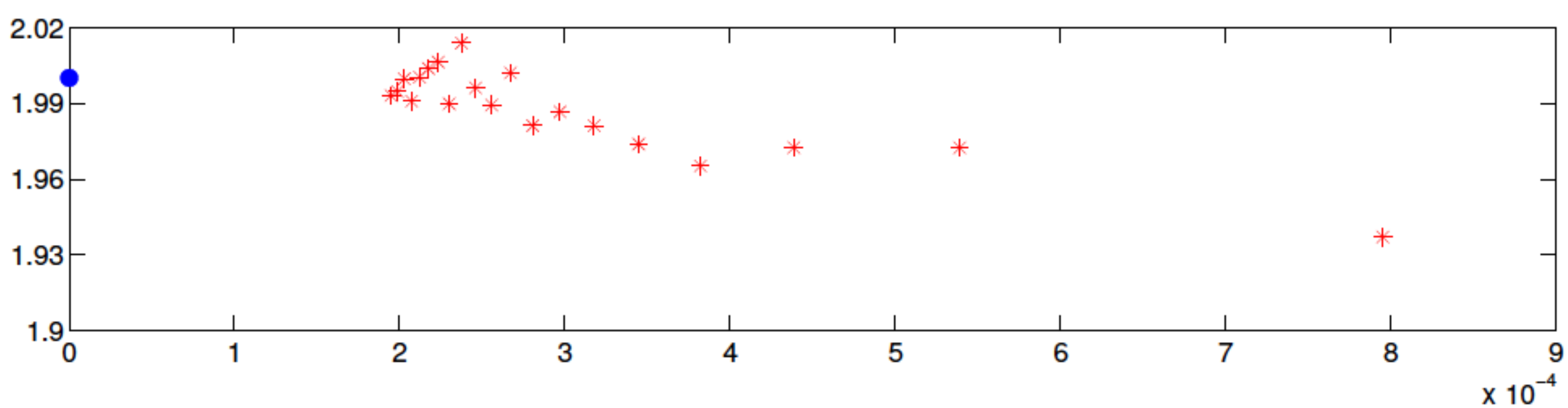
Figure 2. Variance of the position of the maximum of the characteristic polynomial for 20 equally spaced data points corresponding to size $N = 150$ up to size $N = 3000$ GUE matrices with 250, 000 realizations. The x axis has been chosen as $1/[10(\ln N)^3]$. The blue point is the prediction (123)



Kurtosis

prediction -1.067

semi-circle -1



First inverse moment

prediction 2

semi-circle 2

Joint values of Max and Min

on circle

Joint PDF of values of Max and Min for GFF on the circle

X. Cao, PLD, arXiv 1604.02282

(discrete) circular log-REM $1 \leq j \leq M$

$$V_{j,M} = \Re\left[\sum_{k=-\frac{M}{2}}^{\frac{M}{2}-1} \sqrt{\frac{1}{|k|}}(u_k + \mathbf{i} v_k) \exp\left(\frac{2\pi \mathbf{i} k j}{M}\right)\right]$$

i.i.d centered indep. Gaussians

$$\xi_{j,M} = \exp\left(\frac{2\pi \mathbf{i} j}{M}\right)$$

$$\overline{V_{j,M}^2} = 2(\ln M + W)$$

$$\overline{V_{j_M,M} V_{k_M,M}} \to 2 \ln |\xi - \eta|$$

$$(\xi_{j_M,M}, \xi_{k_M,M}) \to (\xi, \eta)$$

$$W \to \gamma_E - \ln 2$$

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$$(\xi_{j_M,M}, \xi_{k_M,M}) \to (\xi, \eta)$$

$$W \to \gamma_E - \ln 2$$

discrete partition sum

$$\mathcal{Z}_{M\pm} = \sum_{j=1}^{M} \exp\left(\mp \beta V_{j,M}\right)$$

$$Z_{\pm} = \frac{\mathcal{Z}_{M\pm}}{M^{1+\beta^2} e^{\beta^2 W}}$$

moments are CG integrals on unit circle with two types of charges

$$\overline{Z_+^m Z_-^n} = \int \mu_n^\alpha(\underline{\xi}) \mu_m^\alpha(\underline{\eta}) \prod_{a,b} |1 - \xi_a^* \eta_b|^{-2/\alpha}$$

infinite M

$$1/\alpha = -\beta^2$$

$$\mu_n^\alpha(\underline{\xi}) = \prod_{a=1}^{n} \frac{d\xi_a}{2\pi \mathbf{i} \xi_a} \prod_{a<a'} |\xi_a - \xi_{a'}|^{2/\alpha}$$

Result for JPDF of Min/Max

$$V_{M\pm} = \pm \min_{j=1}^{M} (\pm V_{j,M})$$

for large M $\quad V_{M\pm} = \mp 2 \ln M \pm \frac{3}{2} \ln \ln M + v_{\pm} \pm c_M$

Carpentier,PLD

v_+ $\quad$ same
$-v_-$ marginals $\quad P(v_+ > y) = 2e^{\frac{y}{2}} K_1(2e^{\frac{y}{2}}) \longleftrightarrow \overline{\exp(t v_\pm)} = \Gamma^2(1 \pm t)$

Fyodorov-Bouchaud

Laplace transform of Joint PDF

$$\overline{\exp(t_1 v_+ - t_2 v_-)} = S_1(t_1, t_2) \prod_{i=1}^{2} \Gamma^2(1 + t_i)$$

$$S_\beta(t_1, t_2) = \sum_{\lambda} \prod_{\substack{(x,y)\in\lambda \\ i=1,2}} \frac{x\beta^{-1} + y\beta + t_i}{(x+1)\beta^{-1} + (y+1)\beta + t_i}$$

provides correlations between max and min

Min-Max correlation and numerical check

$$-\overline{V_{M+}V_{M-}}^{c} \xrightarrow{M\to\infty}$$

$$-\overline{v_+ v_-}^{c} = \left.\frac{\partial^2 S_1}{\partial t_1 \partial t_2}\right|_{t_1,t_2=0}$$

$$= \sum_{\lambda\neq\emptyset} \frac{1}{4} \prod_{x,y\neq(0,0)} \frac{(x+y)^2}{(x+y+2)^2}$$

$$= 0.338\,..$$

$$\overline{v_{\max}^2}^{c} = \pi^2/3 = 3.29\,..$$

Fine test of freezing-duality conjecture

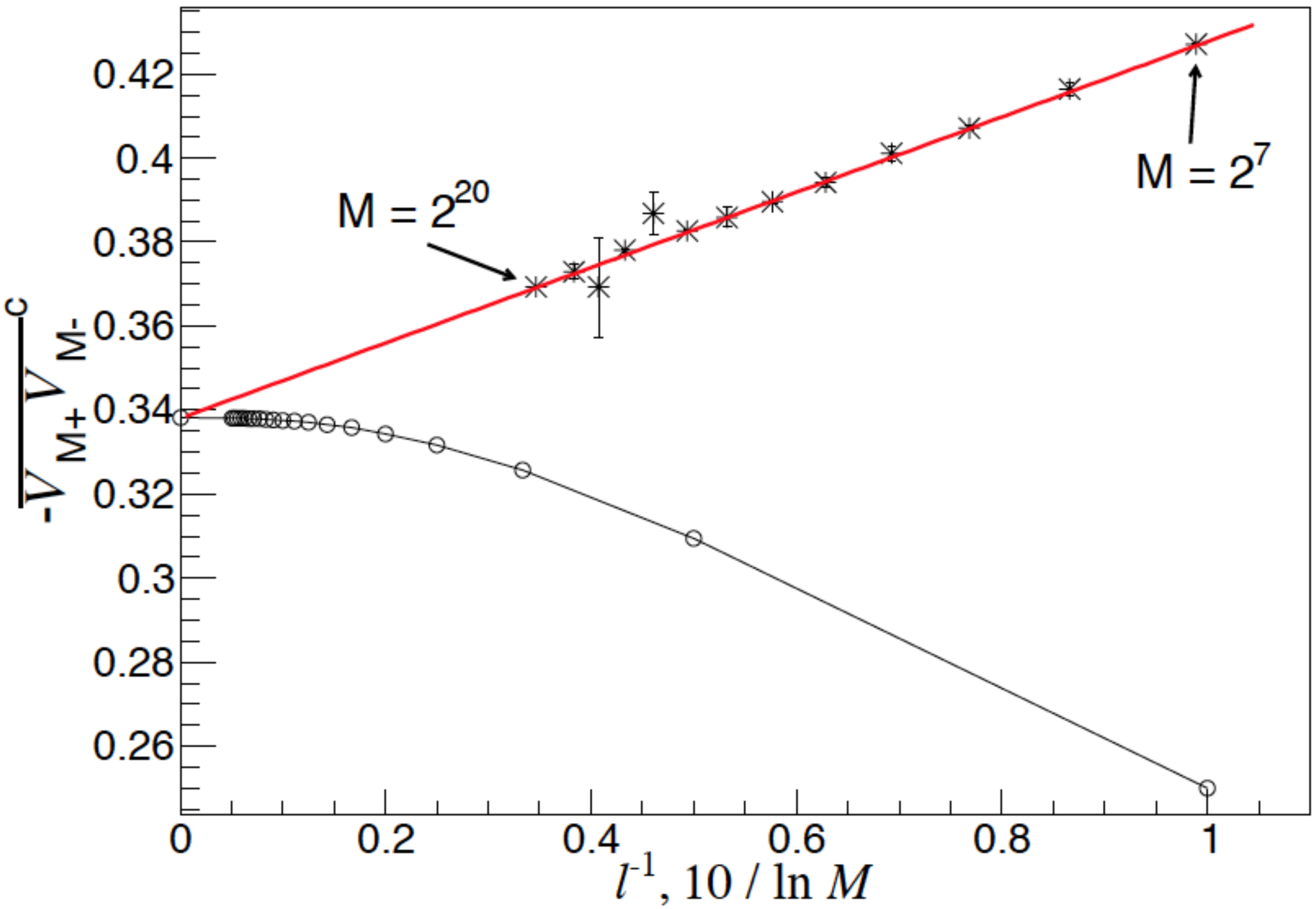

Figure 1: Numerical check of the (minus) min-max covariance $-\overline{V_{M+}V_{M-}}^{c}$. The $1/f$–noise (2) is generated using Fast Fourier transform, with $\geq 10^6$ independent realisations for each size. The numerical data $(*)$ $2^7 \leq M \leq 2^{20}$ are consistent finite-size scaling $a + b/\ln M$, with $b = 0.89(1)$ and $a = 0.338(1)$, in 3-digit agreement with (32). The sums over partitions in this work are all convergent and calculated by the method of [35], which involves a truncation size l. The sum (32) truncated to $l = 1, \ldots, 20$ are ploted $(\circ)$ to appreciate convergence; in all cases $l \sim 10^2$ yields sufficient precision.

1) use square of Cauchy identity

$$\prod_{a,b} |1-\xi_a^*\eta_b|^{-2/\alpha} = \sum_{\lambda,\mu} P_\lambda^{(\alpha)}(\underline{\xi})Q_\lambda^{(\alpha)}(\underline{\eta^*})P_\mu^{(\alpha)}(\underline{\eta})Q_\mu^{(\alpha)}(\underline{\xi^*})$$

dual basis of Jack polynomials

$$P_\lambda^{(\alpha)}(\underline{\xi}) \quad Q_\lambda^{(\alpha)}(\underline{\xi})$$

2) use orthogonality on unit circle

$$\int \mu_n^\alpha(\underline{\xi})P_\lambda^{(\alpha)}(\underline{\xi})Q_\mu^{(\alpha)}(\underline{\xi^*}) = \delta_{\lambda\mu}p_n^\lambda(\alpha)c_n(\alpha)$$

$$\longrightarrow \quad \overline{Z_+^n Z_-^m} \stackrel{!}{=} \frac{\Gamma(1-n\beta^2)\Gamma(1-m\beta^2)}{\Gamma(1-\beta^2)^{m+n}} \sum_\lambda p_n^\lambda(\alpha)p_m^\lambda(\alpha)$$

$$f_\pm := F_\pm \pm \frac{1}{\beta}\ln\Gamma(1-\beta^2) \qquad p_n^\lambda(\alpha) = \prod_{(x,y)\in\lambda} \frac{\alpha x + n - y}{\alpha(x+1)+n-(y+1)}$$

$$F_\pm := -\beta^{-1}\ln Z_\pm$$

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$$p_n^\lambda(\alpha) = \prod_{(x,y)\in\lambda} \frac{\alpha x + n - y}{\alpha(x+1) + n - (y+1)}$$

→ LT of JPDF of free energies for $\beta < 1$

$$t_1 = -n\beta \qquad t_2 = -m\beta$$

$$\overline{\exp(t_1 f_+ - t_2 f_-)} \overset{!}{=} S_\beta(t_1, t_2) \prod_{i=1}^{2} \Gamma(1+\beta t_i)$$

needs extra factor

$$\Gamma(1 + t_i/\beta)$$

$$S_\beta(t_1,t_2) = \sum_\lambda \prod_{\substack{(x,y)\in\lambda \\ i=1,2}} \frac{x\beta^{-1} + y\beta + t_i}{(x+1)\beta^{-1} + (y+1)\beta + t_i}$$

S is clearly duality invariant

exchange x and y

i.e. exchange partition
and its dual partition

obtained defining

$$y_{\pm\beta} := f_\pm \mp \beta^{-1} g_\pm$$

two independent Gumbel variables

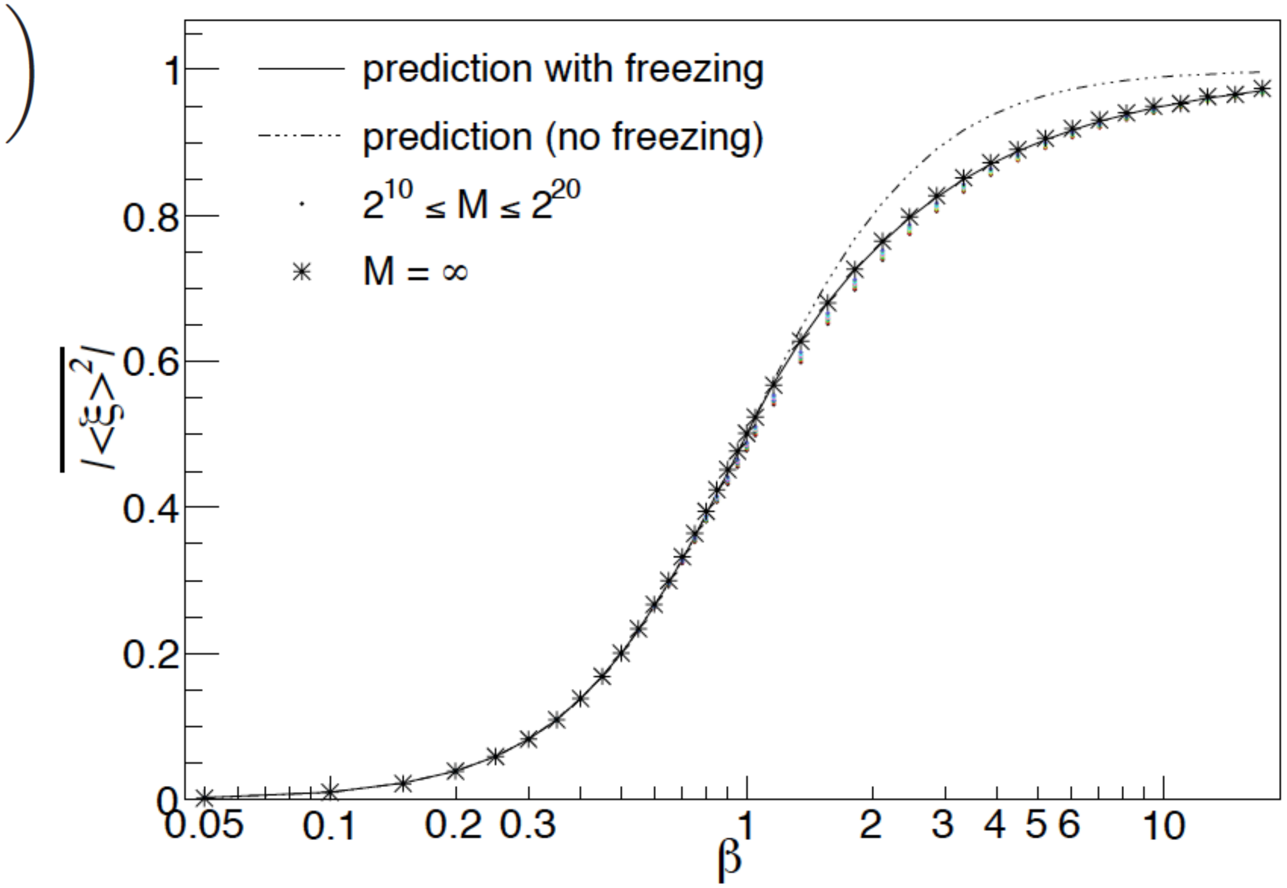
Edwards-Anderson order parameter

O(2) spin glass

thermal average $\xi_{j,M} = \exp\left(\frac{2\pi \mathbf{i} j}{M}\right)$

$$\langle \xi \rangle = \frac{\sum_{j=1}^{M} \exp(-\beta V_{j,M}) \xi_{j,M}}{\sum_{j=1}^{M} \exp(-\beta V_{j,M})}$$

$$\overline{|\langle \xi \rangle|^2} \overset{M\to\infty}{=} \begin{cases} \frac{\beta^2}{1+\beta^2}, & \beta \le 1 \\ \frac{2\beta - 1}{2\beta}, & \beta \ge 1 \end{cases}$$



glass (disorder relevant) at all T

if short-range correlated RP -> it would be zero

freezing: change of nature of glass, RS to RSB

Value of second minimum

What about PDF of value of second minimum ?

non-universal (UV-dependent) BUT

X. Cao, Y. Fyodorov, PLD,
SciPost Phys. 1, 011 (2016)

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given the PDF of value of minimum known+universal for circle

$$\overline{\delta(V_{\min} - y)} = -\partial_y G_\infty(y) = 2e^y K_0(2e^{y/2})$$

then $\overline{\delta(V_{\min,1} - y)} = -G'_\infty(y) + \overline{g} G''_\infty(y)$

for all log-rem's

$\overline{g} =$ mean gap in [0,1]

gap PDF depends on UV details

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k-th order stat ?

gap PDF depends on UV details

SDPPP: IR randomly shifted, UV decorated, Gumbel Poisson point process

JPDF positions min,min1

$$P(\xi_1, \xi_2) = c_0 \delta(\xi_1 - \xi_2) \overline{p_1(\xi_1)} + (1 - c_0) \overline{p_1(\xi_1) p_1(\xi_2)}$$

c_0 is proba same cluster

$$\overline{g} = 1 - c_0$$

What about PDF of value of second minimum ?

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X. Cao, Y. Fyodorov, PLD, SciPost Phys. 1, 011 (2016)

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k-th order stat ?

gap PDF depends on UV details

SDPPP: IR randomly shifted, UV decorated, Gumbel Poisson point process

JPDF positions min,min1 $P(\xi_1, \xi_2) = c_0 \delta(\xi_1 - \xi_2) \overline{p_1(\xi_1)} + (1 - c_0) \overline{p_1(\xi_1) p_1(\xi_2)}$

c_0 is proba same cluster

$$\overline{g} = 1 - c_0$$

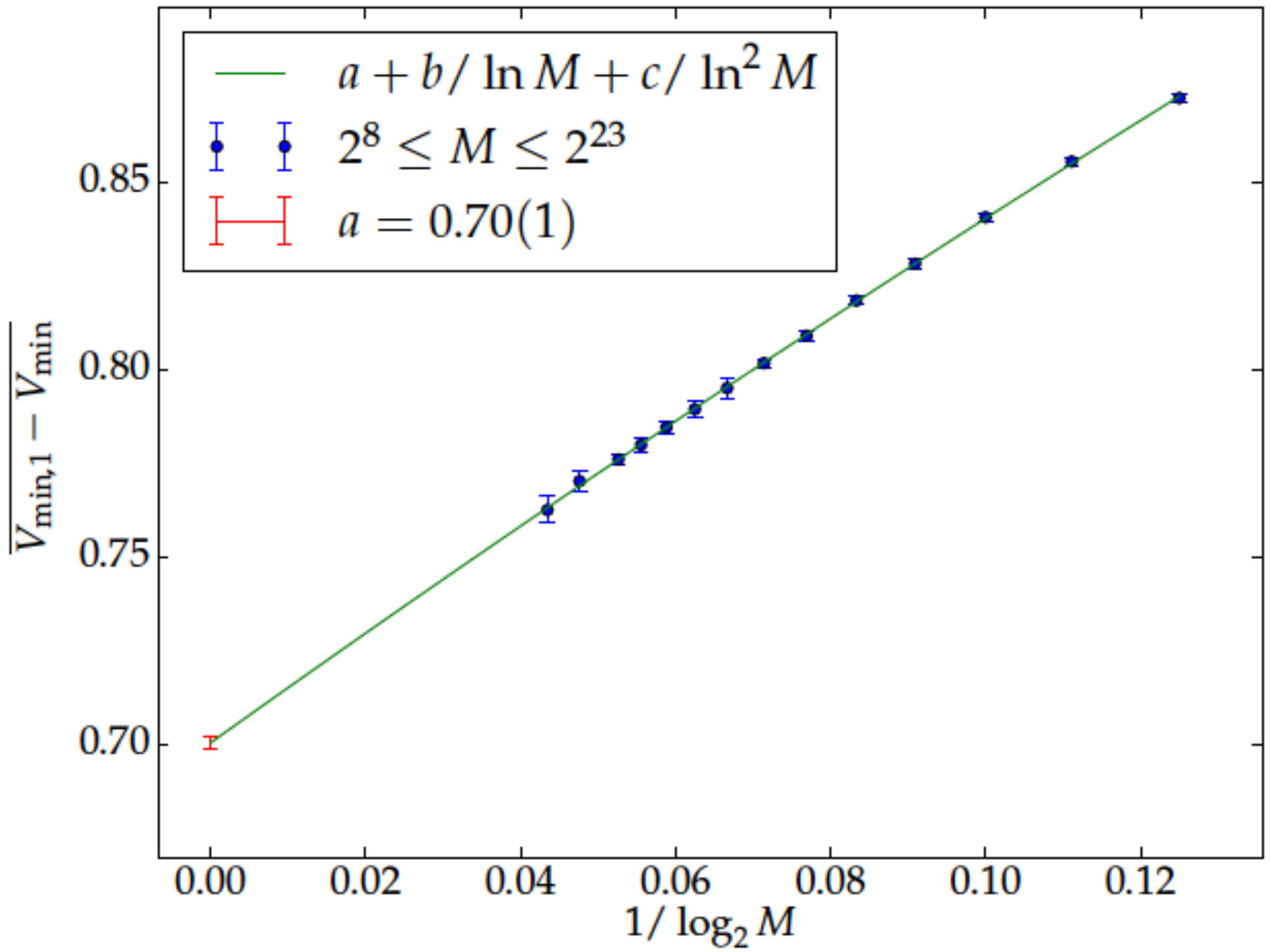
far second min $\min(V_{j,M}, |j - j_{\min}| > N/2)$

$\overline{\theta(g_1^{\text{far}} - \Delta)} \to \exp(-\Delta)$ $M \to \infty$ $\overline{g} = 1$

numerical checks for circular log-REM

measure of mean gap

check of prediction for PDF of Vmin1



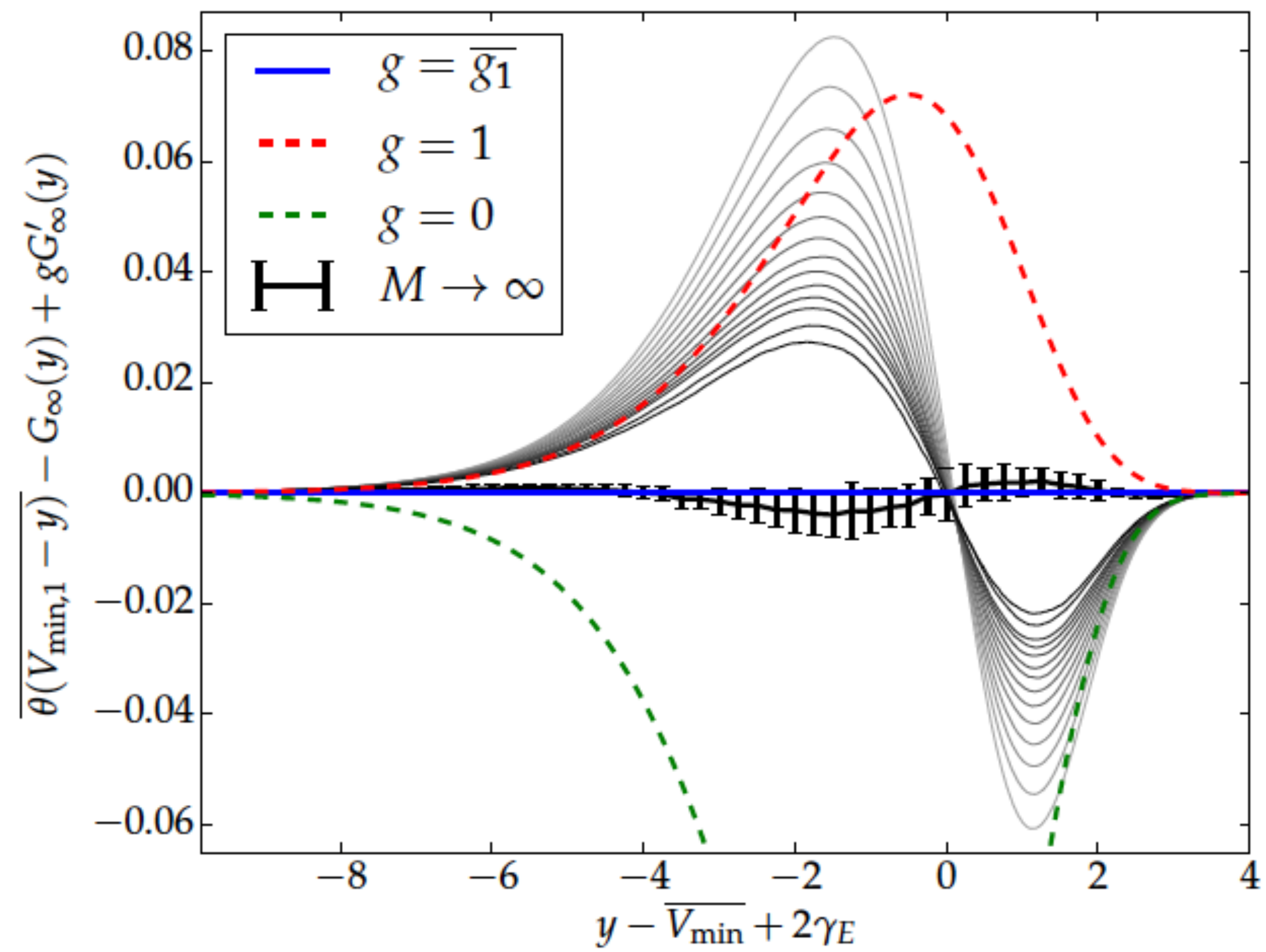


Figure 2. Left: the numerical measure of the mean of the first gap, as a function of the system size (points), is well described by a quadratic finite-size Ansatz $a + b/\ln M + c/\ln^2 M$. We use it to extract the $M \to \infty$ value $\overline{g_1} = a = 0.70(1)$. Right: The cumulative distribution function of the second minimum $V_{\min,1}$ of the circular $1/f$-noise model, with the theoretical prediction (39) subtracted, and the parameter $g = \overline{g_1}$ fed by the previous measurement. Grey curves are numerical data with system sizes $2^8 \le M \le 2^{23}$, and the extrapolation to $M \to \infty$ (thick black curve with error bars) is performed by applying the quadratic Ansatz pointwise. The error bars combine the error in the distribution with that in $\overline{g_1}$. For comparison we plot in dash lines (39) with other values of g.

Exact results in D=2 ?

Extrema (PDF of value, position..)

GFF in domain (e.g. disk..) ???

PDF of position in plane + confining charges

can be obtained from Liouville field theory

Kogan, Mudry, Tsvelik (1996)

Carpentier, PLD (2001)

Liouville field theory and log-REMs

X. Cao, A. Rosso, R. Santachiara, PLD, arXiv1611.02193

SETTING: 2D GFF plane $z \in \mathbb{C}$

$$\overline{\phi(z)\phi(w)} = 4\ln(R/|z-w|)$$

$$\overline{\phi(z)^2} = 4\ln(R/\epsilon) \qquad \epsilon \to 0, R \to \infty$$

GFF + background potential $a_1, a_2 > 0$

$$U(z) \stackrel{\text{def}}{=} 4a_1 \ln|z| + 4a_2 \ln|z-1|$$

Gibbs measure for a particle

$$p_\beta(z) \stackrel{\text{def}}{=} \frac{1}{Z} e^{-\beta(\phi(z)+U(z))}$$

$$Z \stackrel{\text{def}}{=} \int_{\mathbb{C}} e^{-\beta(\phi(z)+U(z))} \mathrm{d}^2 z$$

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$$U(z) \stackrel{\text{def}}{=} 4a_1 \ln|z| + 4a_2 \ln|z-1|$$

$$\overline{F} = -Q\ln M + x\ln\ln M + O(1)\,,\; M = (R/\epsilon)^2$$
$$Q = b + b^{-1}\,,\; b = \min(1,\beta)\,.$$

$a_1 + a_2 > Q/2$ confine the particle finite region

$a_1, a_2 < Q/2$ avoid collapse

→ random Gibbs measure $p_\beta(z)$ well defined limit $\epsilon \to 0, R \to \infty$

Dotsenko-Fateev integrals: value of max is plagued by IR divergences

Main Claim:

$$\overline{p_\beta(z)} \overset{\beta<1}{\propto} \langle \mathcal{V}_{a_1}(0)\mathcal{V}_{a_2}(1)\mathcal{V}_b(z)\mathcal{V}_{a_3}(\infty)\rangle_b$$

$$a_3 = Q - a_1 - a_2 \qquad Q = b + b^{-1}$$

$\mathcal{V}_a(z)$ vertex operator

primary field $\Delta_a = a(Q-a)$

Liouville FT $c = 1 + 6Q^2$

defined from (i) axioms (ii) measure (i) S.Ribault et al. (ii) V. Vargas et al.

$$\mathcal{S}_b = \int_\Sigma \left[\frac{1}{16\pi}(\nabla\varphi)^2 - \frac{1}{8\pi}Q\hat{R}\varphi + \mu e^{-b\varphi}\right] \mathrm{d}A \qquad \mathcal{V}_a(w) \rightsquigarrow e^{-a\varphi(w)}$$

$$\Sigma = \mathbb{C}\cup\{\infty\}$$

$$\hat{R}(z) = 8\pi\delta^2(z-\infty)$$

$$\mathrm{d}A = \mathrm{d}^2 z$$

Main Claim:

$$\overline{p_\beta(z)} \overset{\beta<1}{\propto} \langle \mathcal{V}_{a_1}(0)\mathcal{V}_{a_2}(1)\mathcal{V}_b(z)\mathcal{V}_{a_3}(\infty)\rangle_b$$

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$$\Sigma = \mathbb{C} \cup \{\infty\}$$

$$\hat{R}(z) = 8\pi\delta^2(z-\infty)$$

$$\mathrm{d}A = \mathrm{d}^2 z$$

integrate over zero-mode $\varphi(z) = \varphi_0 + \tilde{\varphi}(z)$

$$\int d\varphi_0 e^{-\mu Z_0 e^{-b\varphi_0} - (b+\sum_i a_i - Q)\varphi_0} \sim Z_0^{-1}$$

($b+\sum_i a_i - Q \to 0$)

$$K_4 \overset{\text{def}}{=} \int \mathcal{D}\varphi\, e^{-\mathcal{S}_b - b\varphi(z) - a_1\varphi(0) - a_2\varphi(1) - a_3\varphi(\infty)}$$

$$\int \mathcal{D}\tilde{\varphi} e^{-\int \frac{1}{16\pi}(\nabla\tilde{\varphi})^2 \mathrm{d}^2 z}\mathcal{O}[\tilde{\varphi}] = \overline{\mathcal{O}[\phi]}$$

$$= \overline{e^{-a_1\phi(0) - a_2\phi(1) + (a_1+a_2)\phi(\infty) - b\phi(z)}/Z_0}$$

$$= \overline{e^{-b(\phi(z)+U(z))}/Z}$$

$$= \overline{p_\beta(z)}$$

$$Z_0 = \int_{\mathbb{C}} e^{-b\phi(z)}\mathrm{d}^2 z$$

$$Z \overset{\text{def}}{=} \int_{\mathbb{C}} e^{-\beta(\phi(z)+U(z))}\mathrm{d}^2 z$$

four-point function of LFT known from conformal bootstrap:

Zamolodchikov^2 algorithm: S. Ribault R. Santachiara

integral involving (i) structure factors (ii) conformal blocks

1) check in high-T phase of

$$\overline{p_\beta(z)} \overset{\beta<1}{\propto} \langle \mathcal{V}_{a_1}(0)\mathcal{V}_{a_2}(1)\mathcal{V}_b(z)\mathcal{V}_{a_3}(\infty)\rangle_b$$

2) check in low-T phase

freezing-duality conjecture $\overline{p_{\beta>1}} = \overline{p_1}$

distribution of the position of the minimum !

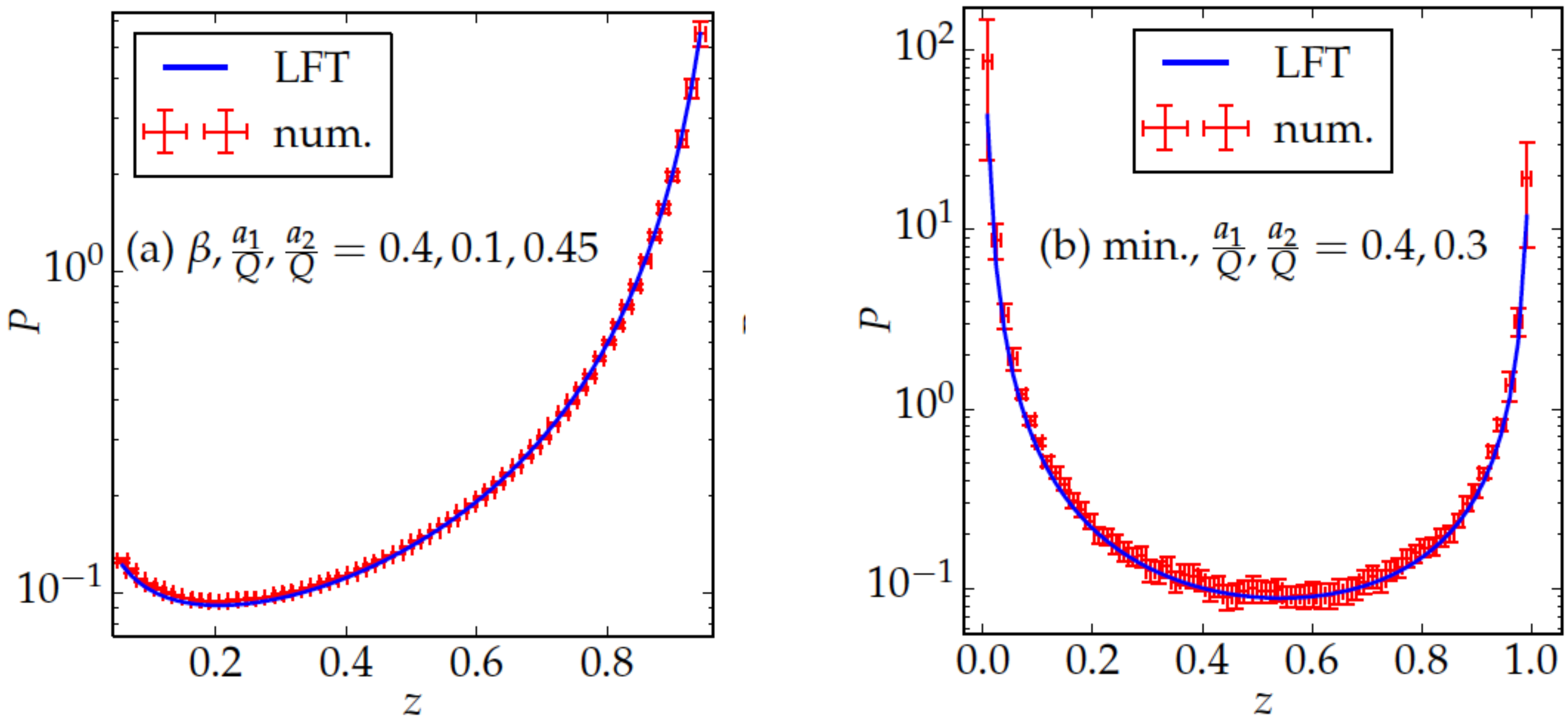

Figure 2. (Color online) Test of (9) on the segment $z \in [0, 1]$. (a) High-T regime ($\beta = .4$). (b) Minimum position distribution versus LFT with $b = 1$. Numerical parameters: $L = 2^{12}, \ell = 2^9$, 5×10^6 independent samples.

generalization and applications to log-REM's

$$\overline{\prod_{i=1}^{n} p_\beta^{q_i}(z_i)} \overset{\beta<1}{\propto} \left\langle \prod_{j=1}^{k+1} \mathcal{V}_{a_j}(w_j) \prod_{i=1}^{n} \mathcal{V}_{\beta q_i}(z_i) \right\rangle_b$$

$$U(z) = \sum_{j=1}^{k} 4a_j \ln|z - w_j| \qquad \forall a_j < Q/2$$

$$a_{k+1} \overset{\text{def}}{=} Q - \sum_{j=1}^{k} a_j < Q/2$$

Liouville OPE

$$\langle \mathcal{V}_a(0)\mathcal{V}_{a'}(z)\ldots\rangle_b \overset{z\to 0}{\sim} \begin{cases} |z|^{-2\delta_0}, \; a'' \overset{\text{def}}{=} a + a' < \frac{Q}{2}, \\ |z|^{-2\delta_0} \ln^{-\frac{1}{2}}|1/z|, \; a'' = \frac{Q}{2}, \\ |z|^{-2\delta_1} \ln^{-\frac{3}{2}}|1/z|, \; a'' > \frac{Q}{2}, \end{cases}$$

$$\delta_0 = 2aa', \; \delta_1 = \Delta_a + \Delta_{a'} - \Delta_{\frac{Q}{2}}, \; \Delta_a = a(Q-a) \qquad (16)$$

$$\overline{p_\beta(w)p_\beta(z+w)} \overset{z\to 0}{\sim} \begin{cases} |z|^{-4\beta^2} & \beta < 3^{-\frac{1}{2}} \\ |z|^{-4/3} \ln^{-\frac{1}{2}}|1/z| & \beta = 3^{-\frac{1}{2}} \\ |z|^{-3+\frac{\beta^2+\beta-2}{2}} \ln^{-\frac{3}{2}}|1/z| & \beta \in (3^{-\frac{1}{2}}, 1] \\ c'T\,|z|^{-2} \ln^{-\frac{3}{2}}|1/z| + (1-T)\delta(z) & \beta > 1 \end{cases}$$

"prefreezing" "termination point" (for $\beta = 3^{-\frac{1}{2}}$)

"freezing" (for $\beta > 1$)

generalization and applications to log-REM's

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"prefreezing" "termination point"

"freezing"

DP Cayley Tree

overlap of 2 DP length t in same sample $\quad q = \hat{q}/t$

$$|z| = r = \kappa^{-\hat{q}/2}$$

$$P(\hat{q})\mathrm{d}\hat{q} = \overline{p_\beta(w)p_\beta(w+r)}2\pi r \mathrm{d}r$$

$$P(\hat{q}) \sim \begin{cases} \kappa^{(2\beta^2-1)\hat{q}}, & \beta < 3^{-\frac{1}{2}}, \\ \kappa^{-\hat{q}/3}\hat{q}^{-\frac{1}{2}}, & \beta = 3^{-\frac{1}{2}}, \\ \kappa^{-(\beta-\beta^{-1})^2\hat{q}/4}\,\hat{q}^{-\frac{3}{2}}, & \beta \in (3^{-\frac{1}{2}}, 1) \\ \hat{q}^{-\frac{3}{2}}\beta^{-1}, & \beta \geq 1, q \ll t \end{cases}$$

Derrida-Mottishaw EPL 2016

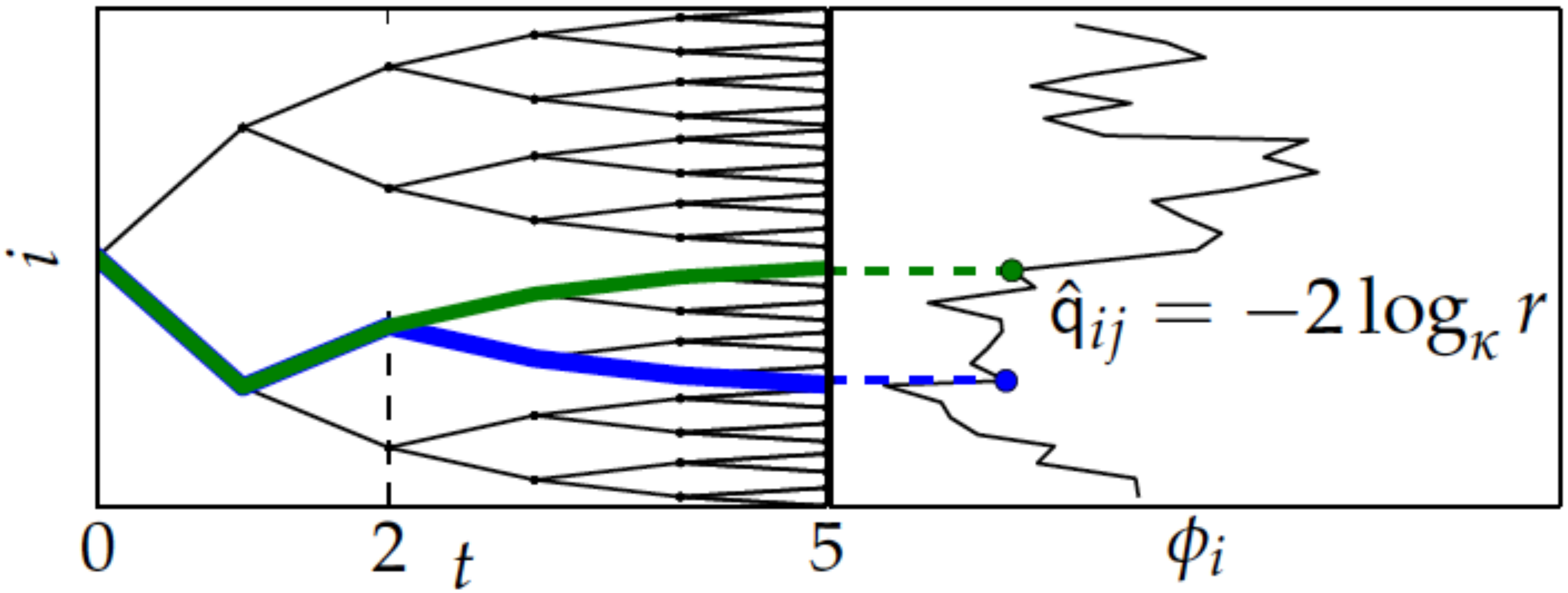

Figure 3. A Cayley tree with $\kappa = 2$ and $t = 5$. Two directed polymers are drawn in bold; they have common length $\hat{\mathsf{q}} = 2$
The energies of the DP's are plotted on the right. The common length–distance mapping is illustrated.

Conclusion

- predictions for exact results for extreme value stat. for log correlated fields (log-REM's)

- some features are the same for all log-REM's 1step RSB

also, tree, BBrowM k-th order stat RSDPP

detailed PDF depend on IR and in some cases (Vmink>1) on UV details

- in some cases exact results from duality-freezing conjecture

d=1 integrability (Selberg, Jack) relations to RMT ensembles

d=2 Liouville field theory exact results T>Tc => conjecture min

- position of minima: for interval, in 2D plane+charges
- value for circle, interval, other cases: unbounded problematic

most are still conjectures: high precision numerical tests !

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most are still conjectures: high precision numerical tests !

- **Remaining Questions:** - on interval get PDF position from moments

- get joint position/value PDF $\overline{(y_m^2 - \overline{y_m^2})(V_m - \overline{V_m})} = \frac{9}{686}$ GUE-CP

- PDF of value fBm0 does not work! how to get it? are moments correct?

- Extent of universality of predictions for Liouville 2D

subdominant terms,back to 1D? what is role of duality? integrability?

Non exhaustive!

Freezing

T. Madaule, R. Rhodes, V. Vargas. Glassy phase and freezing of log-correlated Gaussian potentials. *Ann. Appl. Probab.* **26** Number 2, 643-690 (2016)

E. Subag and O. Zeitouni. Freezing and decorated Poisson point processes. *Commun. Math. Phys.* **337**, Issue 1, pp 55-92 (2015)

3/2 log log N

M. Bramson and O. Zeitouni. Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field *Comm. Pure Appl. Math.* **65** 1-20 (2012)

GUE-CP

Y. V. Fyodorov and N. J. Simm. On the distribution of maximum value of the characteristic polynomial of GUE random matrices. e-preprint arXiv:1503.07110 (2015)

Selberg-Barnes

D. Ostrovsky. Mellin Transform of the Limit Lognormal Distribution, *Comm. Math. Phys.* **288**, 287-310 (2009).

D. Ostrovsky. Selberg Integral as a Meromorphic Function. *Int. Math. Res. Not.* **2012** 41 pp (2012).

D. Ostrovsky. Theory of Barnes Beta Distributions. *Electron. Commun. Prob.* **18**, no. 59, 116, (2012).

D. Ostrovsky. On Barnes Beta Distributions, Selberg Integral and Riemann Xi. *Forum Mathematicum.* DOI: 10.1515/forum-2013-0149, September 2014.

Fyodorov, PLD, Rosso J. Stat. Mech.(2009)

- duality $$g_\beta(y) = g_{1/\beta}(y)$$ $$\sum_{n=1}^{\infty} \frac{s^n}{n!} \overline{y^n}^c = \ln\Gamma(1+s\beta) + \ln\Gamma(1+\frac{s}{\beta})$$

inside high-T phase only!

found to be exact for interval and circle

=> $$\partial_\beta g_\beta(y)|_{\beta=\beta_c^-} = 0 \quad , \quad \text{for all } y$$

$$t = -n\beta \qquad \overline{e^{t f_\beta}} = \Gamma(1+\beta t)$$

define the random variable

$$y_\beta = f_\beta - G/\beta \qquad \overline{e^{t y_\beta}} = \Gamma(1+\beta t)\Gamma(1+t/\beta)$$

G is independent Gumbel

$$g_\beta(x) = \overline{e^{-e^{\beta(x-f_\beta)}}} =< Prob(y_\beta > x) >_G$$

$$\overline{e^{-nV_m}} = \Gamma(1-n)S(n) \qquad V_m =_{\text{in law}} f_1 - G$$

$$S(n) = \frac{G(1)G(2+a)G(2+b)G(4+a+b-2n)}{G(1-n)G(2+a-n)G(2+b-n)G(4+a+b-n)}$$

Borodin-Gorin (nested) contour integral formula

in Appendix A of
Y. V. Fyodorov, PLD, arXiv 1511.04258

$$< y^k >_{\beta,a,b,n} = \frac{1}{n\beta^2} \int \prod_{i=1}^{k} \frac{du_i}{2i\pi} \prod_{1\le i<j\le k} \frac{u_j - u_i}{(u_j - u_i + \beta^2)(u_j - u_i + 1)}$$

$$\times \prod_{1\le i+1<j\le k} (u_j - u_i + 1 + \beta^2) \prod_{i=1}^{k} \frac{u_i + \beta^2}{u_i + \beta^2(1-n)} \times \frac{u_i - 1 - a}{u_i - 2 - a - b - \beta^2(1-n)}$$

$|u_1| \ll |u_2| \ll .. \ll |u_k|$ contours enclose do not enclose

$$< y >_{\beta,a,b,n} = \frac{1}{n\beta^2} \int \frac{du_1}{2i\pi} \frac{u_1 + \beta^2}{u_1 + \beta^2(1-n)} \frac{u_1 - 1 - a}{u_1 - 2 - a - b - \beta^2(1-n)}$$

residue at $u_1 = -(1-n)\beta^2$

$$= \frac{1 + a - \beta^2(n-1)}{2 + a + b - 2\beta^2(n-1)}$$

Moments for other ensembles

In Section 5 of Y. V. Fyodorov, PLD, arXiv 1511.04258

- Laguerre-Wishart ensemble GFF on positive axis + edge charge at 0

$b \to +\infty$

- Gaussian ensemble GFF on the real axis + gaussian weight

$a = b \to +\infty$ Burgers equation with 1/x^2 random initial velocity

- inverse Jacobi y -> 1/y

- **Lift from interval to circle:** ⟶

Moments for the circular ensemble (CUE) with weights

GFF on circle + background potential

$$\frac{1}{\mathcal{Z}_n^C} \prod_{i=1}^{n} \frac{d\theta_i}{2\pi} |1 + e^{i\theta_i}|^{2\mu} \prod_{1 \le i < j \le n} |e^{i\theta_i} - e^{i\theta_j}|^{2\kappa}$$

$$< \cos(k\theta) >_{\text{circular}} = (-1)^k < y^k >_{\kappa,a,b,n} \Big|_{a=-\mu-1-\kappa(n-1),\, b=2\mu}$$

- also: Cauchy ensemble

Exact solutions from high-temperature

discrete model $Z_M = \sum_{j=1}^{M} e^{-\beta V_i}$ $C_{jk} = \overline{V_j V_k}$

$\beta = 1/T$ $C_{jj} = 2\ln M + W$ $W \geq 0$

- CIRCLE (circular log-REM) $C_{j\neq k} = -2\ln|e^{i\theta_j} - e^{i\theta_k}|$ $\theta_j = \frac{2\pi j}{M}$

$M \to \infty$ periodic 1/f noise in $[0, 2\pi[$

$$V_l = \sqrt{\frac{2}{M}} \sum_{k=1}^{M/2} \sqrt{\lambda_k} \left[x_k \cos\{\frac{2\pi}{M}kl\} + y_k \sin\{\frac{2\pi}{M}kl\} \right]$$

- INTERVAL $C_{j\neq k} = -2\ln|x_j - x_k|$ $x_j = \frac{j}{M} \in [0,1]$

$$\overline{Z_M^n} \simeq M^{n(1+\beta^2)} I_n(\beta) \qquad n\beta^2 < 1$$

$$I_n(\beta) = \prod_{i=1}^{n} \int_0^{2\pi} \frac{d\theta_i}{2\pi} \prod_{j<k} |e^{i\theta_j} - e^{i\theta_k}|^{-2\beta^2} = \frac{\Gamma(1-n\beta^2)}{\Gamma(1-\beta^2)^n}$$ Dyson

$$I_n(\beta) = \prod_{i=1}^{n} \int_0^{1} dx_i \prod_{j<k} \frac{1}{|x_j - x_k|^{2\beta^2}} = \prod_{j=1}^{n} \frac{\Gamma(1-(j-1)\beta^2)^2 \Gamma(1-j\beta^2)}{\Gamma(2-(n+j-2)\beta^2)\Gamma(1-\beta^2)}$$ Selberg

convergent for $n\beta^2 < 1$

Conclusion

- **predictions for Log-CF from duality-freezing conjecture**

  high precision numerical tests

  what is role of duality? integrability?

- **get PDF of position of maximum from moments?**

  study of high moments

- **what is information in n dependence?**

  moments of argmax conditioned to value of max

  joint PDF max/argmax

Appendix H of
Y. V. Fyodorov, PLD, arXiv 1511.04258

GUE-CP $\overline{(y_m^2 - \overline{y_m^2})(V_m - \overline{V_m})} = \frac{9}{686}$

- **Black sheep:** fBm0

  problem in PDF of V_m ,moments seem fine