Deriving the SBE from weakly asymmetric interacting particle systems

Patrícia Gonçalves

Joint with Cedric Bernardin (Nice), Oriane Blondel (Lyon), Tertuliano Franco (Bahia), Milton Jara (Rio de Janeiro), Nicolas Perkowski (Berlin), Adriana Neumann (Rio Grande do Sul), Sunder Sethuraman (Arizona) and Marielle Simon (Lille)

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The *universal* law behind growth patterns: bacterial growth, coffee ring effects, freezing rain deposition, tumor growth...

(KPZ in nature)

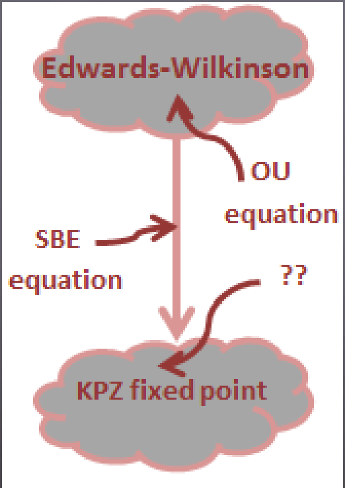
The *universal* law behind growth patterns: bacterial growth, coffee ring effects, freezing rain deposition, tumor growth...

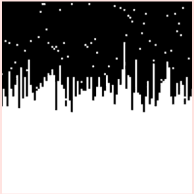
(KPZ in nature)

Analogy with a game that everybody knows...

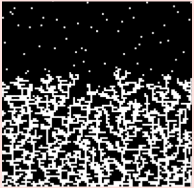
(tetris) (stickytetris)

Universality classes:





No sticky deposition of blocks.



Sticky deposition of blocks.

The **KPZ/SBE** *equation*

- Let $h(t,u)$ be the height of an interface
  $dh(t,u) = \kappa_1 \Delta h(t,u)dt + \kappa_2(\nabla h(t,u))^2 dt + \kappa_3 W_t$ (KPZ).
- If $Y_t = \nabla h_t$, then
  $dY_t = \kappa_1 \Delta Y_t dt + \kappa_2 \nabla Y_t^2 + \kappa_3 \nabla W_t$ (SBE).
- 1st a mathematical challenge:
  - meaning of solution;
  - Cole-Hopf solutions: $Z(t,u) = e^{\frac{\kappa_3}{\kappa_1} h(t,u)}$.
- 2nd a physical challenge:
  - derive KPZ/SBE from microscopic models;
  - many microscopic models *do not* satisfy Cole-Hopf.

Which microscopic models are described by KPZ/SBE?

Slowed exclusion processes: the dynamics

• η_t is an exclusion process, $\Omega = \{0,1\}^{\mathbb{Z}}$,
for $x \in \mathbb{Z}$, $\eta(x) = 1$ if the site is occupied, otherwise $\eta(x) = 0$.

• The rates are given by

$$\left(\tfrac{1}{2} - \tfrac{a}{2n^{\gamma}}\right) \curvearrowleft \qquad \curvearrowright \left(\tfrac{1}{2} + \tfrac{a}{2n^{\gamma}}\right)$$

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• At the slow bond $\{-1,0\}$ the rates are given by

$$\curvearrowright \left(\tfrac{\alpha}{2n^{\beta}} + \tfrac{a}{2n^{\gamma}}\right) \qquad\qquad \curvearrowleft \left(\tfrac{\alpha}{2n^{\beta}} - \tfrac{a}{2n^{\gamma}}\right)$$

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We assume $\gamma > \beta$ or $\beta = \gamma$ and $\alpha \geq a$ (in last case if $a = \alpha$ then $\{-1,0\}$ is totally asymmetric).

• For $a = 0$, we obtain the SSEP with a slow bond.

• For $\alpha = 1$ and $\beta = 0$ we obtain the WASEP - weak asymmetry.

• ν_ρ the Bernoulli product measure of parameter ρ is invariant.

Hydrodynamic limit: the case $a = 0$

- For $\eta \in \Omega$, let $\pi_t^n(\eta; du) = \frac{1}{n} \sum_{x\in\mathbb{Z}} \eta_{tn^2}(x)\delta_{\frac{x}{n}}(du)$.
- Fix $\rho_0 : \mathbb{R} \to [0,1]$ and μ_n such that for every $\delta > 0$ and every continuous function $H : \mathbb{R} \to \mathbb{R}$,

$$\frac{1}{n}\sum_{x\in\mathbb{Z}} H(\tfrac{x}{n})\,\eta(x) \to_{n\to\infty} \int_{\mathbb{R}} H(u)\,\rho_0(u)du,$$

wrt μ_n. Then for any $t > 0$, $\pi_t^n \to \rho(t,u)du$, as $n \to \infty$, where $\rho(t,u)$ evolves according to the heat equation $\partial_t \rho(t,u) = \Delta\rho(t,u)$:

  - $\beta < 1$: no boundary conditions.
  - $\beta = 1$: with linear **Robin** boundary conditions $\partial_u\rho(t,0^-) = \partial_u\rho(t,0^+) = \alpha(\rho(t,0^+) - \rho(t,0^-))$.
  - $\beta > 1$: with **Neumann** boundary conditions $\partial_u\rho(t,0^-) = \partial_u\rho(t,0^+) = 0$.

Equilibrium density fluctuations: $a = 0$

- Fix $\rho \in (0,1)$ and start the process from ν_ρ.
- The *density fluctuation field* $\{\mathcal{Y}_t^{\beta,\gamma,n}\,;\; t \in [0, T]\}$ is given on functions $H \in \mathcal{S}_\beta(\mathbb{R})$ by

$$\mathcal{Y}_t^{\beta,\gamma,n}(H) := \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} H\Big(\tfrac{x}{n}\Big)(\eta_{tn^2}(x) - \rho).$$

Definition (Space of test functions)

Let $\mathcal{S}(\mathbb{R}\backslash\{0\})$ be the space of functions $H : \mathbb{R} \to \mathbb{R}$ such that:

1. H is smooth in $\mathbb{R}\backslash\{0\}$,
2. H is continuous from the right at 0,
3. for all non-negative integers k, ℓ, the function H satisfies

$$\|H\|_{k,\ell} := \sup_{u \neq 0} \Big|(1 + |u|^\ell)\frac{d^k H}{du^k}(u)\Big| < \infty.$$

Space of test functions

1. For $\beta < 1$, $\mathcal{S}_\beta(\mathbb{R}) := \mathcal{S}(\mathbb{R})$, the usual Schwartz space $\mathcal{S}(\mathbb{R})$.
2. For $\beta = 1$, $\mathcal{S}_\beta(\mathbb{R})$ is the subset of $\mathcal{S}(\mathbb{R}\backslash\{0\})$ composed of functions H such that

$$\frac{d^{2k+1}H}{du^{2k+1}}(0^+) = \frac{d^{2k+1}H}{du^{2k+1}}(0^-) = \alpha\Big(\frac{d^{2k}H}{du^{2k}}(0^+) - \frac{d^{2k}H}{du^{2k}}(0^-)\Big)$$

   for any integer $k \geq 0$.
3. For $\beta > 1$, $\mathcal{S}_\beta(\mathbb{R})$ is the subset of $\mathcal{S}(\mathbb{R}\backslash\{0\})$ composed of functions H such that

$$\frac{d^{2k+1}H}{du^{2k+1}}(0^+) = \frac{d^{2k+1}H}{du^{2k+1}}(0^-) = 0$$

   for any integer $k \geq 1$.

Density fluctuation field for $a = 0$

Theorem (Franco, G., Neumann, 13')

If $a = 0$, the sequence of processes $\{\mathcal{Y}_t^{\beta,\gamma,n}\ ;\ t \in [0, T]\}_{n\in\mathbb{N}}$ converges to the Ornstein-Uhlenbeck process given by

$$d\mathcal{Y}_t^{\beta} = \tfrac{1}{2}\Delta_\beta \mathcal{Y}_t^{\beta} dt + \sqrt{\chi(\rho)}\nabla_\beta d\mathcal{W}_t^{\beta},$$

where $\{\mathcal{W}_t^{\beta}\ ;\ t \in [0, T]\}$ is an $\mathcal{S}'_\beta(\mathbb{R})$-valued Brownian motion and $\chi(\rho) = \rho(1 - \rho)$.

Density fluctuation field for $a \neq 0$: removing the drift

We redefine for any $H \in \mathcal{S}_\beta(\mathbb{R})$

$$\mathcal{Y}_t^{\beta,\gamma,n}(H) = \frac{1}{\sqrt{n}} \sum_{x \in \mathbb{Z}} H\Big(\frac{x - n^{2-\gamma} a(1-2\rho)t}{n}\Big)(\eta_{tn^2}(x) - \rho).$$

To simplify take $\rho = 1/2$.

Theorem (Franco, G., Simon, 16')

If one of these two conditions are satisfied:

- $\beta \leq 1/2$ *and* $\gamma > 1/2$,
- $\beta > 1/2$ *and* $\gamma \geq \beta$

then $\{\mathcal{Y}_t^{\beta,\gamma,n}\ ;\ t \in [0,T]\}_{n \in \mathbb{N}}$ *converges to the Ornstein-Uhlenbeck process as in the case* $a = 0$.

The influence of the asymmetry is **NOT SEEN** in the limit.

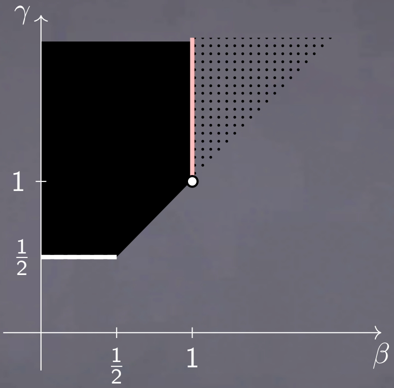
Effect of a stronger asymmetry $a \neq 0$: the KPZ scaling

Theorem (Franco, G., Simon, 16')

Fix $\rho = 1/2$. For $\beta \leq 1/2$ and $\gamma = 1/2$, $\{\mathcal{Y}_t^{\beta,\gamma,n}\ ;\ t \in [0, T]\}_{n\in\mathbb{N}}$ converges to the stationary energy solution (SES) of the **stochastic Burgers equation (SBE)**

$$d\mathcal{Y}_t = \tfrac{1}{2}\Delta\mathcal{Y}_t dt + a\nabla(\mathcal{Y}_t)^2 dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t,$$

where $\{\mathcal{W}_t\ ;\ t \in [0, T]\}$ is an $\mathcal{S}'(\mathbb{R})$-valued Brownian motion.



- Stochastic Burgers equation (KPZ regime)
- OU process with no boundary conditions
- OU process with Robin boundary conditions
- OU process with Neumann boundary conditions
- OU process with Robin boundary conditions, stronger noise

On the definition of SES of the SBE

Definition (Controlled process)

A pair of stochastic processes $\{(\mathcal{Y}_t, \mathcal{A}_t); t \in [0, T]\}$ with trajectories in $\mathcal{C}([0, T]; \mathcal{S}'(\mathbb{R}))$ is controlled by the Ornstein-Uhlenbeck process given by $d\mathcal{Y}_t = \frac{1}{2}\Delta\mathcal{Y}_t dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t$, if:

i) For each $t \in [0, T]$, $\mathcal{Y}_t$ is a white noise of variance $2\chi(\rho)$,

ii) for each $H \in \mathcal{S}(\mathbb{R})$, the process $\mathcal{M}_t(H) = \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \int_0^t \mathcal{Y}_s(\frac{1}{2}\Delta H)ds - \mathcal{A}_t(H)$ is a Brownian motion of variance $\chi(\rho)\|\nabla H\|_2^2 t$,

iii) for each $H \in \mathcal{S}(\mathbb{R})$ $\{\mathcal{A}_t(H); t \geq 0\}$ is a.s. of zero quadratic variation: $E[\lim_{\varepsilon \to 0} \int_0^t \frac{1}{\varepsilon}(\mathcal{A}_{s+\varepsilon}(H) - \mathcal{A}_s(H))^2 ds] = 0$.

iv) for each $T > 0$, $\{(\mathcal{Y}_{T-t}, -(\mathcal{A}_{T-t} - \mathcal{A}_T)); t \in [0, T]\}$ satisfies ii).

Proposition (Gubinelli and Perkowski, 18')

Let $\{(\mathcal{Y}_t, \mathcal{A}_t); t \geq 0\}$ *be a controlled process and let* $\{\iota_\epsilon; \epsilon \in (0,1)\}$ *be an approximation of the identity. Then, for any* $H \in \mathcal{S}(\mathbb{R})$ *the limit*

$$\mathcal{B}_t(H) = \lim_{\epsilon \to 0} \mathcal{B}_t^\epsilon(H) := \lim_{\epsilon \to 0} \int_0^t \int_{\mathbb{R}} \big(\mathcal{Y}_s(\iota_\epsilon(u))\big)^2 H'(u)\, du\, ds$$

exists in $\mathbb{L}^2$*.*

Definition (Stationary energy solution)

A controlled process $\{(\mathcal{Y}_t, \mathcal{A}_t); t \geq 0\}$ is a SES of the SBE

$$d\mathcal{Y}_t = \tfrac{1}{2}\Delta \mathcal{Y}_t dt + a\nabla(\mathcal{Y}_t)^2 dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t,$$

if $\mathcal{A}_t(H) = a\mathcal{B}_t(H)$ a.s. for all $H \in \mathcal{S}(\mathbb{R})$ and $t \in [0, T]$.

Definition (Stationary energy solution)

We say that a process $\{\mathcal{Y}_t \ ; \ t \in [0, T]\}$ with trajectories in $\mathcal{C}([0, T], \mathcal{S}'(\mathbb{R}))$ is a *stationary energy solution* of the SBE

$$d\mathcal{Y}_t = \frac{1}{2}\Delta\mathcal{Y}_t dt + a\nabla(\mathcal{Y}_t)^2 dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t,$$

if:

(i) for each $t \in [0, T]$, $\mathcal{Y}_t$ is a white noise of variance $2\chi(\rho)$,

(ii) there exists $\kappa > 0$ s.t. for any $H \in \mathcal{S}(\mathbb{R})$ and $0 < \delta < \epsilon < 1$

$$\mathbb{E}\Big[\Big(\mathcal{B}^{\epsilon}_{s,t}(H) - \mathcal{B}^{\delta}_{s,t}(H)\Big)^2\Big] \le \kappa\epsilon(t-s)\|\nabla H\|_2^2, \qquad \text{(energy estimate)}$$

where

$$\mathcal{B}^{\epsilon}_{s,t}(H) := \int_s^t \int_{\mathbb{R}} \Big(\mathcal{Y}_r(\iota_\epsilon(u))\Big)^2 H'(u)\,du\,dr$$

and for $u \in \mathbb{R}$ the function $\iota_\varepsilon(u) : \mathbb{R} \to \mathbb{R}$ is given by $\iota_\varepsilon(u)(v) := \varepsilon^{-1}\,\mathbf{1}_{]u,u+\varepsilon]}(v)$,

(iii) for any $H \in \mathcal{S}(\mathbb{R})$ the process

$$\mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \int_0^t \mathcal{Y}_s(\tfrac{1}{2}\Delta H)ds + a\mathcal{B}_t(H)$$

is a Brownian motion of variance $\chi(\rho)\|\nabla H\|_2^2 t$, where $\mathcal{B}_t(H) = \lim_{\epsilon\to 0}\mathcal{B}^{\epsilon}_{0,t}(H)$ in $\mathbb{L}^2$,

(iv) the reversed process $\{\mathcal{Y}_{T-t} \ ; \ t \in [0, T]\}$ also satisfies (iii) with a replaced by $-a$.

Proposition (Gubinelli and Perkowski, 18')

*There exists only **ONE** stationary energy solution of the stochastic Burgers equation.*

How do we prove the results?

(1) First, we prove tightness.
(2) Second, we characterize the limit point.

The KPZ scaling: how to get the SES

To show that $\mathcal{Y}_t$ is a stationary energy solution of the SBE

$$d\mathcal{Y}_t = \tfrac{1}{2}\Delta\mathcal{Y}_t dt + a\nabla(\mathcal{Y}_t)^2 dt + \sqrt{\chi(\rho)}\nabla d\mathcal{W}_t,$$

we need to prove that $\{\mathcal{M}_t : t \in [0, T]\}$ given by

$$\mathcal{M}_t(H) \ := \ \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \tfrac{1}{2}\int_0^t \mathcal{Y}_s(\Delta H)ds + a\mathcal{B}_t(H)$$

is a continuous martingale with quadratic variation

$$\langle \mathcal{M}(H)\rangle_t = \chi(\rho)\|\nabla H\|_2^2 t,$$

where

$$\mathcal{B}_t(H) = \lim_{\varepsilon\to 0} \int_0^t \int_{\mathbb{R}} \left(\mathcal{Y}_s(\iota_\epsilon(u))\right)^2 H'(u)\, du\, ds$$

in $\mathbb{L}^2$, and $\iota_\varepsilon(u)(v) = \frac{1}{\varepsilon}\mathbf{1}_{u<v\leq u+\varepsilon}$, for $v \in \mathbb{R}$.

Features of the models: the instantaneous current

It is simple to check that $\mathcal{L}\eta(x) = j^n_{x-1,x}(\eta) - j^n_{x,x+1}(\eta)$, where

$$j^n_{x,x+1}(\eta) = j^{n,S}_{x,x+1}(\eta) + j^{n,A}_{x,x+1}(\eta)$$

with

$$\begin{aligned} j^{n,A}_{x,x+1}(\eta) &= \frac{an^2}{2n^\gamma}(\eta(x+1)-\eta(x))^2, & x\in\mathbb{Z},\\ j^{n,S}_{x,x+1}(\eta) &= \frac{n^2}{2}(\eta(x)-\eta(x+1)), & x\neq -1,\\ j^{n,S}_{-1,0}(\eta) &= \frac{\alpha n^2}{2n^\beta}(\eta(-1)-\eta(0)). \end{aligned}$$

Important:
(1) $j^{n,S}_{x,x+1}(\eta)$ is a gradient!
(2) $j^{n,A}_{x,x+1}(\eta) = -2(\eta(x)-\rho)(\eta(x+1)-\rho) + \frac{1}{2}$.

Associated martingales

Simple computations show that

$$\mathcal{M}_t^n(H) := \mathcal{Y}_t^n(H) - \mathcal{Y}_0^n(H) - \mathcal{I}_t^n(H) - \mathcal{B}_t^n(H),$$

plus some negligible term, where

$$\mathcal{I}_t^n(H) := \frac{1}{2}\int_0^t \frac{1}{\sqrt{n}} \sum_{x\in\mathbb{Z}} (\eta_{sn^2}(x) - \rho)\Delta H\Big(\frac{x}{n}\Big)\, ds = \frac{1}{2}\int_0^t \mathcal{Y}_s^n(\Delta H)\, ds,$$

(Note that $\mathcal{I}_t^n(H)$ is written in terms of the density field!) and

$$\mathcal{B}_t^n(H) = -a\frac{\sqrt{n}}{n^\gamma}\int_0^t \sum_{x\in\mathbb{Z}} (\eta_{sn^2}(x+1) - \rho)(\eta_{sn^2}(x) - \rho)\nabla H\Big(\frac{x}{n}\Big)\, ds.$$

Last term is the hard one since it is not written in terms of the density field! (Attention to the dependence on γ!)

The 2nd order Boltzmann-Gibbs Principle

Theorem (Boltzmann-Gibbs Principle)

Let v be a function such that $\|v\|_{2,n}^2 := \frac{1}{n}\sum_{x\in\mathbb{Z}} v^2\left(\frac{x}{n}\right) < \infty$.
Then, there exists $C > 0$ such that for any $t > 0$ and $\ell \in \mathbb{N}$:

$$\mathbb{E}_\rho\left[\left(\int_0^t \sum_{x\in\mathbb{Z}} v\left(\tfrac{x}{n}\right)\left\{\bar{\eta}_{sn^2}(x)\bar{\eta}_{sn^2}(x+1)-\left(\left(\overrightarrow{\eta}^{\ell}_{sn^2}(x)\right)^2-\frac{\chi(\rho)}{\ell}\right)\right\}ds\right)^2\right]$$

$$\leq Ct\left\{\frac{\ell}{n}+\frac{n^\beta}{\alpha n}+\frac{tn}{\ell^2}\right\}\|v\|_{2,n}^2 + Ct\left\{\frac{n^\beta(\log_2(\ell))^2}{\alpha n}\right\}\frac{1}{n}\sum_{x\neq -1} v^2\left(\tfrac{x}{n}\right),$$

where $\bar{\eta}(x) = \eta(x) - \rho$ and

$$\overrightarrow{\eta}^{\ell}(x) = \frac{1}{\ell}\sum_{y=x+1}^{x+\ell}\bar{\eta}(y).$$

Consequences of the Boltzmann-Gibbs Principle

It shows that for $\gamma > 1/2$ the field $\mathcal{B}^n_t(H)$ vanishes, as $n \to \infty$ but for $\gamma = 1/2$ and for $\ell = \epsilon n$ it can be replaced by

$$-a \int_0^t \sum_{x\in\mathbb{Z}} \left(\overrightarrow{\eta}^{\epsilon n}_{sn^2}(x)\right)^2 \nabla H\Big(\tfrac{x}{n}\Big)\, ds,$$

which is equal to

$$-a \int_0^t \frac{1}{n} \sum_{x\in\mathbb{Z}} \left(\mathcal{Y}^n_s(\iota_\epsilon(x))\right)^2 \nabla H\Big(\tfrac{x}{n}\Big)\, ds,$$

where $\iota_\varepsilon(x)(y) = \frac{1}{\varepsilon}\mathbf{1}_{x<y\leq x+\varepsilon}$, for $y \in \mathbb{R}$, and in the limit as $n \to \infty$ it converges to

$$\int_0^t \int_{\mathbb{R}} \left(\mathcal{Y}_s(\iota_\epsilon(x))\right)^2 H'(x)\, dx\, ds.$$

The idea of the proof of the Boltzmann-Gibbs

The idea consists in using the following decomposition of the local function

$$\bar\eta(x)\bar\eta(x+1) - (\overrightarrow{\eta}^{L}(x))^2 + \frac{\chi(\rho)}{L}$$

$$= \bar\eta(x)\Big(\bar\eta(x+1) - \overrightarrow{\eta}^{\ell_0}(x)\Big)$$

$$+ \overrightarrow{\eta}^{\ell_0}(x)\Big(\eta(x) - \overleftarrow{\eta}^{\ell_0}(x)\Big)$$

$$+ \overleftarrow{\eta}^{\ell_0}(x)\Big(\overrightarrow{\eta}^{\ell_0}(x) - \overrightarrow{\eta}^{L}(x)\Big)$$ **(needs a multi-scale analysis)**

$$+ \overrightarrow{\eta}^{L}(x)\Big(\overleftarrow{\eta}^{\ell_0}(x) - \eta(x)\Big)$$

$$+ \overrightarrow{\eta}^{L}(x)\bar\eta(x) - (\overrightarrow{\eta}^{L}(x))^2 + \frac{(\bar\eta(x) - \bar\eta(x+1))^2}{2L}$$

$$- \frac{(\bar\eta(x) - \bar\eta(x+1))^2}{2L} + \frac{\chi(\rho)}{L}.$$

Averaging over a box:

Proposition (Replacing occupation sites by averages)
Let $\ell_0 \in \mathbb{N}$ and $\psi : \Omega \to \mathbb{R}$ a local function whose support does not intersect the set of points $\{1, \dots, \ell_0\}$. We assume that ψ has mean zero with respect to ν_ρ and we denote by $\mathrm{Var}(\psi)$ its variance. Then, for any $t > 0$:

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} v\Big(\tfrac{x}{n}\Big)\tau_x\psi(\eta_{sn^2})(\bar{\eta}_{sn^2}(x+1) - \overrightarrow{\eta}^{\ell_0}_{sn^2}(x))\, ds\Big)^2\Big]$$
$$\leq C(\rho)t\mathrm{Var}(\psi)\Big(\frac{\ell_0^2}{n}\|v\|^2_{2,n} + \frac{\ell_0 n^\beta}{n^2\alpha}\sum_{x\in\Lambda_1^{\ell_0-1}} v^2\big(\tfrac{x}{n}\big)\Big),$$

where $\Lambda_1^{\ell_0-1} = \{-\ell_0, \dots, -2\}$.

Doubling the size of the box:

Proposition (Doubling the box)

Let $\ell_k \in \mathbb{N}$, $\ell_{k+1} = 2\ell_k$ and $\psi : \Omega \to \mathbb{R}$ a local function whose support does not intersect the set of points $\{1, \dots, \ell_{k+1}\}$. We assume that ψ has mean zero with respect to ν_ρ and we denote by $\mathrm{Var}(\psi)$ its variance. Then, for any $t > 0$:

$$\mathbb{E}_\rho\Big[\Big(\int_0^t \sum_{x\in\mathbb{Z}} v\big(\tfrac{x}{n}\big)\tau_x\psi(\eta_{sn^2})\big(\overrightarrow{\eta}^{\ell_k}_{sn^2}(x) - \overrightarrow{\eta}^{\ell_{k+1}}_{sn^2}(x)\big)\, ds\Big)^2\Big]$$

$$\leq C(\rho)t\mathrm{Var}(\psi)\Big(\frac{\ell_k^2}{n}\|v\|^2_{2,n} + \frac{n^\beta \ell_k}{n^2\alpha}\sum_{x\neq -1} v^2\big(\tfrac{x}{n}\big)\Big).$$

On the UNIVERSALITY of the stationary energy solution of the SBE from microscopic stochastic dynamics

Exclusion processes (with M. Jara)

• η_t a Markov process with space state $\Omega := \{0,1\}^{\mathbb{Z}}$.

• Jump rates: $p_n r(\tau_x\eta)\eta(x)(1-\eta(x+1))$ and $(1-p_n)r(\tau_x\eta)\eta(x+1)(1-\eta(x))$, where $p_n = \frac{1}{2}+\frac{a}{2n^\gamma}$,

• Where $r:\Omega\to\mathbb{R}$ is a local function that satisfies:

[i] There exists $\varepsilon_0>0$ such that $\varepsilon_0 < r(\eta) < \varepsilon_0^{-1}$ for any $\eta\in\Omega$.

[ii] There exists $\omega:\Omega\to\mathbb{R}$ s. t. $r(\eta)(\eta(1)-\eta(0)) = \tau_1\omega(\eta)-\omega(\eta)$, for any $\eta\in\Omega$.

Ref: Gonçalves, P., Jara, M. (2014): Nonlinear fluctuations of weakly asymmetric interacting particle systems, Archive for Rational Mechanics and Analysis, Volume 212, Issue 2, 597 - 644.

Zero-range processes (with M. Jara and S. Sethuraman)

- η_t a Markov process with space state $\Omega := \mathbb{N}^{\mathbb{Z}}$.

- for $x \in \mathbb{Z}$, $\eta(x)$ counts the number of particles at the site x, the jump rate of a particle at the site x only depends on the number of particles at x and is given by a function $g : \mathbb{N}_0 \to \mathbb{R}_+$ such that $g(0) = 0$, $g(k) > 0$ for $k \geq 1$ and g is Lipschitz:

$$\sup_{k \geq 0} |g(k+1) - g(k)| < \infty.$$

- Jump rates: $p_n g(\eta(x))$ and $(1 - p_n) g(\eta(x+1))$, where $p_n = \frac{1}{2} + \frac{a}{2n^\gamma}$.

- Spectral gap condition: restrict the dynamics to configurations with k particles on a box of size ℓ, then if $W(k, \ell)$ denotes the inverse of the spectral gap, we need $E[(W(k,\ell))^2] \leq C\ell^4$.

Kinetically constrained (with M. Jara and S. Sethuraman)

- η_t is a Markov process with space state $\Omega = \{0,1\}^{\mathbb{Z}}$.

- here particles more likely hop to unoccupied nearest-neighbor sites when at least $m-1 \geq 1$ other neighboring sites are full.

- Jump rates: $p_n c^{m,n}_{x,x+1}(\eta)\eta(x)(1-\eta(x+1))$ and $(1-p_n)c^{m,n}_{x+1,x}(\eta)\eta(x+1)(1-\eta(x))$, where $p_n = \frac{1}{2} + \frac{a}{2n^\gamma}$ and

$$c^{m,n}_{x,x+1}(\eta) = c^{m,n}_{x+1,x}(\eta) = \sum_{k=1}^{m} \prod_{\substack{j=-(m-k) \\ j\neq 0,1}}^{k} \eta(x+j) + \frac{\theta}{2n},\ \theta > 0.$$

- For $m=2$, $c^{2,n}_{x,x+1}(\eta) = \left[\eta(x-1) + \eta(x+2) + \frac{\theta}{2n}\right]$.

Degenerate rates (with O. Blondel and M. Simon)

- The previous models with $\theta = 0$. For example, if $m = 2$, then

$$c^2_{x,x+1}(\eta) = \Big[\eta(x-1) + \eta(x+2)\Big].$$

- Existence of **blocked configurations**.

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- Boltzmann-Gibbs Principle works thanks to:

(1) the existence of a mobile cluster.

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(2) the probability to find a blocked configuration in a finite box is exponentially small in the size of the box.

Exclusion with long jumps (with M. Jara)

• We consider transition probabilities $p_n : \mathbb{Z} \to [0,1]$ with $p(0) = 0$ and given by $p_n(z) = s(z) + \gamma_n a(z)$, where:

- $s(z)$ is irreducible, with finite variance: $\sum_{z\in\mathbb{Z}} z^2 s(z) = \sigma^2 < \infty$.
- $a(z)$ satisfies: $|a(z)| \leq Cs(z)$, for any $z \in \mathbb{Z}$.
- $\gamma_n\sqrt{n} \to_{n\to\infty} b \neq 0$.

We get, for $\rho = 1/2$, the SBE given by

$$d\mathcal{Y}_t = \frac{\sigma^2}{2}\Delta\mathcal{Y}_t dt + bm\nabla(\mathcal{Y}_t)^2 dt + \sqrt{\frac{\sigma}{2}}\nabla d\mathcal{W}_t,$$

where $m = \sum_{z\in\mathbb{Z}} za(z)$.

Example: $s(z) = \frac{c}{|z|^{1+\beta}}$, $a(z) = sgn(z)s(z)$ and $\beta > 2$.

Exclusion with reservoirs (with C. Landim and A. Milanes)

(WEAK asymmetry) $\gamma = 1$, $\alpha \neq \beta$. Out of equilibrium and uses the microscopic Cole-Hopf.

Exclusion with reservoirs (with N. Perkowski and M. Simon)

(STRONG asymmetry), $\alpha = \beta = \frac{1}{2}$, $\gamma = 1/2$.

Future directions

• Derive SBE with other types of boundary conditions ?
• Models with several conserved quantities
(work in progress with C. Bernardin and M. Simon):

- Each one lives in its own time scale.
- The quantities depend on each other.
- The picture behind universality classes is richer that in systems with 1 conservation law.

• Obtain other singular SPDEs from IPS...

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Thank you and...