Extreme Value Statistics in Stochastic Processes

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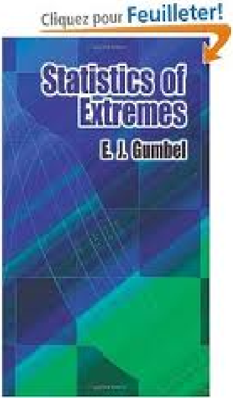
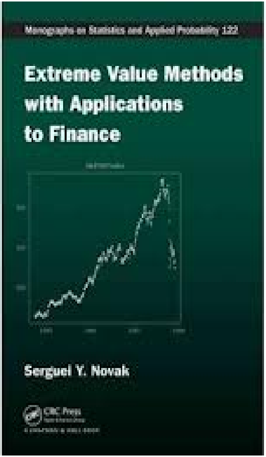
Extreme Events: rare but devastating

Applications

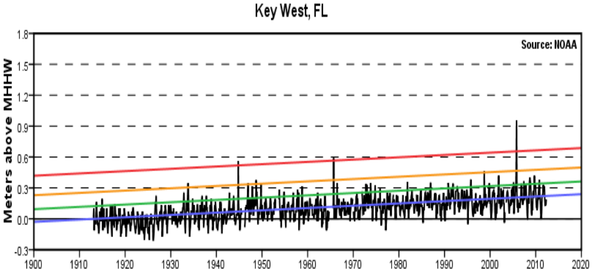


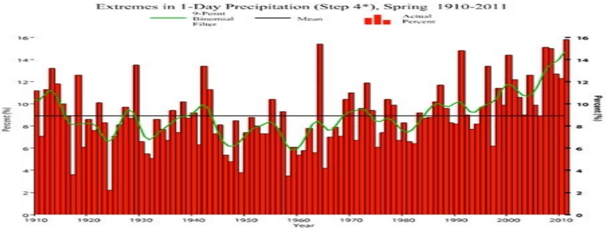


Climate studies, finance and economics, hydrology, sports,....

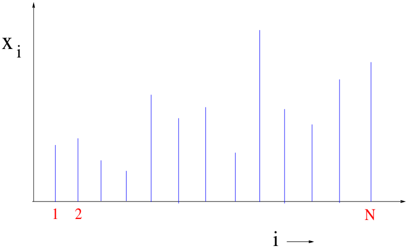
Random walks, disordered systems, random matrices, number theory, ...

Average vs. Extreme





General setting:



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$$P(x_1, x_2, \ldots, x_N)$$

independent or correlated

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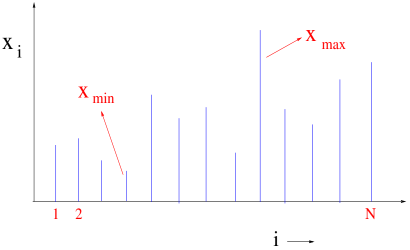

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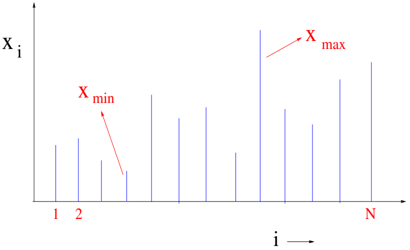

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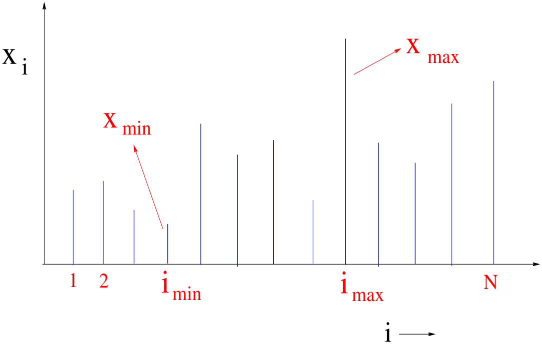
Q: Given $P(x_1, x_2, \ldots, x_N)$, what can we say about the statistics of $x_{\max}$ and $x_{\min}$?

Other extreme observables: times at which extremes occur



Q: Given $P(x_1, x_2, \ldots, x_N)$, statistics of the times $i_{\max}$ and $i_{\min}$?

Other extreme observables: Order and Gap statistics

Order the random variables: $\{x_1, x_2, \ldots, x_N\} \Rightarrow \{y_1, y_2, \ldots, y_N\}$

such that: $\{y_1 > y_2 > y_3 \ldots > y_N\}$

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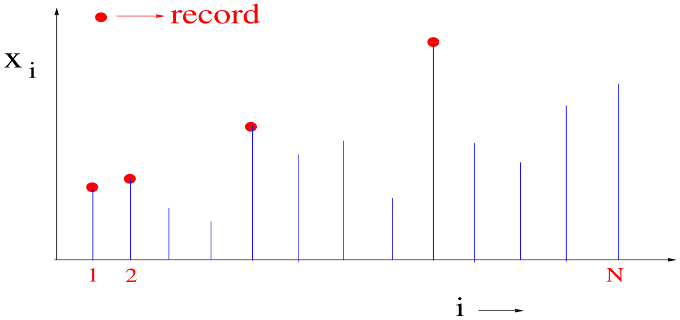
Gap statistics:

$g_k = y_k - y_{k+1} \Rightarrow$ gap between the k-th and (k+1)-th maximum

Other extreme observables: Record statistics



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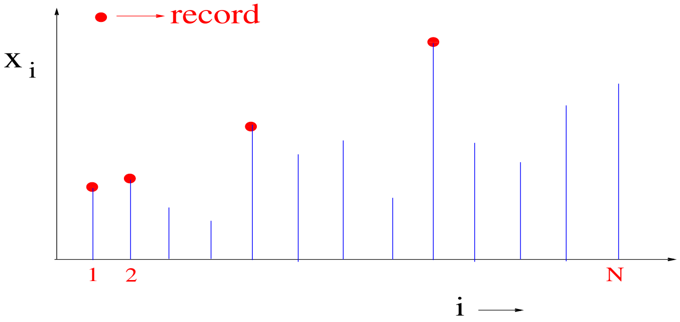

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$$\boxed{x_k > \{x_1, x_2, \ldots, x_{k-1}\}}$$

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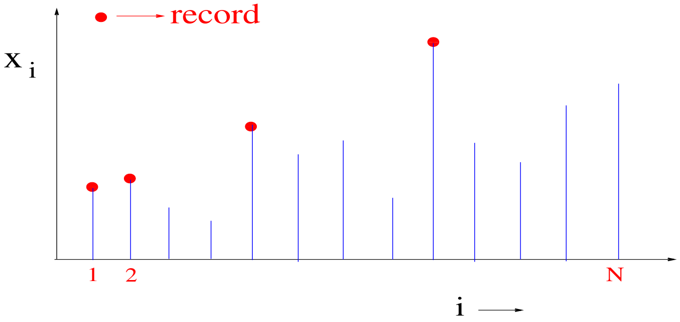

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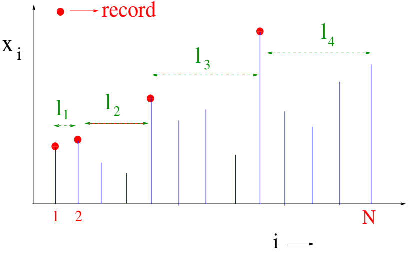
$$\boxed{x_k > \{x_1, x_2, \ldots, x_{k-1}\}}$$

or equivalently: $x_k > \max\left[x_1, x_2, \ldots, x_{k-1}\right]$

Record statistics for a random sequence



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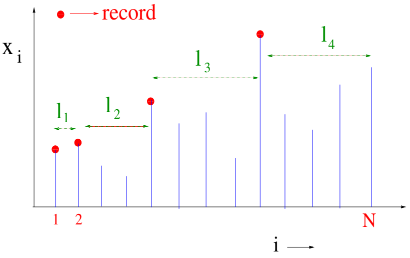

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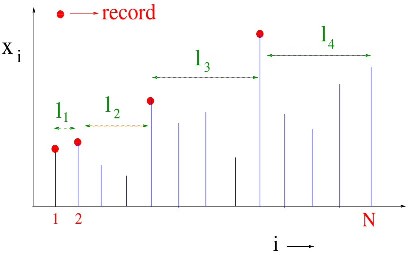

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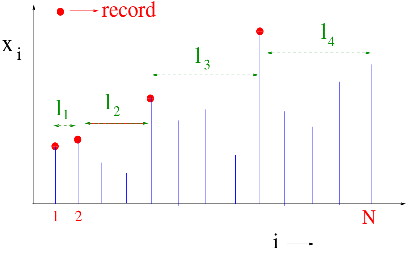

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- How long does a record survive? Ages of records: $\{l_1, l_2, \ldots, l_{R_N}\}$
- Age of the longest (shortest) record?

$$l_{\max} = \max\left[l_1, l_2, \ldots, l_{R_N}\right]$$

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Extreme statistics of i.i.d random variables

A particularly simple case is when

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In this case, several extreme observables can be computed explicitly:

- limiting laws (large N) of $x_{\max}$ and $x_{\min}$ (Fréchet, Gumbel and Weibull)
- statistics of $i_{\max}$ and $i_{\min}$
- Order and Gap statistics
- Record statistics
- ...

For a recent review, see G. Schehr and S.M., arXiv:1305.0639

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Scaling limit: N large, x large: $\boxed{Q_N(x) \to F\left[(x - a_N)/b_N\right]}$

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Several applications $\Longrightarrow$ Climate, Finance, Oceanography, Disordered Systems (Random Energy Model of Derrida),.....

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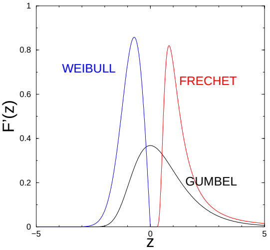

Extreme statistics of correlated variables

In many situations, however, the underlying random variables

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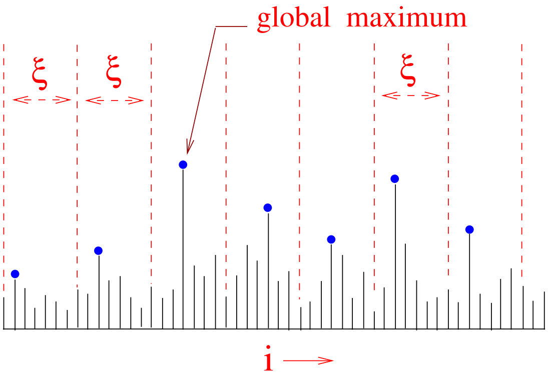
Extreme statistics of correlated variables $\Rightarrow$ nontrivial

Extreme statistics in weakly correlated systems

Weakly correlated variables $\{x_1, x_2, \ldots, x_N\}$
$\rightarrow$ finite correlation length $\xi << N$



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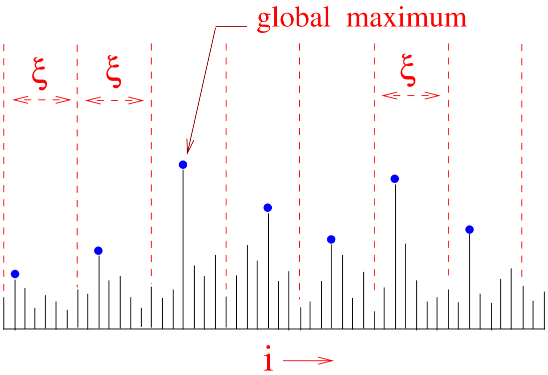

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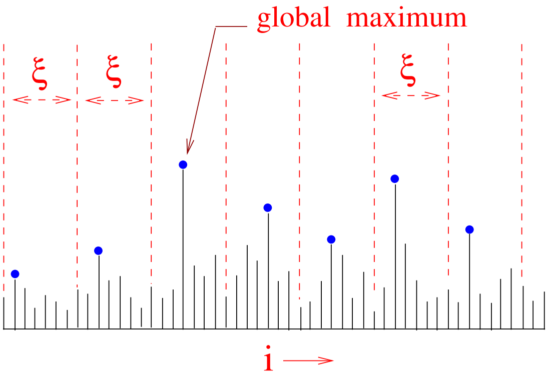

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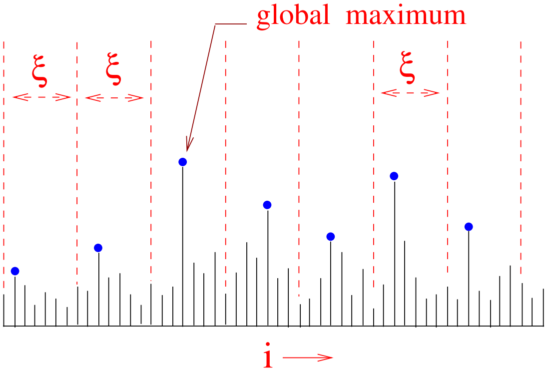

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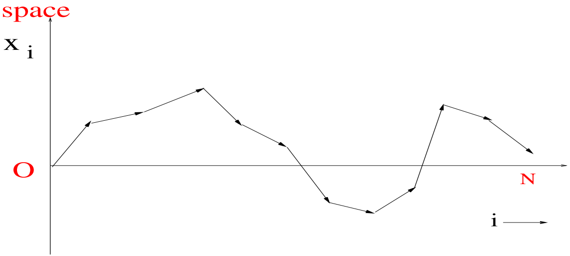
Example: Random Walks/Lévy flights in 1-d



Discrete-time random walk on a line:

$$\boxed{x_i = x_{i-1} + \eta_i, \quad x_0 = 0}$$

$\eta_i \rightarrow$ i.i.d jump lengths, each drawn from a symmetric $p(\eta)$

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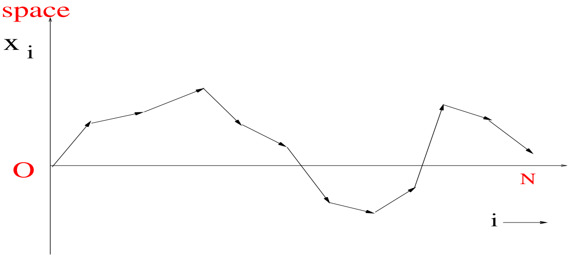

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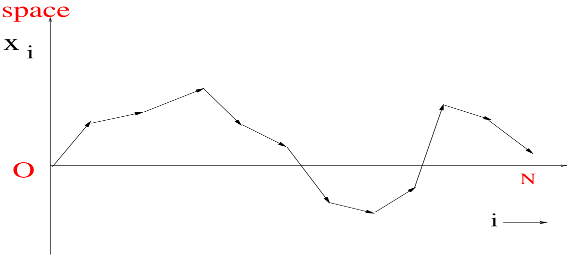

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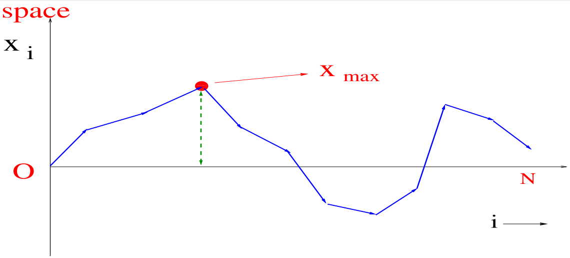
Even though the increments η_i's are uncorrelated, the position x_i's are strongly correlated

Extreme statistics of Random walks/Lévy flights



Applications: fluctuating interfaces, disordered systems (Sinai model), finance, ecology,....

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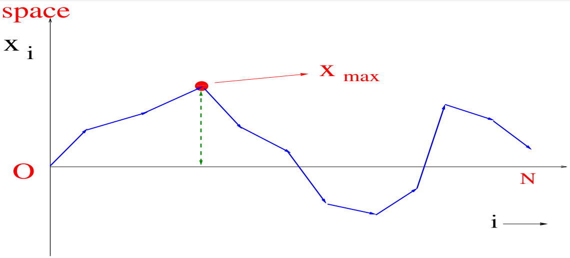

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Extreme statistics and related questions: Record statistics, Order and Gap statistics, statistics of maximal relative height, Convex hulls in 2-d, etc. have been studied recently with many nontrivial results

[Comtet, Dumonteil, Godrèche, Kearney, Mounaix, Rosso, Randon-Furling, Sabhapandit, Schehr, Wergen, Ziff, Zoia...+ S.M.]

for a recent review see, G. Schehr and S.M., arXiv:1305.0639

Extreme statistics in physics of disordered systems

Extreme statistics is also important in disordered systems: spin glasses, polymers in disordered medium, combinatorial optimization, ...

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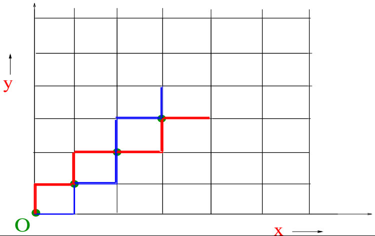
- Example: directed polymer in a random medium

At each site i, $\epsilon_i \rightarrow$ quenched energy (each drawn independently from a distribution $\rho(\epsilon)$) $\Longrightarrow$ defines the random medium

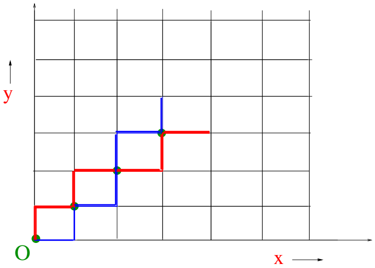
Consider now n-step directed (up-right) polymer paths starting from O



$$E_{\text{path-blue}} = \sum_{j \in \text{path-blue}} \epsilon_j$$

$$E_{\text{path-red}} = \sum_{j \in \text{path-red}} \epsilon_j$$

Directed polymer in a disordered medium



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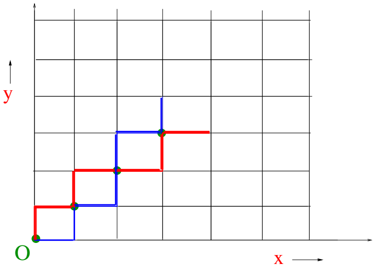
For fixed disorder ϵ_i's: $E_{\rm path} = \sum_{j\in {\rm path}} \epsilon_j$

Ground state energy: $E_0 = \min_{\rm paths} \left[E_{\rm path-1}, E_{\rm path-2}, ..., E_{\rm path-2^n}\right]$

Directed polymer in a disordered medium



$$E_{\rm path-blue} = \sum_{j \in {\rm path-blue}} \epsilon_j$$

$$E_{\rm path-red} = \sum_{j \in {\rm path-red}} \epsilon_j$$

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Clearly, energies of different polymer paths are strongly correlated as they share some common ϵ_i's (sites common to both paths)

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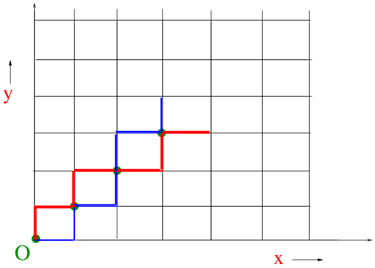

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Baik, Comets, Corwin, Deift, Johansson, Johnstone, Péché, Quastel, Rains,....

Brunet, Calabrese, Derrida, Dotsenko, Le Doussal, Rambeau, Rosso, Sasamoto, Schehr, Spohn,....

Extreme statistics in correlated systems

For correlated $\{x_1, x_2, \ldots, x_N\}$ variables: $P(x_1, x_2, \ldots, x_N) \neq \prod_{i=1}^{N} p(x_i)$

Extreme statistics $\rightarrow$ nontrivial

$\Longrightarrow$ few exact results

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Some examples that will be discussed today:

- Logarithmically correlated Gaussian random variables and Liouville field theory

  talks by Keating, Le Doussal, Rosso

- Extreme statistics in Random Matrices

  talks by Keating, Péché

- Branching Brownian motion in 1-d

  talks by Derrida, Schehr, Shi