A PDE construction of the Euclidean Φ^4_3 quantum field theory

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joint work with Massimiliano Gubinelli

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- **probability measure on** $\mathcal{S}'(\mathbb{R}^3)$
  - given formally by

$$\nu(\mathrm{d}\varphi) \sim \exp\left\{-2\int_{\mathbb{R}^3} \frac{1}{2}|\nabla\varphi|^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4\right\} \prod_{x\in\mathbb{R}^3} \mathrm{d}\varphi(x) \qquad m^2>0,\ \lambda>0$$

  - non-Gaussian + Osterwalder–Schrader axioms
    - Euclidean invariance, reflection positivity, regularity, symmetry, cluster property
  - Schwinger-Dyson equations (integration by parts formula)

- success of the constructive quantum field theory program in the 60' – 80':
  - Glimm, Jaffe, Feldman, Osterwalder, . . . , Brydges, Fröhlich, Sokal, . . .

- Parisi–Wu '81 proposed to study the Φ^4 model

$$(\partial_t - \Delta + m^2)\varphi + \lambda\varphi^3 = \xi \qquad \text{on} \quad \mathbb{R}_+ \times \mathbb{R}^3$$

  - ξ a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^3$
  - ν is the **invariant measure**

- ξ space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^3$ - a random distribution of regularity $-5/2-\kappa$
- Schauder estimates - gain of 2 degrees of regularity
- $\Rightarrow$ φ is a distribution of regularity $-1/2-\kappa$ (renormalization needed to define φ^3)

GAIN: we have a dynamics & tools from PDEs, SPDEs & Gaussian noise

- $m^2 > 0$, $\lambda > 0$ arbitrary
- use the dynamics to construct the measure

$$\nu(\mathrm{d}\varphi) \sim \exp\left\{-2\int_{\mathbb{R}^3} \frac{1}{2}|\nabla\varphi|^2 + \frac{m^2}{2}\varphi^2 + \frac{\lambda}{4}\varphi^4\right\} \prod_{x\in\mathbb{R}^3} \mathrm{d}\varphi(x) \qquad \text{on} \qquad \mathcal{S}'(\mathbb{R}^3)$$

- non-Gaussian
- translation invariance, reflection positivity, regularity, symmetry
- integration by parts formula, Swinger–Dyson equations

Applies to: multicomponent models with $O(N)$ symmetry

Missing: rotation invariance, cluster property

- let $\Lambda_{M,\varepsilon} = (\varepsilon(\mathbb{Z}/M\mathbb{Z}))^3$ – periodic lattice of mesh size ε and side length M
- consider the **Gibbs measure** on $\mathbb{R}^{\Lambda_{M,\varepsilon}}$

$$\nu_{M,\varepsilon}(\mathrm{d}\varphi) \sim \exp\left\{-2\varepsilon^3 \sum_{\Lambda_{M,\varepsilon}} \left[\frac{1}{2}|\nabla_\varepsilon \varphi|^2 + \frac{m^2 - 3a_{M,\varepsilon} + 3b_{M,\varepsilon}}{2}|\varphi|^2 + \frac{1}{4}|\varphi|^4\right]\right\} \prod_{x \in \Lambda_{M,\varepsilon}} \mathrm{d}\varphi(x)$$

  - $a_{M,\varepsilon}, b_{M,\varepsilon}$ renormalization constants
  - translation invariant, reflection positive, ...

- study **stationary solutions** of (finite-dimensional) stochastic quantization equation

$$(\partial_t - \Delta_\varepsilon + m^2)\varphi + (-3a_{M,\varepsilon} + 3b_{M,\varepsilon})\varphi + \varphi^3 = \xi_{M,\varepsilon} \qquad \text{on} \quad \mathbb{R}_+ \times \Lambda_{M,\varepsilon}$$

- find **energy estimates** uniform in both parameters M, ε $\quad \Rightarrow \quad$ **tightness** of $\nu_{M,\varepsilon}$

- for a **nice** f consider

$$(\partial_t - \Delta + m^2)u + u^3 = f$$

- test the equation by u (= multiply by u and integrate wrt x)

$$\langle \partial_t u, u\rangle - \langle \Delta u, u\rangle + m^2\langle u, u\rangle + \langle u^3, u\rangle = \langle f, u\rangle$$

$$\frac{1}{2}\partial_t \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + m^2\|u\|_{L^2}^2 + \|u\|_{L^4}^4 \leqslant \|f\|_{L^2}\|u\|_{L^2} \leqslant C_\delta \|f\|_{L^2}^2 + \delta\|u\|_{L^2}^2$$

- if u **stationary** then

$$\frac{1}{2}\underbrace{\partial_t \mathbb{E}\|u(t)\|_{L^2}^2}_{=0} + \mathbb{E}\|\nabla u(t)\|_{L^2}^2 + (m^2 - \delta)\mathbb{E}\|u(t)\|_{L^2}^2 + \mathbb{E}\|u(t)\|_{L^4}^4 \leqslant C_\delta \mathbb{E}\|f(t)\|_{L^2}^2$$

- $\Rightarrow$ control of the **moments of the invariant measure** depending on f
- other bounds for $\langle f, u\rangle$ possible

$$\langle f, u\rangle \leqslant \|f\|_{L^{4/3}}\|u\|_{L^4}, \qquad \langle f, u\rangle \leqslant \|f\|_{H^{-1}}(\|u\|_{L^2} + \|\nabla u\|_{L^2})$$

- product of distributions are generally not well-defined
- Littlewood–Paley decomposition

$$f = \sum_{i \geqslant -1} \Delta_i f, \qquad g = \sum_{j \geqslant -1} \Delta_j g$$

- leads to

$$fg = \left(\sum_{i \geqslant -1} \Delta_i f \right) \left(\sum_{j \geqslant -1} \Delta_j g \right)$$

$$= \sum_{i,j:i<j-1} \Delta_i f \Delta_j g + \sum_{i,j:j<i-1} \Delta_i f \Delta_j g + \sum_{i,j:|i-j| \leqslant 1} \Delta_i f \Delta_j g$$

$$=: f \prec g + f \succ g + f \circ g$$

- the two paraproducts always well-defined
- the resonant term requires positive sum of the regularities

- let φ be a **stationary** solution to

$$(\partial_t - \Delta_\varepsilon + m^2)\varphi + (-3a + 3b)\varphi + \varphi^3 = \xi \qquad \text{on} \quad \mathbb{R}_+ \times \Lambda_{M,\varepsilon}$$

- **ansatz** $\varphi = X + \eta$ where $(\partial_t - \Delta_\varepsilon + m^2) \underbrace{X}_{-1/2-\kappa} = \underbrace{\xi}_{-5/2-\kappa}$ (stationary) gives

$$(\partial_t - \Delta_\varepsilon + m^2)\eta + 3b\varphi + \underbrace{[\![X^3]\!]}_{-3/2-\kappa} + 3\eta \underbrace{[\![X^2]\!]}_{-1-\kappa} + 3\eta^2 \underbrace{X}_{-1/2-\kappa} + \eta^3 = 0$$

- **instead of** removing X^{Ψ} where $(\partial_t - \Delta_\varepsilon + m^2)X^{\Psi} = -[\![X^3]\!]$
- **let** Y solve $(\partial_t - \Delta_\varepsilon + m^2)Y = -[\![X^3]\!] - 3(\Delta_{>L}[\![X^2]\!]) \succ Y$ (via fixed point)
- and **define** $\varphi = X + Y + \phi$ which leads to

$$(\partial_t - \Delta_\varepsilon + m^2)\phi + \phi^3 = -3[\![X^2]\!] \succ \phi - 3[\![X^2]\!] \circ \phi + \text{better (after renormalization)}$$

$$\frac{1}{2}\partial_t \|\phi\|_{L^{2,\varepsilon}}^2 + \|\phi\|_{L^{4,\varepsilon}}^4 + \langle \phi, (m^2 - \Delta_\varepsilon) \underbrace{\phi}_{1-\kappa} \rangle_\varepsilon$$

$$= \langle \phi, \underbrace{-3[\![X^2]\!] \succ \phi}_{-1-\kappa} \rangle_\varepsilon + \langle \phi, -3\underbrace{[\![X^2]\!]}_{-1-\kappa} \circ \phi \rangle_\varepsilon + \langle \phi, \text{better (after renormalization)} \rangle_\varepsilon$$

- **approximate duality**

$$\langle \phi, -3[\![X^2]\!] \circ \phi \rangle_\varepsilon - \langle -3[\![X^2]\!] \succ \phi, \phi \rangle_\varepsilon =: D(\phi, -3[\![X^2]\!], \phi)$$

bounded if the **sum** of the regularities of $\phi, -3[\![X^2]\!], \phi$ **positive!**

- combine with the Laplace term

$$\langle \phi, (m^2 - \Delta_\varepsilon)\phi + 2 \cdot 3[\![X^2]\!] \succ \phi \rangle_\varepsilon$$

- complete the square using **elliptic paracontrolled ansatz** (ψ is more regular than ϕ)

$$(m^2 - \Delta_\varepsilon)\psi := (m^2 - \Delta_\varepsilon)\phi + 3[\![X^2]\!] \succ \phi$$

- include a polynomial weight $\rho(x) = (1 + |x|^2)^{-\theta/2} \in L^4$ (= test by $\rho^4\phi$ instead of ϕ)
- denote $\mathbb{X}_{M,\varepsilon} = (X_{M,\varepsilon}, [\![X^2_{M,\varepsilon}]\!], X^{\Psi}_{M,\varepsilon}, \ldots)$
- we obtain – **uniformly** in M, ε

$$\frac{1}{2}\partial_t \|\rho^2\phi_{M,\varepsilon}\|^2_{L^{2,\varepsilon}} + \|\rho\phi_{M,\varepsilon}\|^4_{L^{4,\varepsilon}} + \|\rho^2\phi_{M,\varepsilon}\|^2_{H^{1-2\kappa,\varepsilon}} + \|\rho^2\psi_{M,\varepsilon}\|^2_{L^{2,\varepsilon}} + \|\rho^2\nabla_\varepsilon\psi_{M,\varepsilon}\|^2_{L^{2,\varepsilon}}$$

$$\leqslant (|\log t| + 1) Q_\rho(\mathbb{X}_{M,\varepsilon}).$$

- the resonant product $[\![X^2]\!] \circ \phi$ not controlled; $[\![X^2]\!] \circ \psi$ also not
- analogy with PDE **weak solutions** (equation interpreted in a suitable duality sense)
- cf. **strong solutions** approach Gubinelli, H. [arXiv:1804.11253]

- slightly higher regularity possible to recover the continuum equation
- $\Rightarrow$ integration by parts + Schwinger–Dyson equations

- recall
  - $\varphi_{M,\varepsilon} = X_{M,\varepsilon} + Y_{M,\varepsilon} + \phi_{M,\varepsilon}$ is stationary with law $\nu_{M,\varepsilon}$
  - $X_{M,\varepsilon}$ stationary, $Y_{M,\varepsilon}$ **not** stationary $\Rightarrow$ $\phi_{M,\varepsilon}$ **not** stationary
- let $\varphi_{M,\varepsilon} = X_{M,\varepsilon} + X^{\Psi}_{M,\varepsilon} + \zeta_{M,\varepsilon} \quad \Rightarrow \quad$ all components stationary

Theorem

- *The family of joint laws of* $(\varphi_{M,\varepsilon}, \mathbb{X}_{M,\varepsilon})$ *evaluated at some* $t \geqslant 0$ *is tight.*
- *Let* $\zeta := \varphi - X - X^{\Psi}$. *Then any limit measure* μ *satisfies for all* $p \in [1,\infty)$

$$\mathbb{E}_\mu \|\varphi\|^{2p}_{H^{-1/2-2\kappa}(\rho^2)} + \mathbb{E}_\mu \|\zeta\|^{2p}_{L^2(\rho^2)} + \mathbb{E}_\mu \|\zeta\|^2_{H^{1-2\kappa}(\rho^2)} + \mathbb{E}_\mu \|\zeta\|^4_{B^0_{4,\infty}(\rho)} < \infty.$$

Osterwalder–Schrader axioms and nontriviality

- denote ν the limit of $\nu_{M,\varepsilon}$ (projection of μ on the first component)

Nontriviality = non-Gaussianity *Show that the* ***connected four-point function***

$$U_{4,\nu}(x_1,\ldots,x_4) := \langle\varphi(x_1)\cdots\varphi(x_4)\rangle - \langle\varphi(x_1)\varphi(x_2)\rangle\langle\varphi(x_3)\varphi(x_4)\rangle$$

$$-\langle\varphi(x_1)\varphi(x_3)\rangle\langle\varphi(x_2)\varphi(x_4)\rangle - \langle\varphi(x_1)\varphi(x_4)\rangle\langle\varphi(x_2)\varphi(x_3)\rangle, \qquad x_1,\ldots,x_4 \in \mathbb{R}^3$$

is ***nonzero***

Idea:

- use the decomposition $\varphi = X + X^{\Psi} + \zeta$
- together with the moment estimates

- let $K_i * \cdot = \Delta_i \cdot$ be the Littlewood–Paley projector on the scale 2^i
- study $U_{4,\nu}$ convolved with (K_i, K_i, K_i, K_i) and evaluated at $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$

$$U_{4,\nu} * (K_i, \ldots, K_i)(0, \ldots, 0) = \langle (\Delta_i \varphi)^4(0) \rangle - 3\langle (\Delta_i \varphi)^2(0) \rangle^2 =: L(\varphi, \varphi, \varphi, \varphi)$$

- use the decomposition $\varphi = X + X^{\Psi} + \zeta$ to get

$$L(\varphi, \varphi, \varphi, \varphi) = L(X, X, X, X) + 4L(X, X, X, X^{\Psi}) + R$$

- show that

$$|R| \lesssim 2^{i(1/2 + 5\kappa)}$$

- whereas explicit computation yields

$$L(X, X, X, X^{\Psi}) \approx -2^i$$

Integration by parts and Schwinger–Dyson equations

$$\nu_{M,\varepsilon}(\mathrm{d}\varphi) \sim \exp\left\{-2\varepsilon^3 \sum_{\Lambda_{M,\varepsilon}} \left[\frac{1}{2}|\nabla_\varepsilon \varphi|^2 + \frac{m^2 - 3a_{M,\varepsilon} + 3b_{M,\varepsilon}}{2}|\varphi|^2 + \frac{1}{4}|\varphi|^4\right]\right\} \prod_{x \in \Lambda_{M,\varepsilon}} \mathrm{d}\varphi(x)$$

- F a cylinder functional on $\mathcal{S}'(\Lambda_{M,\varepsilon})$: $F(\varphi) = \Phi(\varphi(f_1), \ldots, \varphi(f_n))$
- **integration by parts** gives

$$\int \mathrm{D}F(\varphi)\nu_{M,\varepsilon}(\mathrm{d}\varphi) = 2\int F(\varphi)[\varphi^3 + (-3a_{M,\varepsilon} + 3b_{M,\varepsilon})\varphi + (m^2 - \Delta_\varepsilon)\varphi]\nu_{M,\varepsilon}(\mathrm{d}\varphi)$$

To pass to the limit:

- use the decomposition $\varphi = X + X^{\Psi} + \zeta$
- problematic
  - $[\![X^2]\!] \circ \zeta$ – not well-defined based on the energy estimates so far
  - $[\![X^3]\!]$ – only exists as a space-time distribution

- let $h\colon \mathbb{R}\to\mathbb{R}$ smooth with $\operatorname{supp} h\subset\mathbb{R}_+$ and $\int_{\mathbb{R}} h\,\mathrm{d}t=1$
- then letting $[\![\varphi^3]\!] := \varphi^3 + (-3a_{M,\varepsilon}+3b_{M,\varepsilon})\varphi$ we get

$$\int F(\varphi)[\![\varphi^3]\!]\nu_{M,\varepsilon}(\mathrm{d}\varphi) = \mathbb{E}[F(\varphi_{M,\varepsilon}(t))[\![\varphi^3_{M,\varepsilon}(t)]\!]] = \mathbb{E}\left[\int_{\mathbb{R}} h(t)F(\varphi_{M,\varepsilon}(t))[\![\varphi^3_{M,\varepsilon}(t)]\!]\,\mathrm{d}t\right]$$

- prove tightness of $(\varphi_{M,\varepsilon},\mathbb{X}_{M,\varepsilon})$

Theorem *Any limit measure ν satisfies*

$$\int \mathrm{D}F(\varphi)\nu(\mathrm{d}\varphi) = 2\int F(\varphi)[(m^2-\Delta)\varphi]\nu(\mathrm{d}\varphi) + 2\mathcal{J}(F),$$

$$\mathcal{J}(F) = \mathbb{E}\left[\int_{\mathbb{R}} h(t)F(\varphi(t))[\![\varphi^3]\!](t)\,\mathrm{d}t\right]$$

$$[\![\varphi^3]\!] = [\![X^3]\!] + 3[\![X^2]\!]\succ(-X^{\Psi}+\zeta) + 3[\![X^2]\!]\prec(-X^{\Psi}+\zeta)+\cdots.$$

- operator product expansion, Schwinger–Dyson equations follow

Thank you for your attention!