Equilibrium Fluctuations in Maximally Noisy Extended Quantum Systems

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Plan

Motivations

Motivations

- What would a quantum macroscopic fluctuation theory look like ?
- Go beyond (statistical) average behavior in quantum systems subject to statistical noise.
  - Two sources of statistical noise
    - Via monitoring, information read-out and random back-action (Quantum trajectories)
    - Via coupling to environments/baths/reservoirs

Prototype: $U_{t+dt} = e^{-idH_t} U_t \quad dH_t = H_0 dt + \sum_\alpha L_\alpha dB_t^\alpha$

  - Needs to go beyond the Lindblad formalism
  - Goal: study in detail a simple specific model

The model

- A $1d$ chain of fermions, with hopping triggered by external noise (with periodic boundary conditions)
- Unitary but random evolution

Hamiltonian: $dH_t = \sqrt{D} \sum_{j=1}^{L} \left(c_{j+1}^\dagger c_j \, dW_t^j + c_j^\dagger c_{j+1} \, d\overline{W}_t^j \right)$

Evolution operator: $U_{t+dt} = e^{-idH_t} U_t$ (Itō convention)

Brownians: $dW_t^j d\overline{W}_t^k = \delta^{jk} \qquad dW_t^j dW_t^k = d\overline{W}_t^j d\overline{W}_t^k = 0$

Fermions: $\{c_j, c_k^\dagger\} = \delta_{jk} \qquad \{c_j, c_k\} = \{c_j^\dagger, c_k^\dagger\} = 0$

  - Beware : $c_j^\dagger, c_j$ live at site j, but $dW_t^j, d\overline{W}_t^j$ live on the edge $(j, j+1)$.

Origin of the model (I)

- Classical macroscopic fluctuation theory equations

$$\begin{aligned} \partial_t n(x,t) + \partial_x j(x,t) &= 0 \\ j(x,t) &= -D(n)\partial_x n(x,t) + \sqrt{L^{-1}\sigma(n)}\,\xi(x,t) \end{aligned}$$

are hard to quantize (no time derivative of j)

- So replace them with dissipative equations

$$\begin{aligned} \partial_t n + \partial_x j &= 0 \\ \tau_f\, \partial_t j &= -D(n)\partial_x n - \eta\, j + \sqrt{\eta L^{-1}\sigma(n)}\,\xi \end{aligned}$$

with $\eta \to +\infty$ and rescaling of t and j

- Discretize space and look at quantum dissipative $1d$ systems

Origin of the model (II)

- Simple example[1]: XX model with dissipation

$$\begin{aligned} dH_t = & \qquad dH_t^{xx} + \qquad dH_t^{noise} \\ = & \; 2\varepsilon \sum_{j=1}^{L} \left(c_{j+1}^\dagger c_j + c_j^\dagger c_{j+1} \right) dt + \; 2\sqrt{\eta \nu_f} \sum_{j=1}^{L} c_j^\dagger c_j dB_t^j \end{aligned}$$

  - Notice B_t^j lives at site j
  - $n_j := c_j^\dagger c_j$ (occupation number) and $J_j := 2i\varepsilon(c_{j+1}^\dagger c_j - c_j^\dagger c_{j+1})$ (current) satisfy

$$dn_j = (J_{j-1} - J_j) dt \qquad dJ_j = -4\eta\nu_f J_j + \text{somewhat an analog of above}$$

  - Take the large η limit (and then the continuum limit)
  - At large η, only states invariant under the rotations induced by the $U(1)$ generators $n_j dB_t^j$ survive.

[1] For more complicated ones, see `SciPost Phys. 3, 033 (2017)`)

Origin of the model (III)

- Effective (slow) dynamics on the invariant states: interaction representation
  - Easy because noises commute

$$
\begin{aligned}
dH_t^{\mathrm{eff}} &= e^{i2\sqrt{\eta\nu_f}\sum_{j=1}^{L} c_j^\dagger c_j B_t^j}\, dH_t^{xy}\, e^{-i2\sqrt{\eta\nu_f}\sum_{j=1}^{L} c_j^\dagger c_j B_t^j} \\
&= 2\varepsilon \sum_{j=1}^{L} \left(c_{j+1}^\dagger c_j \, dW_t^j(\eta) + c_j^\dagger c_{j+1} \, d\overline{W}_t^j(\eta) \right)
\end{aligned}
$$

$$
dW_t^j(\eta) := e^{i2\sqrt{\eta\nu_f}(B_t^{j+1}-B_t^j)}dt \quad d\overline{W}_t^j(\eta) := e^{-i2\sqrt{\eta\nu_f}(B_t^{j+1}-B_t^j)}dt
$$

  - Brownian transmutation (related to the Wong-Zakai theorem): at large η with rescaled time $s \propto t/\eta$ (slow dynamics)

$$
W_t^j(\eta) \to W_s^j \quad \overline{W}_t^j(\eta) \to \overline{W}_s^j \text{ indep. complex Brownian motions}
$$

- Our, incoherently diffusive, model of interest is retrieved.
  - In this context only meaningful on the invariant states

Back to the model

Fermion bilinears

- We study the statistical fluctuations of

$$(G_t)_{ij} = \mathrm{Tr}(\rho_t c_j^\dagger c_i)$$

  - G_t still depends on the external noise
  - Enough to study the one particle sector in general
  - Enough to compute everything in special cases
    - Special class of density matrices preserved by the evolution

$$\rho_t = Z_t^{-1} \exp(c^\dagger M_t c),\ Z_t =: \mathrm{Tr}\exp(c^\dagger M_t c) = \det\left(1 + e^{M_t}\right)$$

    - Matrix M_t can be retrieved from G_t

$$G_t = \frac{1}{1 + e^{-M_t}}$$

    - General correlation functions via Wick's theorem
- Even simpler, completely solvable case (left as an exercise): joint statistical fluctuations of

$$(C_t)_i := \mathrm{Tr}(\rho_t c_i) \quad (\overline{C}_t)_j := \mathrm{Tr}(\rho_t c_j^\dagger)$$

Goals

- Solving the equations of motion...

$$dG_{i,j} = -2D\, G_{i,j} dt + D\, \delta_{i,j} (G_{i+1,j+1} + G_{i-1,j-1}) dt +$$

$$i\sqrt{D} \left(G_{i,j-1} d\overline{W}^{j-1} + G_{i,j+1} dW^{j} - G_{i-1,j} dW^{i-1} - G_{i+1,j} d\overline{W}^{i} \right)$$

  - Diffusion along the diagonal lead to contact terms proportional to dt

... is too difficult

- Average large time stationary regime...

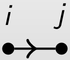
$$0 = -2D\, G_{i,j} + D\, \delta_{i,j} (G_{i+1,j+1} + G_{i-1,j-1})$$

... is trivial : $G \propto \mathrm{Id}$

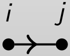
- Higher moments of G in a stationary regime...
  - Do the first, second, third, fourth moment
  - Express the results in terms of $N_k := \mathrm{tr}\, G_t^k$, $k = 1, 2, 3, 4$... which are time-independent because evolution is unitary
  - Prove results for arbitrary moments

... are accessible, and (hopefully) interesting

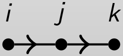
Diagrammatic representation

- Write $\mathbb{E}_\infty[\cdots]$ for the average in the stationary regime
- Represent G_{ij} by 

  - For instance:

$G_{ij} \Leftrightarrow$ 

for $i \neq j$

$G_{ij}G_{jk} \Leftrightarrow$ 

for $i \neq j \neq k \neq i$

$G_{ij}G_{ji} \Leftrightarrow$ 

for $i \neq j$

- Work out on examples (and prove later in general) that
  - $\mathbb{E}_\infty$[Graph] vanishes unless the *in* and *out* degrees are the same at each vertex (Eulerian graphs).
  - $\mathbb{E}_\infty$[Graph] does not depend on the vertex labels (as long as they are distinct for distinct vertices), i.e. correlators are topological
    - Remove vertex labels from the notation

First and second moment

- First moment

$$\mathbb{E}_\infty[$$  $$] = \frac{N_1}{L}$$

- Second moment
  - Use that $N_1 := \operatorname{tr} G_t,\ N_2 := \operatorname{tr} G_t^2$ are conserved quantities

$N_1 = L\,\mathbb{E}_\infty[$  $]$ $\qquad N_2 = L(L-1)\,\mathbb{E}_\infty[$  $] + L\,\mathbb{E}_\infty[$  $]$

  - Impose vanishing of contact terms

$\mathbb{E}_\infty[$  $] = \mathbb{E}_\infty[$  $] + \mathbb{E}_\infty[$  $]$

  - Solve to get

$\mathbb{E}_\infty[$  $] = \dfrac{N_1^2 + N_2}{L(L+1)}$

$\mathbb{E}_\infty[$  $] = \dfrac{LN_1^2 - N_2}{L(L^2-1)}$

$\mathbb{E}_\infty[$  $] = \dfrac{LN_2 - N_1^2}{L(L^2-1)}$

Third moment

- Three conservation laws, plus three contact relations, lead to

$\mathbb{E}_\infty\big[$  $\big] = \dfrac{2N_3 + N_1^3 + 3N_2N_1}{L(L+1)(L+2)}$

$\mathbb{E}_\infty\big[$  $\big] = \dfrac{-2N_3 + (L+1)N_1^3 + (L-1)N_1N_2}{(L-1)L(L+1)(L+2)}$

$\mathbb{E}_\infty\big[$  $\big] = \dfrac{4N_3 + \left(L^2-2\right)N_1^3 - 3LN_1N_2}{(L-2)(L-1)L(L+1)(L+2)}$

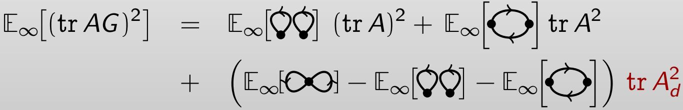
$\mathbb{E}_\infty\big[$  $\big] = \dfrac{LN_3 + (L-1)N_1N_2 - N_1^3}{(L-1)L(L+1)(L+2)}$

$\mathbb{E}_\infty\big[$  $\big] = \dfrac{\left(L^2+2\right)N_1N_2 - L\left(2N_3 + N_1^3\right)}{(L-2)(L-1)L(L+1)(L+2)}$

$\mathbb{E}_\infty\big[$  $\big] = \dfrac{N_3L^2 - 3LN_1N_2 + 2N_1^3}{(L-2)(L-1)L(L+1)(L+2)}$

Unitary invariance (I)

- Introduce spectator matrix A, with A_d diagonal part of A and check
  - Second moments



    - This computation does not use the explicit values of the correlators, just that they are topological
    - The coefficient of the non unitary invariant $\operatorname{tr} A_d^2$ vanishes because of the contact relation
  - Third moments
    - The coefficient of the non unitary invariant contributions vanish because of the contact relations
  - We prove in general that at large time there is convergence to a $\mathbb{E}_\infty[\cdots]$ that is unitary invariant

Unitary invariance (II)

- Evolution preserves total fermion number
  - Restrict to the 1 fermion sector
  - Study the evolution $V_{t+dt} = e^{-idh_t} V_t$ on $SU(L)$ where

$$dh_t = \sqrt{D} \sum_j (E_{j+1,j}\, dW_t^j + E_{j,j+1}\, d\overline{W}_t^j)$$

    with $E_{i,j}$ usual elementary matrices
- "Proof" that stationary measures are $SU(L)$ invariant
  - The $E_{j+1,j}$ and $E_{j,j+1}$ generate $\mathfrak{su}(L) \otimes \mathbb{C} = \mathfrak{sl}(L)$ $(\star)$
  - An invariant measure must be invariant under the corresponding transformations, hence under all of $\mathfrak{su}(L)$, i.e. must be Haar QED
    - This intuitive argument can be turned into a rigorous proof
- "Proof" of large time convergence towards Haar
  - Either via Hörmander's theorem (relies on $(\star)$) plus a variant of Doeblin's theory ...
  - ... or the existence of a spectral gap for elliptic operators QED
- $(\star)$ means maximal noise in the 1 fermion sector
  - To be contrasted with maximal noise in the full Fock space

Unitary invariance (III)

- Possible generalizations of these arguments when

$$dH_t \to H_0 dt + dH_t$$

  - Works smoothly for H_0 a spatially inhomogeneous chemical potential
  - Preliminary weak coupling computations suggest that more robustness, even when there in no 1 fermion dynamics anymore
- This hints at a possible large degree of universality for the infinite time statistics of G.

Unitary invariance (IV)

- Unitary invariance means that $Z(A) := \mathbb{E}_\infty\left[e^{\mathrm{tr}\, AG}\right]$ satisfies

$$Z(A) = Z(VAV^\dagger) \text{ for } V \in U(L)$$

- If $d\eta$ denotes the normalized Haar measure on $U(L)$

$$\begin{aligned} Z(A) &= \mathbb{E}_\infty\left[e^{\mathrm{tr}\, AG}\right] \\ &= \int_{U(L)} d\eta(V)\mathbb{E}_\infty\left[e^{\mathrm{tr}\, VAV^\dagger G}\right] \\ &= \mathbb{E}_\infty\left[\int_{U(L)} d\eta(V) e^{\mathrm{tr}\, AV^\dagger GV}\right] \\ &= \mathbb{E}_\infty\left[\int_{U(L)} d\eta(V) e^{\mathrm{tr}\, AV^\dagger G_{t=0} V}\right] \\ &= \int_{U(L)} d\eta(V) e^{\mathrm{tr}\, AV^\dagger G_{t=0} V} \end{aligned}$$

HarishChandra-Itzykson-Zuber integral

- If $a_i, g_j,\ i,j = 1, \cdots, L$ denote the eigenvalues of A and $G_{t=0}$

$$Z(A) = \int_{U(L)} d\eta(V)\, e^{\operatorname{tr} AV^\dagger G_{t=0} V} = (\prod_{k=1}^{L-1} k!)\, \frac{\det\,(e^{a_i g_j})_{i,j=1}^{L}}{\Delta(a)\Delta(g)}$$

  - This is in principle a complete, non perturbative, description of the correlations in the stationary distributions...
    - Perturbative expansion in terms of Young diagrams and characters matches
  - ... but there is a denominator whose 0s are canceled by 0s of the numerator !
  - For example, take A to be diagonal with only one nonzero matrix element so that $Z(A)$ describes the statistics of the fermion number at a single site (see later)

Extensive scaling limit

- Large L scaling limit, $G_{t=0}$ such that each $N_k := \text{tr}\, G_{t=0}^k$, $k = 0, 1, \cdots$ scales like $\sim L$.
  - Arises for instance when $\rho_{t=0}$ describes independent sites with one site density matrix diagonal in the fermion number basis.
  - There is a concentration phenomenon : $G_{t=\infty}$ is non random $G_{t=\infty} = G_{eq} = \rho_1 \text{Id}$
    - ... where we set $\rho_k := \lim_{L\to\infty} \frac{N_k}{L}$.
  - There are interesting corrections:
    - For instance $\lim_{L\to\infty} L\mathbb{E}_\infty\Big[$  $\Big] = \rho_2 - \rho_1^2$: for a factorized diagonal but inhomogeneous density matrix at $t = 0$, the coherences at $t = \infty$ have non-trivial fluctuations
  - There is a large deviation function (and principle) $w(A) := \lim_{L\to\infty} \frac{1}{L} \log Z(LA)$

Conjecture: $$w(A) = \sum_{k\geq 1} \frac{1}{k} \Big(\lim_{L\to\infty} \mathbb{E}_\infty[\text{cycle of length } k] \Big)\, \text{tr}\, A^k$$

$$w(A) = \rho_1\ \text{tr}\, A + \frac{1}{2}(\rho_2 - \rho_1^2)\ \text{tr}\, A^2 + \frac{1}{3}(\rho_3 - 3\rho_2\rho_1 + 2\rho_1^3)\ \text{tr}\, A^3 + \cdots$$

Other explicit results via invariant theory

- One-site fermion number fluctuations

$$\mathbb{E}_\infty\left[\langle c_i^\dagger c_i\rangle^n\right] = \frac{n!(L-1)!}{(L+n-1)!} \sum_{n_k,\, \sum_{k\geq 1} k n_k = n} \prod_k \frac{1}{n_k!}\left(\frac{N_k}{k}\right)^{n_k}$$

  - Relation with cumulant formulæ
- Large L scaling limit, $G_{t=0}$ such that each $N_k := \operatorname{tr} G_{t=0}^k$, $k = 1, 2\cdots$ scales like $\sim L^k$.
  - Arises for instance when $\rho_{t=0}$ describes independent sites with one site density matrix non-diagonal in the fermion number basis.

$$\mathbb{E}_\infty\left[e^{\operatorname{tr} AG}\right] = e^{\sum_{k\geq 1} \frac{N_k \operatorname{tr} A^k}{kL^k}} + o(L^0)$$

  - Can be crosschecked against one-site fermion number fluctuations

Invariant theory and contact terms (I)

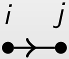
- The relevant invariant theory is that of invariants in the n^{th} symmetric power of the adjoint representation of $U(L)$ because G leaves in the adjoint representation[2]
- Obtained from the invariant theory of the n^{th} tensor power of the adjoint representation of $U(L)$
  - There are $n!$ fundamental invariants indexed by permutations

$$I^{\sigma}_{i_1 j_1, \cdots, i_n j_n} := \delta_{i_{\sigma(1)} j_1} \cdots \delta_{i_{\sigma(n)} j_n}$$

  - A most important object is the matrix $\left(L^{c(\sigma\tau^{-1})}\right)_{\sigma,\tau\in\mathfrak{S}_n}$ where $c(\sigma)$ denotes the number of cycles in the cycle decomposition of σ
    - It gives a natural metric on the space of invariants
    - It simplifies at large $L \Rightarrow$ Large L limits
    - It satisfies two remarkable sum rules $\Rightarrow$ One-site fermion number fluctuations

[2] Case of $(C_t)_i := \mathrm{Tr}(\rho_t c_i)$ and $(\overline{C}_t)_j := \mathrm{Tr}(\rho_t c_j^{\dagger})$: only one invariant remains

Invariant theory and contact terms (II)

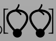
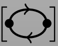
- Recall graphical notation $G_{ij} \Leftrightarrow$ 

- If Γ is any Eulerian graph, it follows from invariant theory that

$$\mathbb{E}_\infty[\Gamma] = \sum_\gamma m_{\Gamma|\gamma} \mathbb{E}_\infty[\gamma]$$

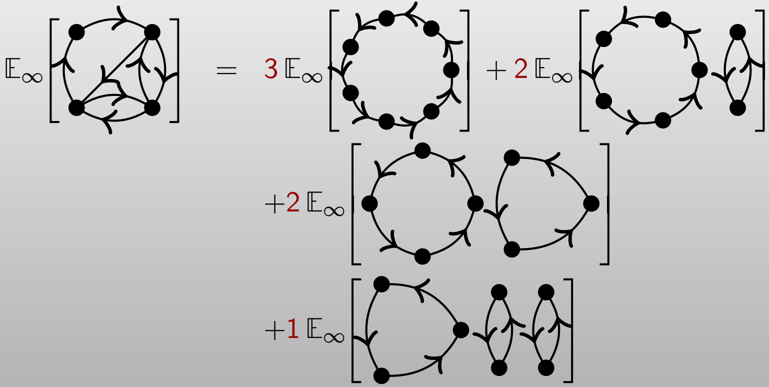
  where

  - The γs are graphs whose connected components are all cycles
  - The decomposition is unique
  - The $m_{\Gamma|\gamma}$s are multiplicities (non-negative integers): $m_{\Gamma|\gamma}$ is the number of times γ arises when the vertices of Γ are blown-up in simple (one *in* one *out* edge) vertices in every possible ways

- Example: $\mathbb{E}_\infty[$  $] = \mathbb{E}_\infty[$  $] + \mathbb{E}_\infty[$  $]$

Invariant theory and contact terms (III)

- More complicated example: decomposition of the sumotori



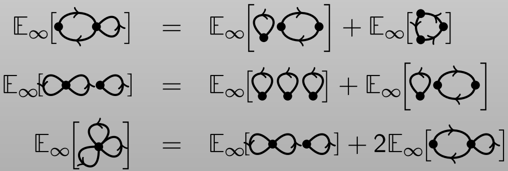
  - Simple sum rule: each $n = n_{in} = n_{out}$ vertex can be blown up in $n!$ ways

$$3 + 2 + 2 + 1 = 8 = (2!)^3$$

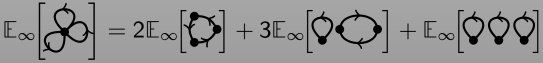
Invariant theory and contact terms (IV)

- The number of contact relations, unitary invariance constraints, and cycle decompositions are all the same
- Going from one set to the other requires a change of basis
- Examples:
  - Second moments contact relation

  - Third moments contact relations



    - The last relation is a combination of the first two and the cycle decomposition



Conclusions

Conclusions

- We studied a simple model of diffusion triggered by external noise
  - The model is kind of a paradigm
  - It has roots as effective dynamics in a class of strongly dissipative models
- The large time behavior of the statistical fluctuations of an important class of quantum averages can be studied in detail
  - Stationary fluctuations have topological nature
  - Perturbative computations
  - Explicit description of the invariant measure (Haar) with the HarishChandra-Itzykson-Zuber formula as a non-perturbative generating function of fluctuations
  - Several large size scaling limits
    - Non-trivial coherences fluctuations, large deviation principle
  - Deep connections with classical representation/invariant theory
    - Access to some finite size observables

Perspectives

- Explore further the universality issue
- Look at open boundary conditions with injection/extraction
  - Find injection/extraction terms that do not kill coherences
- Replace classical (Brownian) noises by quantum noises
- Find a natural physical interpretation of statistical fluctuations
  - Naively, if the noise is not controlled
    - Only joint quantum/statistical averages are accessible
    - Higher statistical moments of quantum averages are out of reach