Renormalisation of geometric SPDEs

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IRS, IHP, January 2019

Curve-shortening flow

We have a Riemannian manifold (M, g) embedded in a space $\mathbb{R}^d$.

A classical deterministic problem is the curve-shortening flow:

$$u : [0,T] \times \mathbb{S}^1 \to M, \qquad \partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\, \partial_x u^\beta \partial_x u^\gamma \ ,$$

where Γ denotes the Christoffel symbols for the Levi-Civita connection associated with the metric $g_{\alpha\beta}$ and we use Einstein's convention of summation over repeated indices.

All we need to know:

- $\Gamma^\alpha_{\beta\gamma} : \mathbb{R}^d \to \mathbb{R}$ is smooth,
- $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$,
- Γ can be written explicitly as a function of g.

Noisy version

For $(\xi_i)_i$ a sequence of independent space-time white noises on $[0, T] \times \mathbb{S}^1$, we want to study

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\, \partial_x u^\beta \partial_x u^\gamma + h^\alpha(u) + \sigma_i^\alpha(u)\, \xi_i \ ,$$

where $h^\alpha, \sigma_i^\alpha : \mathbb{R}^d \to \mathbb{R}^d$ ($\alpha = 1, \ldots, d$, $i = 1, \ldots, m$) are suitable vector fields (tangent to the manifold M) such that

$$\sigma_i^\alpha(u)\sigma_i^\beta(u) = g^{\alpha\beta}(u)$$

and $g^{\alpha\beta}$ is the inverse metric tensor.

In the case of $(\xi_i)_i$ a sequence of independent noises which are white in time but coloured in space, this was studied by Funaki [JFA92]. In this setting we have a standard Stratonovich formulation of the stochastic integral and the solution is smooth in x.

Noisy version

It the $(\xi_i)_i$'s are space-time white noises in

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma + h^\alpha(u) + \sigma_i^\alpha(u)\,\xi_i \ ,$$

the usual problems of singular SPDEs arise: the solution u is expected to be no better than Hölder in space, so that the derivative $\partial_x u$ is a distribution and the above product is ill-defined.

This is a generalised KPZ equation.

Regularisation

Let us fix a space-time mollifier ϱ (compactly supported, integrating to 1, and such that $\varrho(t,-x) = \varrho(t,x)$) and let $\varrho_\varepsilon(t,x) = \varepsilon^{-3}\varrho(t/\varepsilon^2, x/\varepsilon)$, $\xi_i^{(\varepsilon)} = \varrho_\varepsilon * \xi_i$.

We can then study the regularised equation

$$\partial_t u_\varepsilon^\alpha = \partial_x^2 u_\varepsilon^\alpha + \Gamma_{\beta\gamma}^\alpha(u_\varepsilon)\,\partial_x u_\varepsilon^\beta\,\partial_x u_\varepsilon^\gamma + h^\alpha(u_\varepsilon) + \sigma_i^\alpha(u_\varepsilon)\,\xi_i^{(\varepsilon)} \ ,$$

and now everything is smooth. (Since h and σ_i are tangent to the manifold M, then $u_\varepsilon \in M$.)

The question is of course: what happens when $\varepsilon \downarrow 0$?

In analogy with KPZ (and all singular SPDEs) one expects that it is necessary to renormalise this equation:

$$\partial_t \hat{v}_\varepsilon^\alpha = \partial_x^2 \hat{v}_\varepsilon^\alpha + (\partial_x \hat{v}_\varepsilon^\beta)^2 - C_\varepsilon + \xi_i^{(\varepsilon)} \ .$$

Convergence

Theorem

As $\varepsilon \downarrow 0$, u_ε *converges in a suitable Hölder space.*

There is no need to renormalise!

Is this the end of the story?

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As $\varepsilon \downarrow 0$, u_ε converges in a suitable Hölder space.

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Is this the end of the story? Of course it is not...

The limit is not universal, it depends on the choice of mollifier ϱ (and also of the embedding of the manifold M).

This seems to be a very large space of parameters...

Let us take a step back

Let us consider this equation with values in $\mathbf{R}^d$

$$\partial_t u_\varepsilon^\alpha = \partial_x^2 u_\varepsilon^\alpha + \Gamma^\alpha_{\beta\gamma}(u_\varepsilon)\,\partial_x u_\varepsilon^\beta\,\partial_x u_\varepsilon^\gamma + h^\alpha(u_\varepsilon) + \sigma_i^\alpha(u_\varepsilon)\,\xi_i^{(\varepsilon)} \ ,$$

with generic smooth coefficient (there is no manifold anymore). Let $u_\varepsilon := U_\varepsilon(\Gamma, h, \sigma)$.

The theory of regularity structures yields the following result:

Theorem

There is a smooth function $H_{\Gamma,\sigma}$ such that for any mollifier ϱ, there exist unique choice of $\bar{c} \in \mathbf{R}_+$ and of a smooth coefficient F, as well as a choice of $\hat{c} \in \mathbf{R}$, such that for all $b \in \mathbf{R}$

$$\lim_{\varepsilon\to 0} U_\varepsilon\Big(\Gamma, \sigma, h - \frac{\bar{c}}{\varepsilon}\nabla_{\sigma_i}\sigma_i + H_{\Gamma,\sigma}\Big(b + \hat{c} + \log\varepsilon\Big) + F\Big) = U^b(\Gamma, \sigma, h) \ ,$$

for some limit U^b independent of ϱ. Moreover $F \in V$ with $dim(V) = 14$.

Theorem

The solution family $U^b(\Gamma, \sigma, h)$ *has the following property: for any diffeomorphism* $\varphi : \mathbf{R}^d \to \mathbf{R}^d$ *we have*

$$\varphi \cdot U^b(\Gamma, \sigma, h) = U^b(\varphi \cdot \Gamma, \varphi \cdot \sigma, \varphi \cdot h).$$

Moreover we have the Itô isometry

$$U^b(\Gamma, \sigma, h) = U^b(\Gamma, \bar{\sigma}, h) \,,$$

for every $\bar{\sigma}$ *such that* $\bar{\sigma}_i^\alpha \bar{\sigma}_i^\beta = \sigma_i^\alpha \sigma_i^\beta$.

For SDEs one can not have both the above properties simultaneously.

Back to the geometric case

In many cases of interest, one can have

$$\nabla_{\sigma_i}\sigma_i = 0, \qquad H_{\Gamma,\sigma} = 0,$$

and therefore we obtain $U^b = U^0$ for all $b \in \mathbf{R}$, namely there is a single notion of solution.

This should be equal to a Markov process that can be constructed by Dirichlet form methods (see classical works by Driver in the 90's, and recent work by Röckner-Wu-Zhu-Zhu).