Selected topics on the dynamics of quantum walkers

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Prelude

Quantum walks have become popular lately...

- In quantum computing *(mostly in discrete time)*

The natural advanced tool for building quantum algorithms

The Turing machine of the quantum world

Review by S. E. Venegas-Andraca (2012)

- In Physics *(mostly in continuous time)*

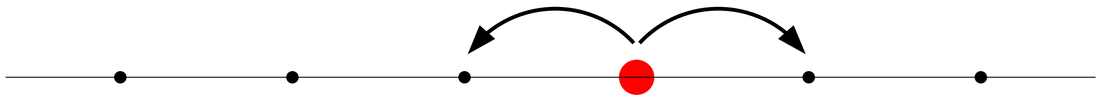
The simplest of all quantum-mechanical dynamical systems

Related to experiments manipulating coherent quantum states

e.g. cold-atom systems or photonic devices

The simplest of all quantum walks

A tight-binding particle hopping on a 1D lattice *(in continuous time)*



$$\mathrm{i}\,\frac{\mathrm{d}\psi_n(t)}{\mathrm{d}t} = \psi_{n+1}(t) + \psi_{n-1}(t)$$

Many properties have been investigated in many situations

I. Elementary properties *(Form of the wavefunction)*

II. One-body effects *(Effect of absorption)*

III. Many-body effects *(Bound-state dynamics)*

This talk will roughly follow the above setup

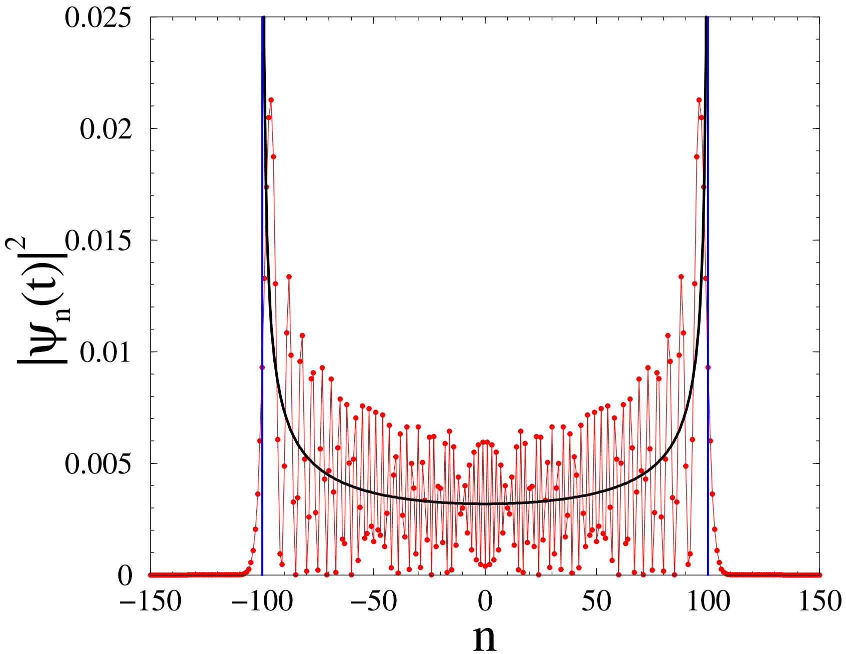
I. Form of the wavefunction

Assume the particle is launched from the origin

$$\mathrm{i}\,\frac{\mathrm{d}\psi_n(t)}{\mathrm{d}t} = \psi_{n+1}(t) + \psi_{n-1}(t) \quad \text{with} \quad \psi_n(0) = \delta_{n0}$$

Then

$$\psi_n(t) = \int \frac{\mathrm{d}q}{2\pi} \mathrm{e}^{\mathrm{i}nq - 2\mathrm{i}t\cos q} = \mathrm{i}^{-n} J_n(2t) \quad \text{(Bessel functions)}$$



$t = 50$

Ballistic spreading

- Ballistic growth of moments

$$\langle n^2 \rangle = 2t^2, \qquad \langle n^4 \rangle = 6t^4 + 2t^2$$

- Marginal recurrence *(dimension of trajectory = space dimension = 1)*

$$T_0(t) = \int_0^t \left|\psi_0(t')\right|^2 \mathrm{d}t' \approx \frac{\ln t}{2\pi}$$

- Dispersion relation $\omega(q) = 2\cos q, \quad v(q) = \mathrm{d}\omega(q)/\mathrm{d}q = -2\sin q$

  Maximum of $v(q)$ yields *spreading velocity* $V = 2$

  ⋆ *Allowed region* $(|n| < 2t)$

$$|\psi_n(t)|^2 \to \frac{1}{\pi\sqrt{4t^2 - n^2}}$$

  ⋆ *Ballistic peaks* $(n \approx \pm Vt \approx 2t)$

$$|n| = 2t + z t^{1/3}$$

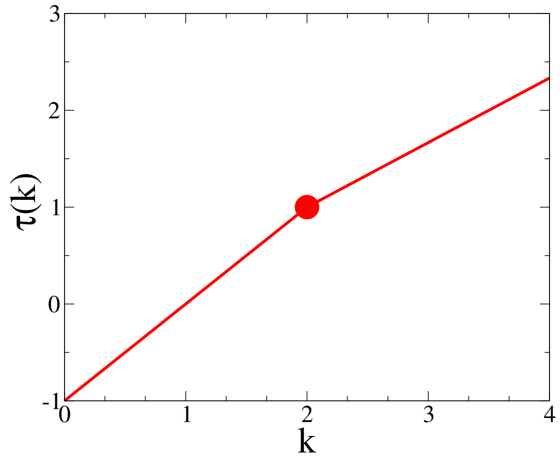
$$\psi_n(t) \approx \mathrm{i}^{-n} t^{-1/3} \mathrm{Ai}(z) \quad \text{(Airy function)}$$

A measurable consequence: *Bifractality*

Dynamical IPRs (inverse participation ratios)

$$I_k(t) = \sum_n |\psi_n(t)|^{2k} \sim t^{-\tau(k)}$$

$$\tau(k) = \begin{cases} k-1 & \text{for } k < 2 \\ \frac{2k-1}{3} & \text{for } k > 2 \end{cases}$$



- *Normal scaling* $(k < 2)$ *due to allowed region*
- *Anomalous scaling* $(k > 2)$ *due to ballistic peaks*
- *Usual IPR* $(k = 2)$ *has logarithmic correction:* $I_2(t) \approx \frac{\ln t}{2\pi^2 t}$

Bifractality persists in the presence of weak diagonal disorder

$$\mathrm{i}\,\frac{\mathrm{d}\psi_n(t)}{\mathrm{d}t} = \psi_{n+1}(t) + \psi_{n-1}(t) + V_n\psi_n(t)$$

$$\langle V_n\rangle = 0, \qquad \langle V_m V_m\rangle = w^2\delta_{mn} \qquad (w \ll 1)$$

- Very same bifractal scaling for asymptotic IPRs $\quad (t \gg \xi_0 \sim 1/w^2)$

  $$I_k(\infty) \sim w^{2\tau(k)}$$

  Due to anomalous band-edge scaling (Halperin, Derrida-Gardner)

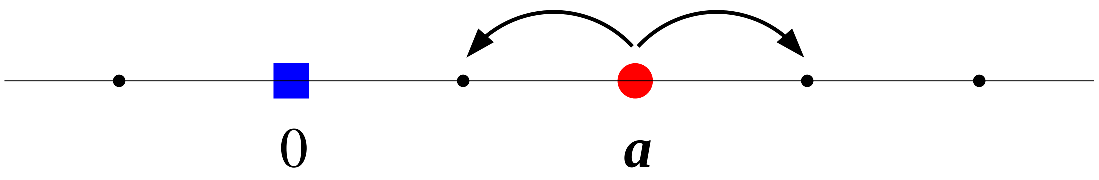
- Scaling law in crossover regime

  $$I_k(t) \approx t^{-\tau(k)} F_k(w^2 t)$$

S. de Toro Arias and JML (1998)

II. Effect of absorption

Warming up: a single trap at the origin



- *Trap modelled as optical potential of strength* γ

$$\mathrm{i}\,\frac{\mathrm{d}\psi_n(t)}{\mathrm{d}t} = \psi_{n+1}(t) + \psi_{n-1}(t) - \mathrm{i}\gamma\delta_{n0}\,\psi_n(t) \quad \text{with} \quad \psi_n(0) = \delta_{na}$$

- *Non-unitary evolution*

- *Survival probability*

$$\Pi(t) = \sum_n |\psi_n(t)|^2 = 1 - 2\gamma \int_0^t |\psi_0(t')|^2\,\mathrm{d}t'$$

Non-trivial asymptotic survival (i.e., escape) probability

$$\Pi_\infty = 1 - 2\gamma \int_0^\infty |\psi_0(t)|^2 \, \mathrm{d}t \; > 0$$

- Purely quantum effect

In classical case (2D Brownian particle): $\quad \Pi(t) \approx \dfrac{2\pi a^2}{\ln t}$

- Explicit expression as a function of initial distance a

(Green's function techniques)

$$\Pi_\infty = 1 - \frac{4\gamma}{\pi} \left[\int_0^{\pi/2} \frac{\sin\theta \, \mathrm{d}\theta}{(\gamma + 2\sin\theta)^2} + \int_0^\infty \frac{\sinh\theta \, \mathrm{d}\theta}{\gamma^2 + 4\sinh^2\theta} \mathrm{e}^{-2a\theta} \right]$$

Features of asymptotic survival (i.e., escape) probability

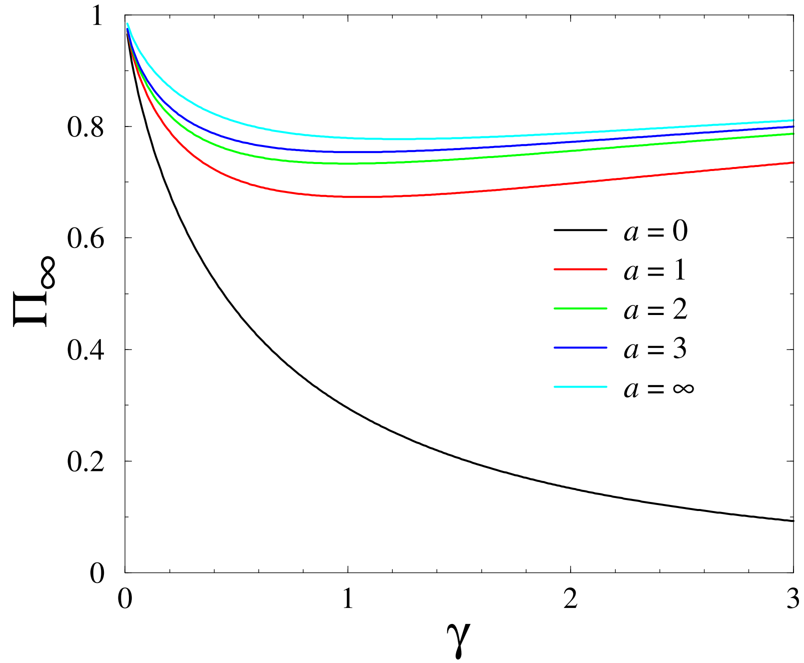
- Monotonic behavior for $a = 0$

$$\Pi_\infty \approx \frac{1}{2\gamma^2}$$

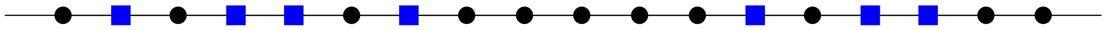
- Non-monotonic behavior for $a > 0$

Paradoxical transparency $(\gamma \to \infty)$

$$\Pi_\infty \approx 1 - \frac{16a^2}{(4a^2 - 1)\pi\gamma}$$



A finite concentration of traps



$$\mathrm{i}\,\frac{\mathrm{d}\psi_n(t)}{\mathrm{d}t} = \psi_{n+1}(t) + \psi_{n-1}(t) - \mathrm{i}\gamma\varepsilon_n\,\psi_n(t)$$

$$\varepsilon_n = \begin{cases} 1 & \text{(trap)} \quad \text{with prob.} \quad c \\ 0 & \text{(no trap)} \quad \text{with prob.} \quad 1-c \end{cases}$$

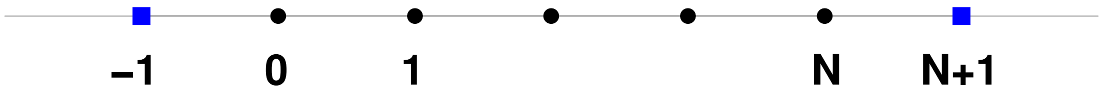
Decay of mean survival probability $\Pi(t)$ *?*

Parris, Edwards & Parris (1989)

KLM (2014)

Approach à la Lifshitz

- *Cluster of* $N+1$ *sites without traps*



- *Stationary problem*

$$\psi_n(t) = \left(A\,\mathrm{e}^{inq} + B\,\mathrm{e}^{-inq}\right)\mathrm{e}^{-\mathrm{i}Et}$$

$$E = 2\cos q \quad (E \text{ and } q \text{ complex})$$

- *Boundary conditions:* $\psi_{-1} = Y_L\psi_0, \qquad \psi_{N+1} = Y_R\psi_N$

- *Lowest mode of large cluster*

$$q_1 = \frac{\pi}{N}\left(1 + \frac{\alpha}{N} + \cdots\right) \qquad \alpha = \frac{1}{Y_L - 1} + \frac{1}{Y_R - 1}$$

- Decay rate

$$\lambda = -2\,\mathrm{Im}\,E_1 \approx \frac{4\pi^2}{N^3}\,\mathrm{Im}\,\alpha$$

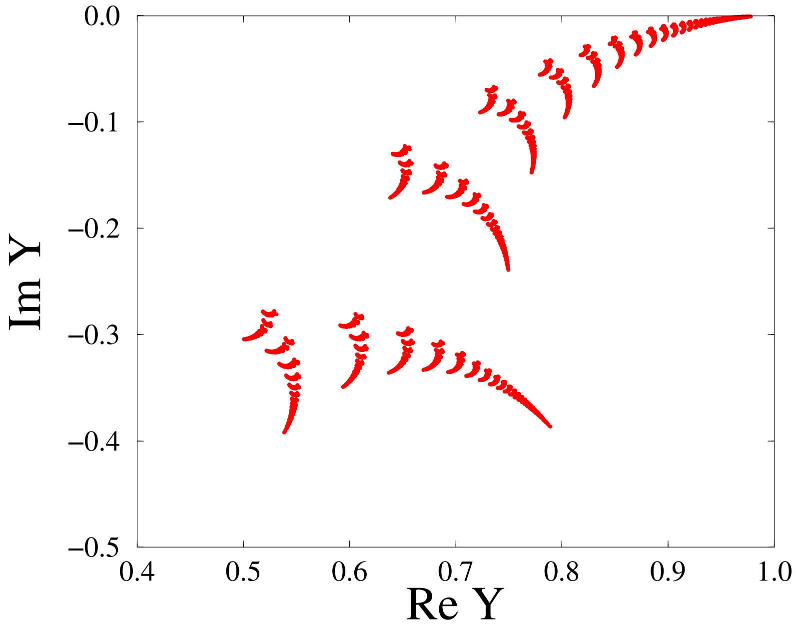
scales as $1/N^3$ and fluctuates (i.e., depends on b.c.)

How is Im α distributed ?

Clue from transfer-matrix approach to 1D disordered systems:

Consider Riccati variables $Y_n = \dfrac{\psi_n}{\psi_{n+1}}$ at band edge ($E = 2$)

- Obey random recursion $Y_n = \dfrac{1}{2 + i\gamma\varepsilon_n - Y_{n-1}}$
- Have invariant distribution whose support is complex fractal



$\gamma = \frac{1}{2}$

- Boundary parameters Y_L and Y_R are independent and drawn from the above law

To conclude

- Large cluster $(N \gg 1)$ has probability $(1-c)^N$ and **optimal** rate $\dfrac{8\pi^2}{N^3}\dfrac{f(\gamma)}{\gamma}$

  $$f(\gamma) = 2\gamma \min \operatorname{Im} \alpha$$

  Minimum taken over fractal invariant set

- Average survival probability on clusters of size N

  $$\Pi(t) \sim \sum_N \exp\left(-\frac{8\pi^2}{N^3}\frac{f(\gamma)}{\gamma}t - |\ln(1-c)|N\right)$$

- Saddle point (**optimal** cluster size)

  $$N \approx \left(24\pi^2 \frac{f(\gamma)}{\gamma}\frac{t}{\ln(1-c)}\right)^{1/4}$$

- Stretched exponential decay à la Lifshitz

  $$\Pi(t) \sim \exp\left[-\frac{4}{3}\left(24\pi^2\frac{f(\gamma)}{\gamma}|\ln(1-c)|^3 t\right)^{1/4}\right]$$

A glimpse into the higher-dimensional situation

- Classical diffusing particles

  Lifshitz (1964), Balagurov & Vaks (1974), Donsker and Varadhan (1975)

$$\Pi(t) \sim \exp\left[-(d+2)\left(\frac{\Omega^2}{4d^d} j^{2d} |\ln(1-c)|^2 t^d\right)^{1/(d+2)}\right]$$

  In $(d+2)$, 2 *is dimension of Brownian trajectory*

  Dependence on parameters explicit and simple

- Quantum walkers

  Parris, Edwards & Parris (1989), KLM (2014)

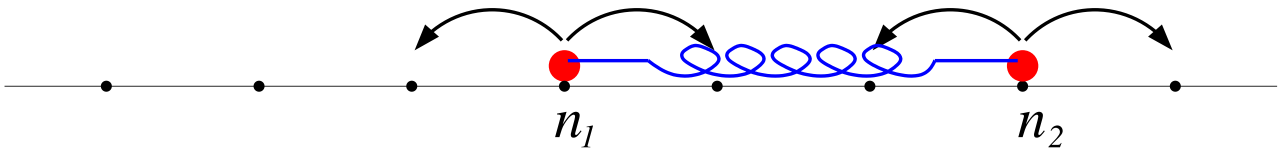
$$\Pi(t) \sim \exp\left[-(d+3)\left(\frac{\Omega^3}{27d^d} A^d |\ln(1-c)|^3 t^d\right)^{1/(d+3)}\right]$$

  In $(d+3)$, 3 *is* **not** *dimension of ballistic trajectory*

  Optimization on b.c. yields non-trivial A

III. Bound state dynamics

- Two identical (fermionic or bosonic) particles at sites $n_1 = n$ and $n_2 = n + m$
- Bound by confining potential W_m



$$i \frac{d\psi_{n,m}(t)}{dt} = \psi_{n,m-1}(t) + \psi_{n+1,m-1}(t) + \psi_{n-1,m+1}(t) + \psi_{n,m+1}(t) + W_m \psi_{n,m}(t)$$

- Basis of plane-wave solutions

$$\psi_{n,m}(t) = e^{i(q n_{\text{com}} - \omega t)} \phi_m$$

- Internal wavefunction ϕ_m is **dispersive**

$$\omega \phi_m = 2 \cos \frac{q}{2} (\phi_{m-1} + \phi_{m+1}) + W_m \phi_m$$

Depends on momentum q *conjugate to* $n_{\text{com}} = \dfrac{n_1 + n_2}{2}$

No separation of center-of-mass dynamics...

Hard-wall potential

$$W_m = \begin{cases} 0 & \text{for } |m| \leq L \\ +\infty & \text{for } |m| > L \end{cases}$$

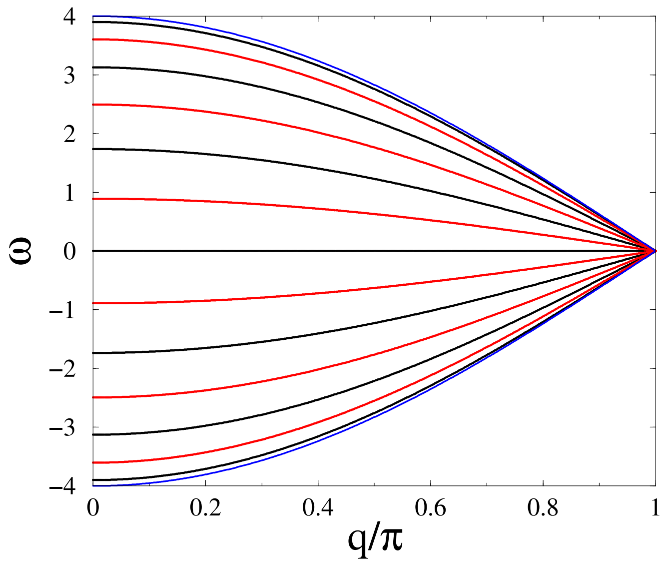
- Relative co-ordinate takes $2L+1$ values: $m = -L, \ldots, L$
- Dispersion relation $\omega(p,q) = 4\cos p \cos \frac{q}{2}$
- Quantization of internal momentum p depends on **statistics**

Bosons (ϕ_m even, *black*)

$$p_k^{(\mathrm{B})} = \frac{(k+\frac{1}{2})\pi}{L+1} \quad (k = 0, \ldots, L)$$

Fermions (ϕ_m odd, *red*)

$$p_k^{(\mathrm{F})} = \frac{k\pi}{L+1} \quad (k = 1, \ldots, L)$$



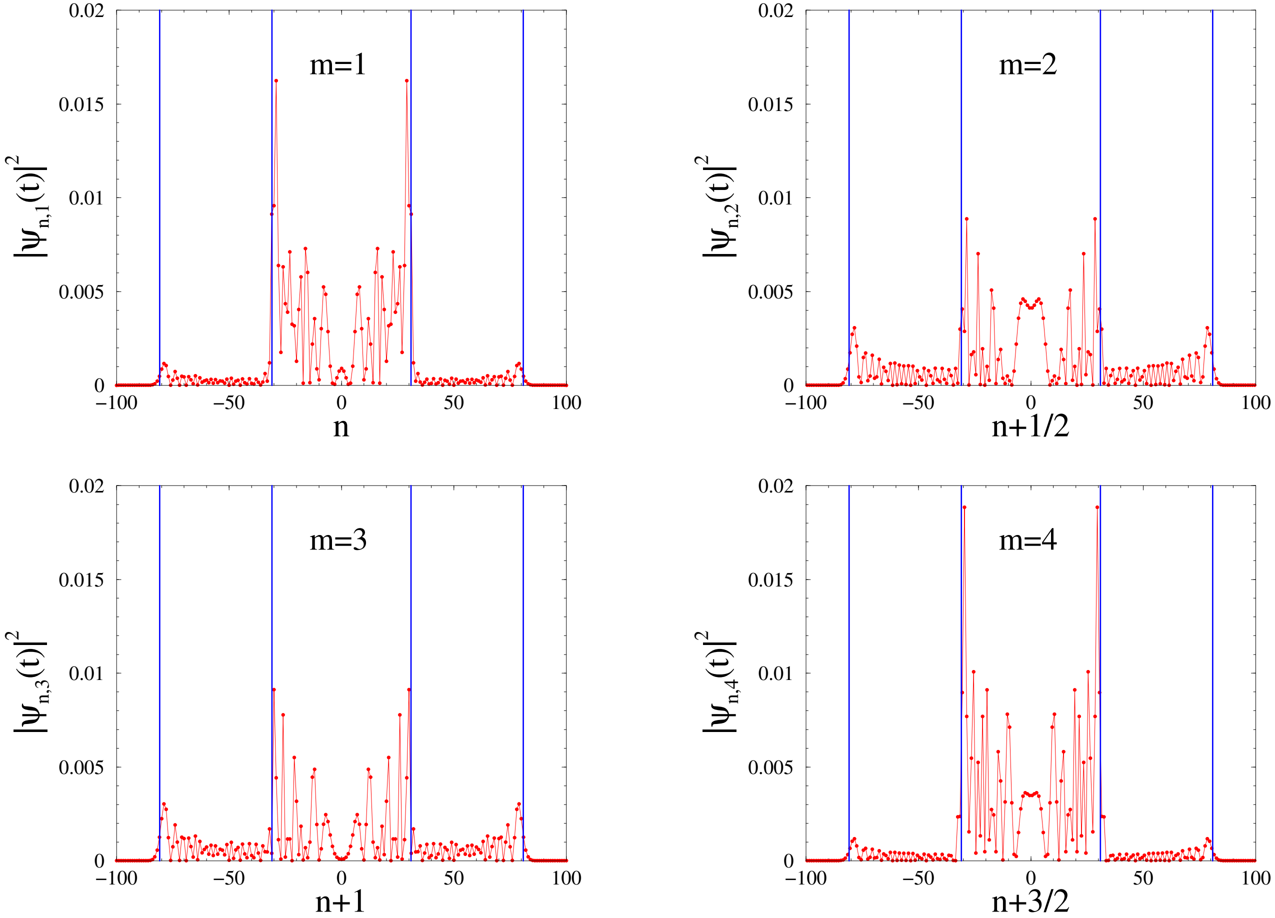
$L = 6$

In the time domain

Each dispersive branch yields a ballistic peak

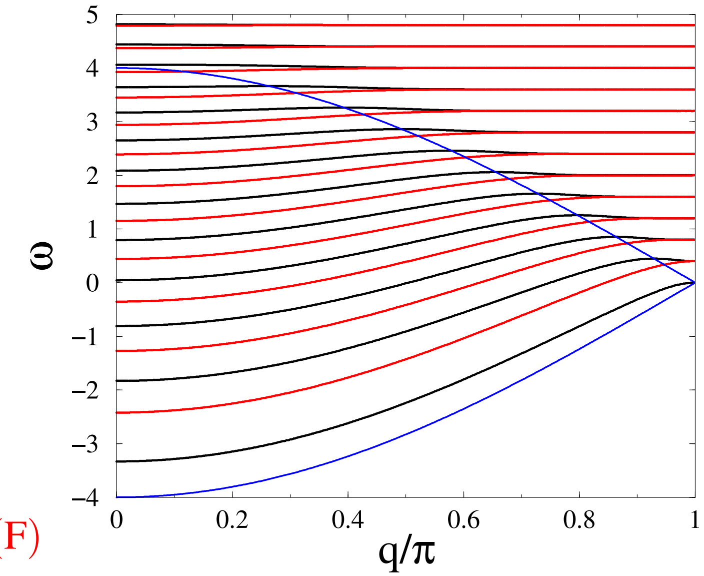
$$V_k^{(\mathrm{B})} = 2\cos\frac{(k+\frac{1}{2})\pi}{L+1}, \qquad V_k^{(\mathrm{F})} = 2\cos\frac{k\pi}{L+1}$$

Example: $L = 4,$ 2 fermions launched from sites 0 and 1



Power-law potential

$$W_m = g\,|m|^{\alpha}$$



- The lowest bands are the most dispersive

$$\alpha = 1, \;\; g = 0.4$$

- Spreading (maximal) velocities $V^{(\mathrm{B})}$ and $V^{(\mathrm{F})}$

  correspond to lowest band in each sector

- Scaling in weak-potential regime $(g \to 0)$

$$V^{(\mathrm{B})} \approx 2 - C^{(\mathrm{B})} g^{2/(3+\alpha)}, \qquad V^{(\mathrm{F})} \approx 2 - C^{(\mathrm{F})} g^{2/(3+\alpha)}$$

Amplitudes $C^{(\mathrm{B})}$ *and* $C^{(\mathrm{F})}$ *universal (depend only on* α*)*

To sum up...

- *Quantum walks are among the simplest of all quantum dynamical systems*
- *Many situations can be studied, often by analytical means*
- *Quite a few surprising quantum features without classical analogues*

I. Form of the wavefunction

Ballistic peaks. Bifractality

II. Effect of absorption

Paradoxical transparency as $\gamma \to \infty$

Fluctuating N^3 *scaling of decay rate. Non-trivial Lifshitz law*

III. Bound-state dynamics

No separation of center-of-mass dynamics

Many internal ballistic peaks besides two extremal ones

References

• P.L. Krapivsky and J.M. Luck, *Dynamics of a quantum particle in low-dimensional disordered systems with extended states*, J. Stat. Mech. P02031 (2011) [arXiv:1101.2101].

• P.L. Krapivsky, J.M. Luck and K. Mallick, *Survival of classical and quantum particles in the presence of traps*, J. Stat. Phys. **154**, 1430–1460 (2014) [arXiv:1311.6298].

• P.L. Krapivsky, J.M. Luck and K. Mallick, *Interacting quantum walkers: Two-body bosonic and fermionic bound states*, J. Phys. A **48**, 475301 (2015) [arXiv:1507.01363].

• P.L. Krapivsky, J.M. Luck and K. Mallick, *Quantum centipedes: collective dynamics of interacting quantum walkers*, J. Phys. A **49**, 335303 (2016) [arXiv:1603.04601].

• P.L. Krapivsky, J.M. Luck and K. Mallick, *Quantum return probability of a system of N non-interacting lattice fermions*, J. Stat. Mech. 023104 (2018) [arXiv:1710.08178].