Extremes and Order Statistics of $1d$ Branching Brownian Motion

Grégory Schehr

Laboratoire de Physique Théorique et Modèles Statistiques,CNRS,
Université Paris-Sud, France

Inhomogeneous random systems, IHP, January 24-25 2017

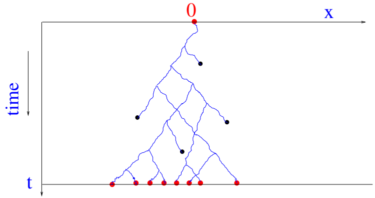
Collaborators: S. N. Majumdar (Orsay) & K. Ramola (Brandeis)

Refs: Phys. Rev. Lett. 112, 210602 (2014)

Longer version: Chaos, Solitons and Fractals 74, 79 (2015)

Span distribution: Phys. Rev. E 91, 042131 (2015)

Branching Brownian Motion with Death



Dynamics: In a small time interval dt, a particle

- branches into 2 with proba. $b\, dt$
- dies with proba. $d\, dt$
- diffuses with proba. $1 - (b + d)\, dt$

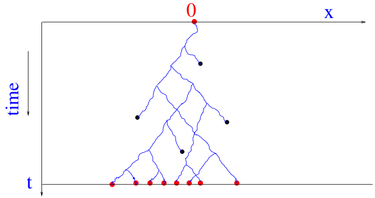
$$x(t + dt) = x(t) + \eta(t)\, dt$$

where $\langle \eta(t) \rangle = 0$

$$\langle \eta(t)\eta(t') \rangle = 2\, D\, \delta(t - t')$$

b, d and D $\Rightarrow$ 3 parameters of the model

Branching Brownian Motion with Death



Dynamics: In a small time interval dt, a particle

- branches into 2 with proba. $b\,dt$
- dies with proba. $d\,dt$
- diffuses with proba. $1-(b+d)\,dt$

$$x(t+dt) = x(t) + \eta(t)\,dt$$

where $\langle \eta(t) \rangle = 0$

$$\langle \eta(t)\eta(t') \rangle = 2\,D\,\delta(t-t')$$

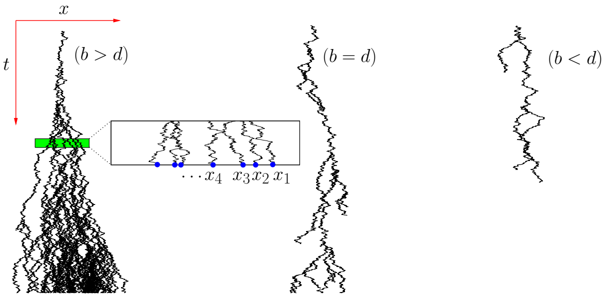
b, d and $D \Rightarrow$ 3 parameters of the model

BBM $\Rightarrow$ prototype model

- Evolutionary system (biology)
- Genealogy
- Cascade model (nuclear physics)
- Directed polymer (statistical physics)
- Epidemic spread, ...etc.

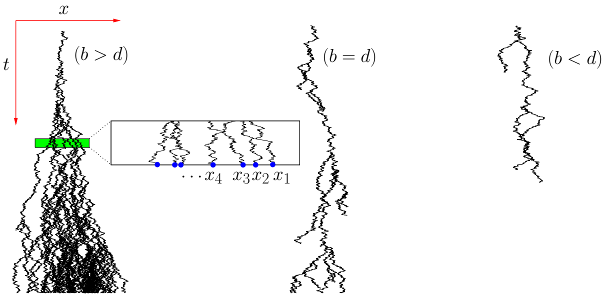
Motivation: extreme and gap statistics

K. Ramola, S. N. Majumdar, G. S., PRL 2014



Motivation: extreme and gap statistics

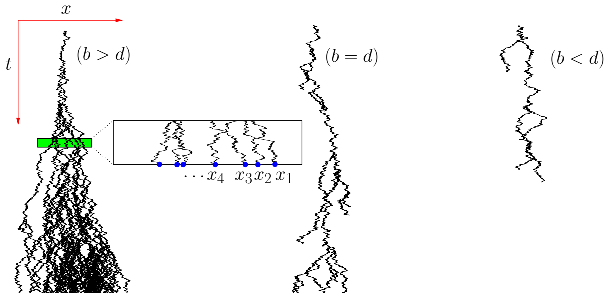
K. Ramola, S. N. Majumdar, G. S., PRL 2014



Q: extreme and gap statistics of BBM ?

Motivation: extreme and gap statistics

K. Ramola, S. N. Majumdar, G. S., PRL 2014

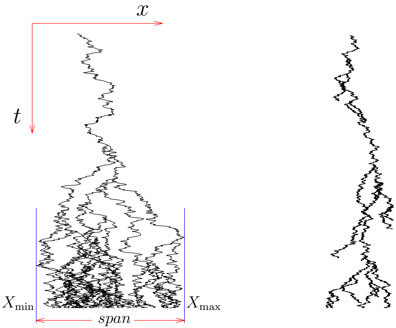


Q: extreme and gap statistics of BBM ?

$\Longrightarrow$ Difficult problem because $x_1(t) > x_2(t) > x_3(t) > \cdots$ are strongly correlated

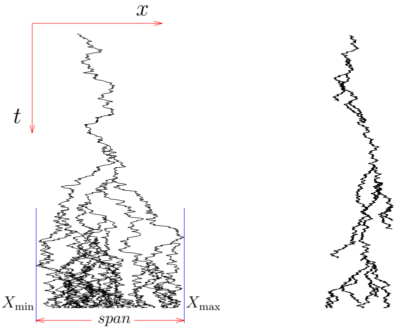
Application of extreme statistics: the span of BBM

K. Ramola, S. N. Majumdar, G. S., PRE 2015



Application of extreme statistics: the span of BBM

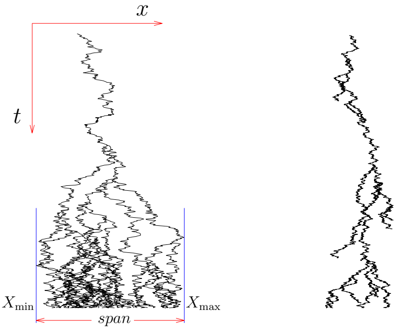
K. Ramola, S. N. Majumdar, G. S., PRE 2015



$\Longrightarrow$ Application to the spatial extent of epidemic spreads

Application of extreme statistics: the span of BBM

K. Ramola, S. N. Majumdar, G. S., PRE 2015



$\Longrightarrow$ Application to the spatial extent of epidemic spreads

see also A. Kundu, S. N. Majumdar, G. S., PRL 2013 (N independent BMs)

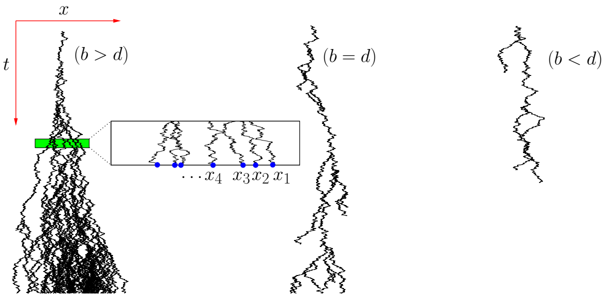
Outline

1. Fluctuations of the particle number
2. Order and gap statistics of BBM with no death (a reminder)
3. Extreme and order statistics with death

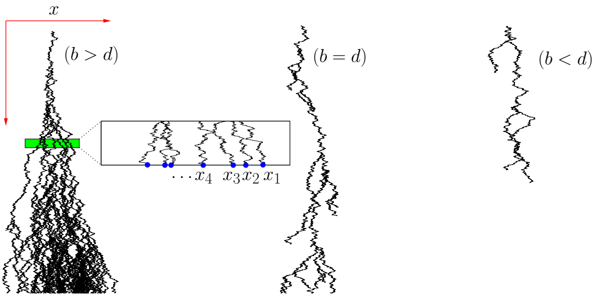
Outline

1. Fluctuations of the particle number
2. Order and gap statistics of BBM with no death (a reminder)
3. Extreme and order statistics with death

Evolution of Population Size

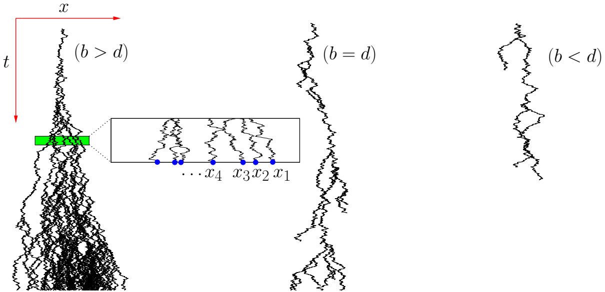


Evolution of Population Size



population size $n(t) \Rightarrow$ random variable

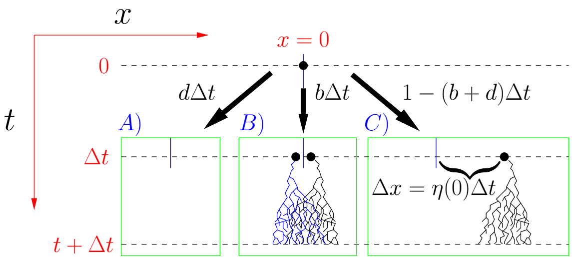
Evolution of Population Size



population size $n(t)$ ⇒ random variable

$P(n, t) = \text{Prob.}[n(t) = n]$ statisfies a backward equation

Evolution of Population Size: backward approach

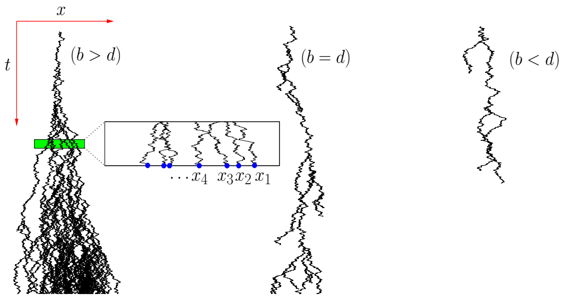


$$P(n,t+\Delta t) = d\Delta t \delta_{n,0} + b\Delta t \sum_{n'=0}^{n} P(n',t)P(n-n',t) + (1-(b+d)\Delta t)P(n,t)$$

$$\frac{dP(n,t)}{dt} = -(b+d)P(n,t) + d\,\delta_{n,0} + b\sum_{n'=0}^{n} P(n',t)P(n-n',t)$$

Exact solution via generating function: $\tilde{P}(z,t) = \sum_{n=0}^{\infty} z^n P(n,t)$

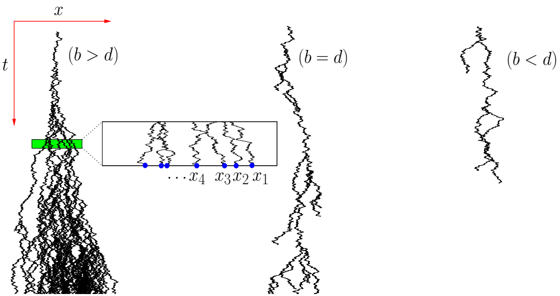
Phase transition at $b = d$



$$P(n,t) = (b-d)^2\, e^{(b+d)t}\, \frac{\left(be^{bt} - be^{dt}\right)^{n-1}}{\left(be^{bt} - de^{dt}\right)^{n+1}} \quad \text{for } n \geq 1$$

$$= d\, \frac{\left(e^{bt} - e^{dt}\right)}{\left(be^{bt} - de^{dt}\right)} \quad \text{for } n = 0$$

Phase transition at $b = d$



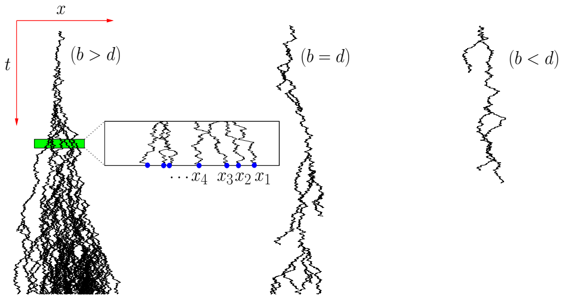
$$P(n,t) = (b-d)^2\, e^{(b+d)t}\, \frac{\left(be^{bt} - be^{dt}\right)^{n-1}}{\left(be^{bt} - de^{dt}\right)^{n+1}} \quad \text{for } n \geq 1$$

$$= d\, \frac{\left(e^{bt} - e^{dt}\right)}{\left(be^{bt} - de^{dt}\right)} \quad \text{for } n = 0$$

Consequently:

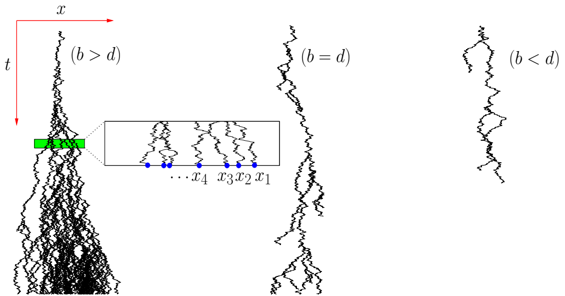
$$\langle n(t) \rangle = e^{(b-d)t} \underset{t\to\infty}{\longrightarrow} \begin{cases} +\infty\ , & b > d \quad \text{supercritical} \\ 1\ , & b = d \quad \text{critical} \\ 0\ , & b < d \quad \text{subcritical} \end{cases}$$

Typical vs. Average Population Size



- Supercritical phase $(b > d)$: $P(n,t) \sim \exp\left[-\frac{n}{n^*(t)}\right]$

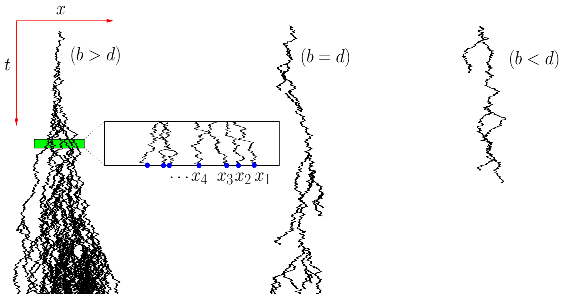
Typical vs. Average Population Size



- Supercritical phase $(b > d)$: $P(n,t) \sim \exp\left[-\frac{n}{n^*(t)}\right]$

  Typical size $n^*(t) \sim e^{(b-d)\,t}$ and Average size $\langle n(t) \rangle = e^{(b-d)\,t}$

Typical vs. Average Population Size

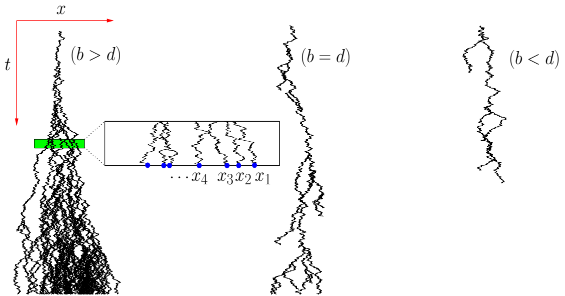


- Supercritical phase $(b > d)$: $P(n,t) \sim \exp\left[-\frac{n}{n^*(t)}\right]$

  Typical size $n^*(t) \sim e^{(b-d)\,t}$ and Average size $\langle n(t)\rangle = e^{(b-d)\,t}$

- Critical point $(b = d)$: $P(n,t) \sim \exp\left[-\frac{n}{n^*(t)}\right]$

Typical vs. Average Population Size



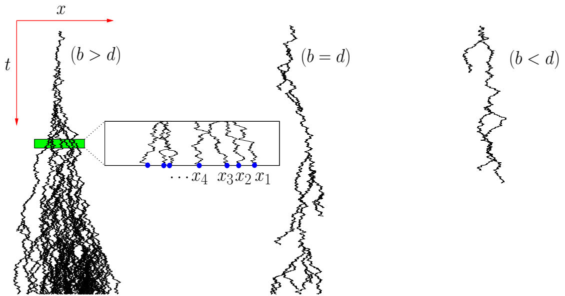
- Supercritical phase $(b > d)$: $P(n,t) \sim \exp\left[-\frac{n}{n^*(t)}\right]$

  Typical size $n^*(t) \sim e^{(b-d)\,t}$ and Average size $\langle n(t)\rangle = e^{(b-d)\,t}$

- Critical point $(b = d)$: $P(n,t) \sim \exp\left[-\frac{n}{n^*(t)}\right]$

  Typical size $n^*(t) \sim b\,t$ while Average size $\langle n(t)\rangle = 1$

Typical vs. Average Population Size



- Supercritical phase $(b > d)$: $P(n,t) \sim \exp\left[-\frac{n}{n^*(t)}\right]$

  Typical size $n^*(t) \sim e^{(b-d)\,t}$ and Average size $\langle n(t)\rangle = e^{(b-d)\,t}$

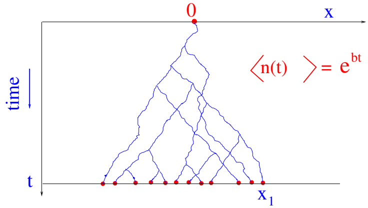
- Critical point $(b = d)$: $P(n,t) \sim \exp\left[-\frac{n}{n^*(t)}\right]$

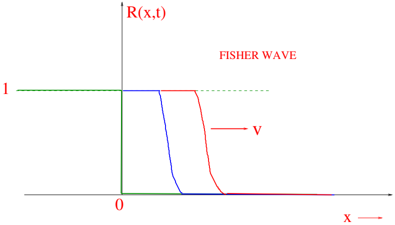
  Typical size $n^*(t) \sim b\,t$ while Average size $\langle n(t)\rangle = 1$

  $\Rightarrow$ Strong fluctuations at the critical point

Outline

Extremal statistics with no death $d = 0$





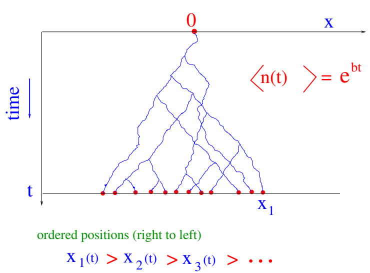
ordered positions (right to left)

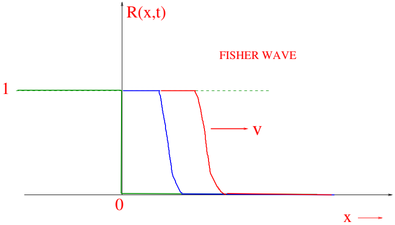
$x_1(t) > x_2(t) > x_3(t) > \cdots$

$R(x,t) = \text{Prob.}[x_1(t) > x]$ satisfies Fisher-KPP equation:

$$\frac{\partial R}{\partial t} = D\,\frac{\partial^2 R}{\partial x^2} + b\,R - b\,R^2 \quad \text{starting from } R(x,0) = \theta(-x)$$

Extremal statistics with no death $d = 0$





$R(x,t) = \text{Prob.}[x_1(t) > x]$ satisfies Fisher-KPP equation:

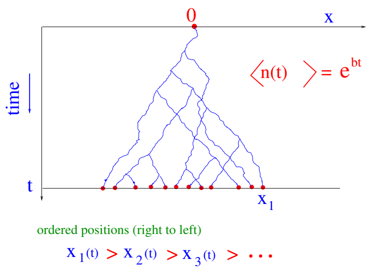
$$\frac{\partial R}{\partial t} = D\,\frac{\partial^2 R}{\partial x^2} + b\,R - b\,R^2 \quad \text{starting from } R(x,0) = \theta(-x)$$

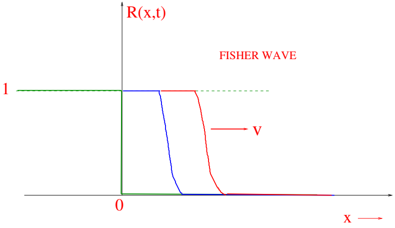
travelling front solution: $R(x,t) \to f(x - vt)$ with speed $v = 2\sqrt{b\,D}$

$$\boxed{x_1(t) \to 2\sqrt{b\,D}\;t + O(\ln t)} \quad \text{at late times}$$

Mckean '75, Bramson '78, Lalley & Selke '87, Kessler et. al. '97,..,Brunet & Derrida '09,...

Extremal statistics with no death $d = 0$





$R(x,t) = \text{Prob.}[x_1(t) > x]$ satisfies Fisher-KPP equation:

$\frac{\partial R}{\partial t} = D\,\frac{\partial^2 R}{\partial x^2} + b\,R - b\,R^2$ starting from $R(x,0) = \theta(-x)$

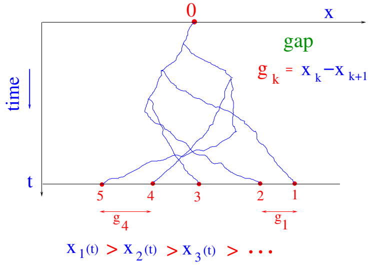
travelling front solution: $R(x,t) \to f(x - vt)$ with speed $v = 2\sqrt{b\,D}$

$$x_1(t) \to 2\sqrt{b\,D}\;t + O(\ln t)$$ at late times

Mckean '75, Bramson '78, Lalley & Selke '87, Kessler et. al. '97,..,Brunet & Derrida '09,...

see also B. Derrida's talk for large deviations

Order and Gap Statistics for $d = 0$



Brunet & Derrida, 2009-2010

$$x_k(t) \xrightarrow[t\to\infty]{} 2\sqrt{b\,D}\,t$$

Gap: $g_k(t) = x_k(t) - x_{k+1}(t)$

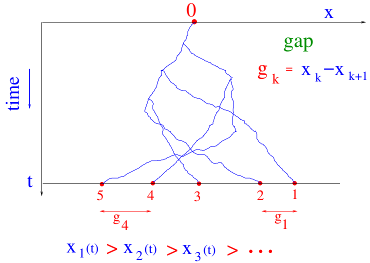
$\longrightarrow$ stationary random variable as $t \to \infty$

In units of $\sqrt{D/b}$:

$\langle g_1 \rangle \approx 0.496$, $\langle g_2 \rangle \approx 0.303$, ...

$$\langle g_k \rangle \xrightarrow[k\to\infty]{} \frac{1}{k} - \frac{1}{k\ln k} + \ldots$$

Order and Gap Statistics for $d = 0$



Brunet & Derrida, 2009-2010

$$x_k(t) \xrightarrow[t\to\infty]{} 2\sqrt{b\,D}\,t$$

Gap: $g_k(t) = x_k(t) - x_{k+1}(t)$

$\longrightarrow$ stationary random variable as $t \to \infty$

In units of $\sqrt{D/b}$:

$\langle g_1 \rangle \approx 0.496$, $\langle g_2 \rangle \approx 0.303$, ...

$$\langle g_k \rangle \xrightarrow[k\to\infty]{} \frac{1}{k} - \frac{1}{k \ln k} + \ldots$$

Heuristic argument for the first gap distribution (Brunet & Derrida, '10):

$$P(g_1, t \to \infty) \approx \exp\left[-(1+\sqrt{2})\sqrt{b/D}\, g_1\right]$$

Order and Gap Statistics for $d > 0 \rightarrow$ this talk

- Question: What happens to order & gap statistics when the death rate $d > 0$ is switched on ? In particular, at the critical point $d = b$

Order and Gap Statistics for $d > 0$ → this talk

- Question: What happens to order & gap statistics when the death rate $d > 0$ is switched on ? In particular, at the critical point $d = b$

- Recall: $\text{Prob.}[n(t) = 0] = P(0, t) = d\, \frac{(e^{bt} - e^{dt})}{(be^{b\,t} - de^{dt})}$

Order and Gap Statistics for $d > 0 \rightarrow$ this talk

- Question: What happens to order & gap statistics when the death rate $d > 0$ is switched on ? In particular, at the critical point $d = b$

- Recall: $\text{Prob.}[n(t) = 0] = P(0,t) = d\,\frac{(e^{bt} - e^{dt})}{(be^{b\,t} - de^{dt})}$

  Consequently, for $d > 0$, with a nonzero proba. there may not be any particle at any given time t!

Order and Gap Statistics for $d > 0 \rightarrow$ this talk

- Question: What happens to order & gap statistics when the death rate $d > 0$ is switched on ? In particular, at the critical point $d = b$

- Recall: $\text{Prob.}[n(t) = 0] = P(0, t) = d \, \frac{(e^{bt} - e^{dt})}{(be^{bt} - de^{dt})}$

  Consequently, for $d > 0$, with a nonzero proba. there may not be any particle at any given time t!

- Naturally leads one to study the Conditioned Ensemble where the system is conditioned to have a fixed n particles at time t.

Order and Gap Statistics for $d > 0 \rightarrow$ this talk

- Question: What happens to order & gap statistics when the death rate $d > 0$ is switched on ? In particular, at the critical point $d = b$

- Recall: $\text{Prob.}[n(t) = 0] = P(0, t) = d \, \frac{(e^{bt} - e^{dt})}{(be^{b\,t} - de^{dt})}$

  Consequently, for $d > 0$, with a nonzero proba. there may not be any particle at any given time t!

- Naturally leads one to study the Conditioned Ensemble where the system is conditioned to have a fixed n particles at time t.

- Prob. distr. of any observable $\hat{O}$ in the full problem

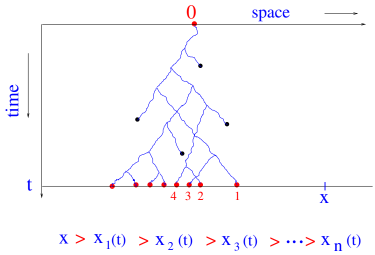
$$P(\hat{O}, t) = \frac{\sum_{n>0} Q(\hat{O}, t|n) \, P(n,t)}{\sum_{n>0} P(n,t)} \approx Q(\hat{O}, t|n^*(t))$$

where $Q(\hat{O}, t|n) \longrightarrow$ proba. of $\hat{O}$ in the Conditioned Ensemble

$n^*(t) \longrightarrow$ typical population size

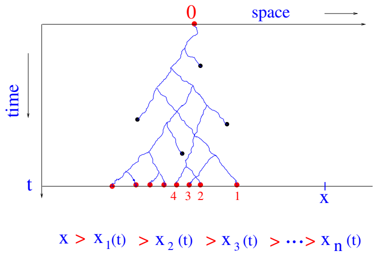
Outline

Extremal statistics for $d > 0$ with fixed nber of part.



$C(n, x, t) \to$ proba. of having n particles at t with all of them to the left of x

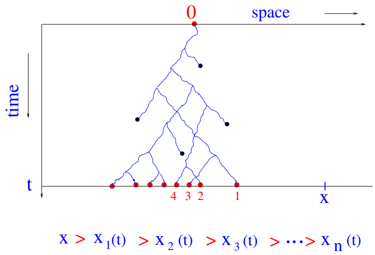
Extremal statistics for $d > 0$ with fixed nber of part.



$C(n, x, t) \rightarrow$ proba. of having n particles at t with all of them to the left of x

$P(n, t) \rightarrow$ Proba. of having exactly n particles at t $\rightarrow$ known exactly

Extremal statistics for $d > 0$ with fixed nber of part.

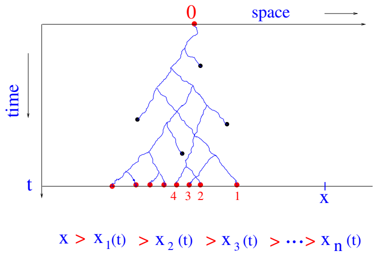


$C(n, x, t) \to$ proba. of having n particles at t with all of them to the left of x

$P(n, t) \to$ Proba. of having exactly n particles at $t \to$ known exactly

Conditional prob.
$Q(x, t|n) = \frac{C(n,x,t)}{P(n,t)} \to$ given that there are n particles at t, the prob. that all of them are to the left of x

Extremal statistics for $d > 0$ with fixed nber of part.



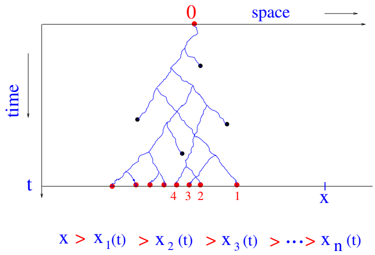
$C(n, x, t) \to$ proba. of having n particles at t with all of them to the left of x

$P(n, t) \to$ Proba. of having exactly n particles at $t \to$ known exactly

Conditional prob.
$Q(x, t|n) = \frac{C(n,x,t)}{P(n,t)} \to$ given that there are n particles at t, the prob. that all of them are to the left of x

Exact backward evolution equation for $C(n, x, t)$ + known $P(n, t) \longrightarrow$

Extremal statistics for $d > 0$ with fixed nber of part.



$C(n,x,t) \rightarrow$ proba. of having n particles at t with all of them to the left of x

$P(n,t) \rightarrow$ Proba. of having exactly n particles at $t \rightarrow$ known exactly

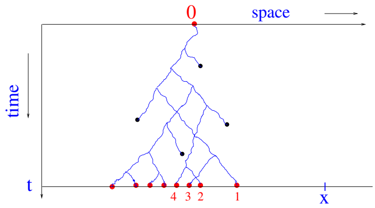
Conditional prob.
$Q(x,t|n) = \frac{C(n,x,t)}{P(n,t)} \rightarrow$ given that there are n particles at t, the prob. that all of them are to the left of x

Exact backward evolution equation for $C(n,x,t)$ + known $P(n,t) \longrightarrow$

$$\frac{\partial Q(x,t|n)}{\partial t} = D\,\frac{\partial^2 Q(x,t|n)}{\partial x^2} + a(t)\sum_{r=1}^{n-1}\left[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)\right]$$

where $a(t) = \frac{(b-d)^2\, e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})}$ and one starts with $Q(x,t|0) = 1$

Extremal statistics for $d > 0$ with fixed nber of part.



$x > x_1(t) > x_2(t) > x_3(t) > \cdots > x_n(t)$

$C(n, x, t) \to$ proba. of having n particles at t with all of them to the left of x

$P(n, t) \to$ Proba. of having exactly n particles at $t \to$ known exactly

Conditional prob.
$Q(x, t|n) = \frac{C(n,x,t)}{P(n,t)} \to$ given that there are n particles at t, the prob. that all of them are to the left of x

Exact backward evolution equation for $C(n, x, t)$ + known $P(n, t) \longrightarrow$

$$\frac{\partial Q(x, t|n)}{\partial t} = D \frac{\partial^2 Q(x, t|n)}{\partial x^2} + a(t) \sum_{r=1}^{n-1} [Q(x, t|r)Q(x, t|n-r) - Q(x, t|n)]$$

where $a(t) = \frac{(b-d)^2 e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})}$ and one starts with $Q(x, t|0) = 1$

$\Rightarrow$ linear in $Q(x, t|n)$ and can be solved recursively

[K. Ramola, S. N. Majumdar & G. S., PRL, 112, 210602 (2014)]

Order Statistics at the Critical Point $b = d$

$$\frac{\partial Q(x,t|n)}{\partial t} = D\,\frac{\partial^2 Q(x,t|n)}{\partial x^2} + a(t)\sum_{r=1}^{n-1}\left[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)\right]$$

Order Statistics at the Critical Point $b = d$

$$\frac{\partial Q(x,t|n)}{\partial t} = D\,\frac{\partial^2 Q(x,t|n)}{\partial x^2} + a(t)\sum_{r=1}^{n-1}\left[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)\right]$$

$$a(t) = \frac{(b-d)^2\, e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})} \sim e^{-(b-d)\,t} \quad \text{for } b > d$$

Order Statistics at the Critical Point $b = d$

$$\frac{\partial Q(x,t|n)}{\partial t} = D\,\frac{\partial^2 Q(x,t|n)}{\partial x^2} + a(t)\sum_{r=1}^{n-1}\left[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)\right]$$

$$a(t) = \frac{(b-d)^2\,e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})} \sim e^{-(b-d)\,t} \quad \text{for } b > d$$

$$\sim 1/t^2 \quad \text{for } b = d$$

Order Statistics at the Critical Point $b = d$

$$\frac{\partial Q(x,t|n)}{\partial t} = D\,\frac{\partial^2 Q(x,t|n)}{\partial x^2} + a(t)\sum_{r=1}^{n-1}\left[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)\right]$$

$$a(t) = \frac{(b-d)^2\, e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})} \sim e^{-(b-d)\,t} \quad \text{for } b > d$$

$$\sim 1/t^2 \qquad \text{for } b = d$$

- The source term $\sim a(t)\, n^*(t) \sim O(1)$ for $b > d$, recall $n^*(t) \sim e^{(b-d)\,t}$

  $\sim 1/t$ for $b = d$, recall $n^*(t) \sim b\,t$

Order Statistics at the Critical Point $b = d$

$$\frac{\partial Q(x,t|n)}{\partial t} = D\,\frac{\partial^2 Q(x,t|n)}{\partial x^2} + a(t)\sum_{r=1}^{n-1}\left[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)\right]$$

$$a(t) = \frac{(b-d)^2\, e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})} \sim e^{-(b-d)\,t} \quad \text{for } b > d$$

$$\sim 1/t^2 \qquad \text{for } b = d$$

- The source term $\sim a(t)\, n^*(t) \sim O(1)$ for $b > d$, recall $n^*(t) \sim e^{(b-d)\,t}$

  $\sim 1/t$ for $b = d$, recall $n^*(t) \sim b\,t$

- Consequently, the source term $\to$ negligible at the critical point $b = d$

Order Statistics at the Critical Point $b = d$

$$\frac{\partial Q(x,t|n)}{\partial t} = D\,\frac{\partial^2 Q(x,t|n)}{\partial x^2} + a(t)\sum_{r=1}^{n-1}\left[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)\right]$$

$$a(t) = \frac{(b-d)^2\,e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})} \sim e^{-(b-d)\,t} \quad \text{for } b > d$$

$$\sim 1/t^2 \qquad \text{for } b = d$$

- The source term $\sim a(t)\,n^*(t) \sim O(1)$ for $b > d$, recall $n^*(t) \sim e^{(b-d)\,t}$

  $\sim 1/t$ for $b = d$, recall $n^*(t) \sim b\,t$

- Consequently, the source term $\rightarrow$ negligible at the critical point $b = d$

Exact late time solution at the critical point $b = d$:

$$Q(x,t|n) \rightarrow \frac{1}{2}\,\mathrm{erfc}\left(-\frac{x}{\sqrt{4Dt}}\right) \text{ for all } 1 \le n \le n^*(t) \sim bt$$

Order Statistics at the Critical Point $b = d$

$$\frac{\partial Q(x,t|n)}{\partial t} = D\,\frac{\partial^2 Q(x,t|n)}{\partial x^2} + a(t)\sum_{r=1}^{n-1}\left[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)\right]$$

$$a(t) = \frac{(b-d)^2\, e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})} \sim e^{-(b-d)\,t} \quad \text{for } b > d$$

$$\sim 1/t^2 \qquad \text{for } b = d$$

- The source term $\sim a(t)\,n^*(t) \sim O(1)$ for $b > d$, recall $n^*(t) \sim e^{(b-d)\,t}$

  $\sim 1/t$ for $b = d$, recall $n^*(t) \sim b\,t$

- Consequently, the source term $\to$ negligible at the critical point $b = d$

  Exact late time solution at the critical point $b = d$:

  $$Q(x,t|n) \to \frac{1}{2}\,\mathrm{erfc}\left(-\frac{x}{\sqrt{4Dt}}\right) \text{ for all } 1 \le n \le n^*(t) \sim bt$$

- Rightmost particle diffuses at the critical point $b = d$ irrespective of n

  $$P(x_1,t|n) \to \frac{1}{\sqrt{4\pi Dt}}\,\exp\left[-\frac{x_1^2}{4Dt}\right]$$

Order Statistics at the Critical Point $b = d$

$$\frac{\partial Q(x,t|n)}{\partial t} = D\,\frac{\partial^2 Q(x,t|n)}{\partial x^2} + a(t)\sum_{r=1}^{n-1}\left[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)\right]$$

$$a(t) = \frac{(b-d)^2\, e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})} \sim e^{-(b-d)\,t} \quad \text{for } b > d$$

$$\sim 1/t^2 \qquad \text{for } b = d$$

- The source term $\sim a(t)\,n^*(t) \sim O(1)$ for $b > d$, recall $n^*(t) \sim e^{(b-d)\,t}$

  $\sim 1/t$ for $b = d$, recall $n^*(t) \sim b\,t$

- Consequently, the source term $\to$ negligible at the critical point $b = d$

  Exact late time solution at the critical point $b = d$:

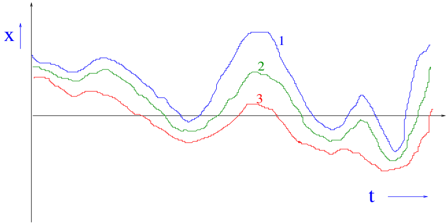
  $$Q(x,t|n) \to \frac{1}{2}\,\mathrm{erfc}\left(-\frac{x}{\sqrt{4Dt}}\right) \text{ for all } 1 \le n \le n^*(t) \sim bt$$

- Rightmost particle diffuses at the critical point $b = d$ irrespective of n

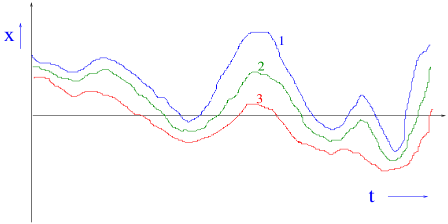
  $$P(x_1,t|n) \to \frac{1}{\sqrt{4\pi Dt}}\,\exp\left[-\frac{x_1^2}{4Dt}\right]$$

- Similarly the second, the third...all diffuse: $P(x_k,t|n) \to \frac{1}{\sqrt{4\pi Dt}}\,\exp\left[-\frac{x_k^2}{4Dt}\right]$

Particles diffuse (as a bunch) when $b = d$

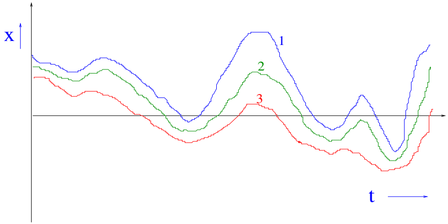


Particles diffuse (as a bunch) when $b = d$



Ordered particles diffuse as a bunch at the critical point $b = d$

Particles diffuse (as a bunch) when $b = d$

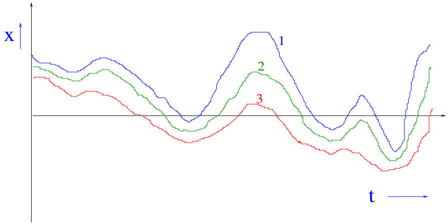


Ordered particles diffuse as a bunch at the critical point $b = d$

While the center of mass $\sim \sqrt{D\,t}$, the gaps $g_k(t) = x_k(t) - x_{k+1}(t)$

$\Longrightarrow$ stationary at late times

Particles diffuse (as a bunch) when $b = d$



Ordered particles diffuse as a bunch at the critical point $b = d$

While the center of mass $\sim \sqrt{D\,t}$, the gaps $g_k(t) = x_k(t) - x_{k+1}(t)$

$\Longrightarrow$ stationary at late times

Exact computation of the gap distribution in the Conditioned Ensemble (fixed n)

Universal gap distribution at the critical point $b = d$

- $n = 2$ sector:

  $\Rightarrow$ Exact solution for the stationary gap distribution

Universal gap distribution at the critical point $b = d$

- $n = 2$ sector:

  $\Rightarrow$ Exact solution for the stationary gap distribution

$$p(g_1|2) = \sqrt{\frac{b}{16D}}\, f\left(\sqrt{\frac{b}{16D}}\, g_1\right)$$

with $\quad f(x) = -4\,x + \sqrt{2\pi}\, e^{2\,x^2}\,(1 + 4\,x^2)\,\mathrm{erfc}\,(\sqrt{2}\,x)$

Universal gap distribution at the critical point $b = d$

- $n = 2$ sector:

  $\Rightarrow$ Exact solution for the stationary gap distribution

$$p(g_1|2) = \sqrt{\frac{b}{16D}}\, f\left(\sqrt{\frac{b}{16D}}\; g_1\right)$$

with $\quad f(x) = -4\,x + \sqrt{2\pi}\, e^{2\,x^2}\,(1 + 4\,x^2)\,\mathrm{erfc}\,(\sqrt{2}\,x)$

$$p(g_1|2) \sim \sqrt{\frac{\pi\, b}{8D}} \qquad \text{as } g_1 \to 0$$

$$\sim \frac{8D}{b}\, g_1^{-3} \qquad \text{as } g_1 \to \infty$$

[K. Ramola, S. N. Majumdar & G. S., PRL **112**, 210602 (2014)]

Universal gap distribution at the critical point $b = d$

- $n = 2$ sector:

  $\Rightarrow$ Exact solution for the stationary gap distribution

$$p(g_1|2) = \sqrt{\frac{b}{16D}}\, f\left(\sqrt{\frac{b}{16D}}\, g_1\right)$$

with $\quad f(x) = -4\,x + \sqrt{2\pi}\, e^{2\,x^2}\,(1 + 4\,x^2)\,\mathrm{erfc}\left(\sqrt{2}\,x\right)$

$$p(g_1|2) \sim \sqrt{\frac{\pi b}{8D}} \qquad \text{as } g_1 \to 0$$

$$\sim \frac{8D}{b}\, g_1^{-3} \qquad \text{as } g_1 \to \infty$$

[K. Ramola, S. N. Majumdar & G. S., PRL **112**, 210602 (2014)]

- $n > 2$ sector: solve by recursion

$$p(g_1|n) \xrightarrow[g_1 \to \infty]{} \frac{8\,D}{b}\, g_1^{-3}$$

Universal gap distribution at the critical point $b = d$

- $n = 2$ sector:

  $\Rightarrow$ Exact solution for the stationary gap distribution

$$p(g_1|2) = \sqrt{\frac{b}{16D}}\, f\left(\sqrt{\frac{b}{16D}}\; g_1\right)$$

with $\quad f(x) = -4\,x + \sqrt{2\pi}\, e^{2\,x^2}\,(1 + 4\,x^2)\,\mathrm{erfc}\left(\sqrt{2}\,x\right)$

$$p(g_1|2) \sim \sqrt{\frac{\pi b}{8D}} \qquad \text{as } g_1 \to 0$$

$$\sim \frac{8D}{b}\, g_1^{-3} \qquad \text{as } g_1 \to \infty$$

[K. Ramola, S. N. Majumdar & G. S., PRL **112**, 210602 (2014)]

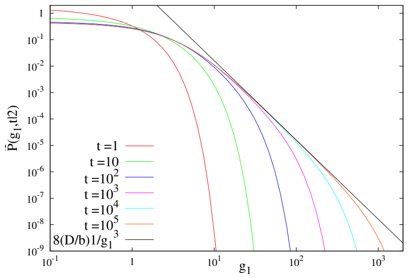
- $n > 2$ sector: solve by recursion

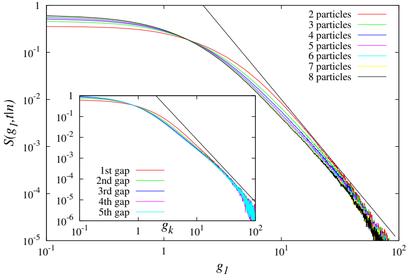
$$p(g_1|n) \xrightarrow[g_1 \to \infty]{} \frac{8\,D}{b}\, g_1^{-3}$$

- Similarly for the k-th gap $\rightarrow$ universal power-law tail for all k and n

$$p(g_k|n) \xrightarrow[g_k \to \infty]{} \frac{8D}{b}\, g_k^{-3}$$

Numerical simulations at the critical point $b = d$





Summary and Conclusion

- Working in the Conditioned Ensemble, we computed the order and the gap statistics for branching Brownian motion with death

Summary and Conclusion

- Working in the Conditioned Ensemble, we computed the order and the gap statistics for branching Brownian motion with death
- At the critical point $b = d$, this provides exact results for the full ensemble

Summary and Conclusion

- Working in the Conditioned Ensemble, we computed the order and the gap statistics for branching Brownian motion with death

- At the critical point $b = d$, this provides exact results for the full ensemble

- At late times gaps become stationary with universal algebraic tail at the critical point $(b = d)$

$$p(g_k) \xrightarrow[g_k \to \infty]{} \frac{8D}{b}\, g_k^{-3}$$

Summary and Conclusion

- Working in the Conditioned Ensemble, we computed the order and the gap statistics for branching Brownian motion with death

- At the critical point $b = d$, this provides exact results for the full ensemble

- At late times gaps become stationary with universal algebraic tail at the critical point $(b = d)$

$$p(g_k) \xrightarrow[g_k \to \infty]{} \frac{8D}{b} \, g_k^{-3}$$

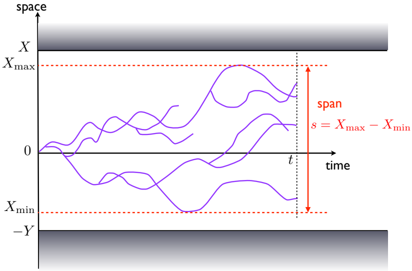
Summary and Conclusion

Other observables at the critical point:

Summary and Conclusion

Other observables at the critical point:

- Joint distribution of the rightmost and leftmost positions of the BBM up to time t

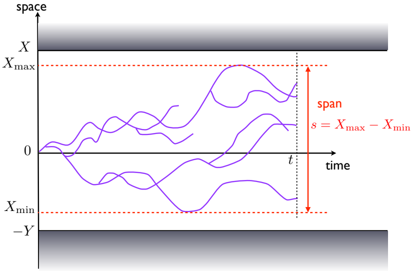


- Full distribution of the span S up to time t as $t \to \infty$

Summary and Conclusion

Other observables at the critical point:

- Joint distribution of the rightmost and leftmost positions of the BBM up to time t



- Full distribution of the span S up to time t as $t \to \infty$

$$p(S) \xrightarrow[S\to\infty]{} (8\,\pi\sqrt{3})\,\frac{D}{b}\,S^{-3}$$

[K. Ramola, S. N. Majumdar & G. S., PRE 2015]