How much can the eigenvalues of a random Hermitian matrix fluctuate?

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Based on joint work with T. Claeys, B. Fahs, and G. Lambert

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Play a role in various disciplines and fields

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We will focus on Hermitian matrices.

The GUE and its eigenvalues

An important prototype model in random matrix theory is the **Gaussian Unitary Ensemble** (GUE):

$$H = (H_{i,j})_{i,j=1}^N \qquad H^* = H,$$

$$(H_{i,i})_{i=1}^N \quad i.i.d. \quad \mathcal{N}(0, 1/(4N)) \qquad (H_{i,j})_{i<j} \quad i.i.d. \quad \mathcal{N}_{\mathbb{C}}(0, 1/(4N)).$$

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Fact

The density of the law of the eigenvalues $(\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$ *of a GUE(N) matrix is given by*

$$\frac{1}{Z_N} \prod_{1 \le i < j \le N} |\lambda_i - \lambda_j|^2 \prod_{j=1}^{N} e^{-2N\lambda_j^2}$$

Wigner's theorem

One of the most classical results in random matrix theory is Wigner's theorem.

Theorem (E. Wigner 1950s)

For bounded continuous $f : \mathbb{R} \to \mathbb{R}$ *and* $\lambda_1, ..., \lambda_N$ *the eigenvalues of a GUE(N) matrix,*

$$\frac{1}{N}\sum_{j=1}^{N} f(\lambda_j) \xrightarrow{\mathbb{P}} \int_{-1}^{1} f(x)\frac{2}{\pi}\sqrt{1-x^2}dx.$$

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Interpretation: for large N, eigenvalues accumulate on $[-1, 1]$ and are distributed according to the semicircle law $\frac{2}{\pi}\sqrt{1-x^2}dx$.

Wigner's theorem

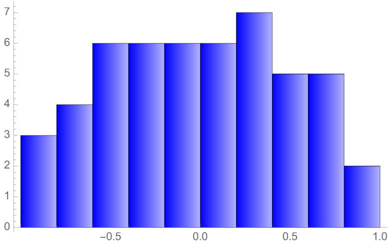

Figure: Histogram for eigenvalues of a 50×50 GUE matrix

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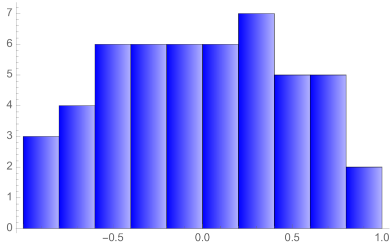

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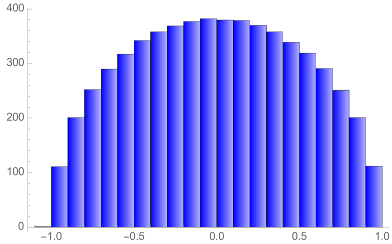

Figure: Histogram for eigenvalues of a 6000×6000 GUE matrix

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Theorem (Gustavsson 2005)

For any fixed $\epsilon > 0$, $j \in (\epsilon N, (1 - \epsilon)N)$ and $\lambda_1 < \lambda_2 < ... < \lambda_N$ the ordered eigenvalues of a GUE(N) matrix,

$$2\sqrt{2}\sqrt{1 - \kappa_j^2}\frac{N}{\sqrt{\log N}}(\lambda_j - \kappa_j) \xrightarrow{d} \mathcal{N}(0, 1)$$

as $N \to \infty$.

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For any fixed $\epsilon, \delta > 0$, there exists a $c > 0$:

$$\mathbb{P}\left(\exists j \in (\epsilon N, (1-\epsilon)N) : \quad |\lambda_j - \kappa_j| \geq N^{-1+\delta}\right) \leq e^{-N^c}.$$

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- In fact, some results replace N^δ by $(\log N)^A$ for some fixed A.
- Can one do better?

Optimal (bulk) rigidity

Theorem (Claeys, Fahs, Lambert, W 2019)

For any $\epsilon > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(1-\epsilon \le 2\frac{N}{\log N} \max_{j=1,\dots,N} \sqrt{1-\kappa_j^2}|\lambda_j - \kappa_j| \le 1+\epsilon\right) = 1.$$

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- Heuristically: if in Gustavsson's theorem one could replace $2\sqrt{2}\frac{N}{\sqrt{\log N}}\sqrt{1-\kappa_j^2}(\lambda_j-\kappa_j)$ by i.i.d. standard Gaussians, one would get this result from classical extreme value statistics results (max of N i.i.d. standard Gaussians $\sim \sqrt{2\log N}$).

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- Unfortunately, proof much more involved than this.

From eigenvalue fluctuations to the counting function

Instead of $(\lambda_j - \kappa_j)_{j=1}^N$, it is more convenient to analyze the eigenvalue counting function

$$h_N(x) = \sqrt{2}\pi \left(\sum_{j=1}^{N} \mathbf{1}\{\lambda_j \leq x\} - N \int_{-1}^{x} \frac{2}{\pi} \sqrt{1 - t^2} dt \right).$$

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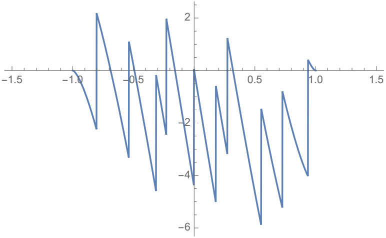
Turns out to be sufficient to understand $\max_{x\in[-1,1]} h_N(x)$ and $\min_{x\in[-1,1]} h_N(x)$.

What kind of beast is the counting function?



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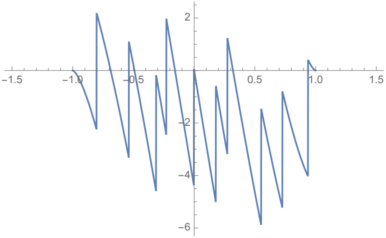

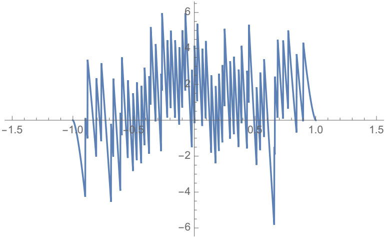

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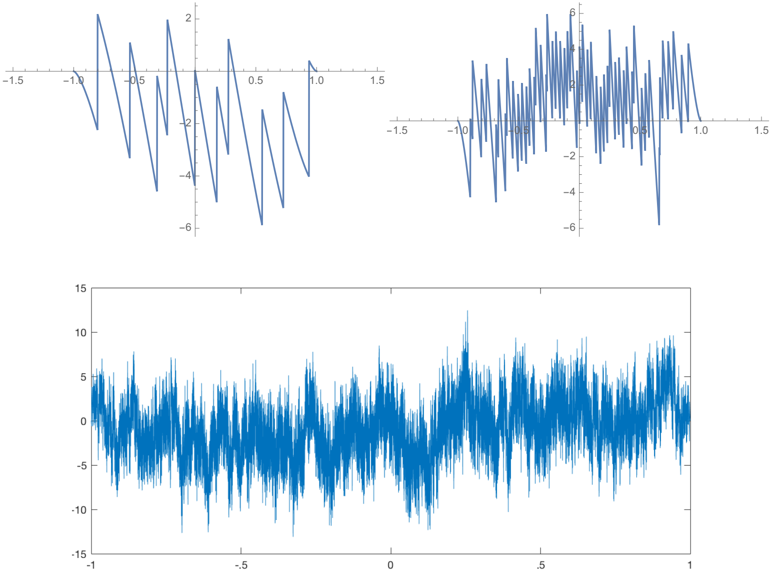

Figure: Realizations of GUE counting functions for $N = 10, 50, 6000$

Facts about the counting function

Theorem (Johansson 1998, ...)

For any $f \in \mathcal{C}_c^\infty(\mathbb{R})$, as $N \to \infty$,

$$\int h_N(x) f(x) dx \xrightarrow{d} \mathcal{N}(0, \sigma_f^2)$$

with

$$\sigma_f^2 = \int_{(-1,1)^2} f(x) f(y) \log \frac{1 - xy + \sqrt{1-x^2}\sqrt{1-y^2}}{|x-y|} dxdy.$$

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Interpretation: as $N \to \infty$, h_N converges to a Gaussian process with covariance

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$C(x,x) = \infty$ – what kind of Gaussian process has infinite variance?

Log-correlated fields

Random generalized functions X on $\Omega \subset \mathbb{R}^d$ with

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Play a role in many models.

- In stat mech, describe fluctuations of the height function in the dimer model (see Chelkak's talk tomorrow), growth models, ...
- In number theory, describes behavior of $\log|\zeta(\frac{1}{2} + i\omega T + ix)|$, where $\omega \sim$Unif$[0,1]$ and $T \to \infty$.
- Describe fluctuations of correctors in 2d stochastic homogenization.
- Important in SLE, construction of Liouville field theory, and random conformal welding (Duplantier-Sheffield; David, Kupiainen, Rhodes, Vargas;...)
- See the other talks of today ...

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Hope: perhaps h_N is kind of like X_ϵ for a suitable $\epsilon = \epsilon(N)$, and maybe similar ideas could work.

Exponential moment estimates

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Examples of required estimates (through asymptotic analysis of Riemann-Hilbert problems)

Proposition (Charlier 2018)

For fixed $x \in (-1, 1)$ *and* $\gamma \in [-\sqrt{2}, \sqrt{2}]$

$$\mathbb{E}e^{\gamma h_N(x)} = (1 + o(1))F(\gamma)N^{\frac{\gamma^2}{2}}(1 - x^2)^{\frac{3\gamma^2}{4}},$$

for a suitable $F(\gamma)$*, and for fixed* $x, y \in (-1, 1)$ *with* $x \neq y$ *and* $\gamma \in [-\sqrt{2}, \sqrt{2}]$*,*

$$\frac{\mathbb{E}e^{\gamma h_N(x)+\gamma h_N(y)}}{\mathbb{E}e^{\gamma h_N(x)}\mathbb{E}e^{\gamma h_N(y)}} = (1 + o(1))\left(\frac{1 - xy + \sqrt{1 - x^2}\sqrt{1 - y^2}}{|x - y|}\right)^{\gamma^2/2}.$$

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Actually need more complicated things: asymptotics for $|x-y| \to 0$, $x \to \pm 1$ at a suitable rate and some more complicated quantities.

Proof of an upper bound

One part of the proof of our main theorem is the following fact:

Proposition

For any $\epsilon, \delta > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{x\in(-1+\epsilon,1-\epsilon)} h_N(x) \geq (\sqrt{2}+\delta)\log N \right) = 0.$$

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- Choosing $\alpha = \sqrt{2}$ yields the claim.

□

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