Non Gaussian Log Correlated fields in RMT

Ofer Zeitouni
Based on joint works with Elliot Paquette, Nick Cook, Fanny Augeri and Raphael Butez

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Asymptotically Gaussian fields in random matrix theory

X_N - random Wigner matrix, e.g. GUE/GOE. in real case, centered independent entries on and above diagonal, variance $1/N$ off diagonal, $2/N$ on diagonal.

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Central limit theorem

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Central limit theorem $f : \mathbb{R} \to \mathbb{R}$ compactly supported, smooth. Consider

$$W_{f,N} = \sum_{i=1}^{N} f(\lambda_i) - N \int f d\sigma.$$

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Theorem (Johansson '98; β ensembles)

$W_{f,N}$ satisfies CLT, mean $(2/\beta - 1)\int f d\nu$, variance

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CLT's of this type go back at least to CLT of Jonsson for moments ('82), Pastur and co-workers, Bai-Silverstein, Shcherbina, ...

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Our basic object of interest: $\log(P_N(z)) = \log(\det(zI - X_N))$.

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If it is log-correlated, what about the extrema?

CUE char poly

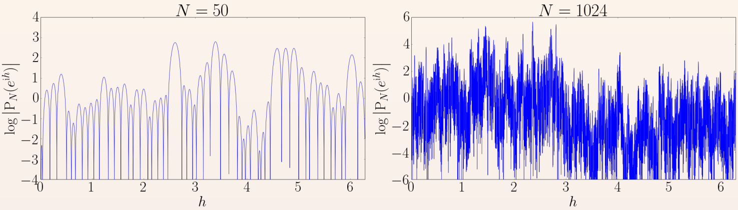

Figure 1: Realizations of $\log |\mathrm{P}_N(e^{ih})|$, $0 \le h < 2\pi$, for $N = 50$ and $N = 1024$. At microscopic scales, the field is smooth away from the eigenvalues, in contrast with the rugged landscape at mesoscopic and macroscopic scales.

(From Arguin, Belius, Bourgade '17)

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where W has the law of the sum of two independent Gumbels.

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Both use in essential way CUE (aka $\beta = 2$), where joint distribution of eigenvalues is

$$\prod_{i<j} |\lambda_i - \lambda_j|^2$$

for which Gaussianity of traces follows from Diaconis-Shashahani and moments ofdeterminant (=exponential moments of $M_N(z)$) are Toeplitz determinants.

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The clincher:

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$\alpha_k = B_k e^{2\pi i \theta_k}$, $EB_k^2 \sim 2/\beta k$, beta variable. $\alpha_k \sim g_k + i g_k'$, Gaussian.

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In addition, $\sup_{|z|=1} |\log|M_N(z)| - \log|\Phi^*_k(z)||$ is tight.

Recursions in the lab

$$\log \Phi_k^*(e^{i\theta}) - \log \Phi_{k-1}^*(e^{i\theta}) = \log(1 - \alpha_j e^{i\Psi_{k-1}}) \sim -\alpha_j e^{i\Psi_{k-1}(\theta)}$$

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Work in progress: Paquette-Z ('20?) Convergence in law of $\max \log|\Phi_N^*(e^{i\theta})|$ to Gumbel shifted by (unknown) r.v..

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where $C_\ell =$ number of cycles of length ℓ, essentially Poisson.

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Theorem (Cook-Z '17)

$M_N^*/\log N \to x_0 \sim 0.65$

Max asymptotics not determined simply by tail of (1), which would give $x_0 = 1$.

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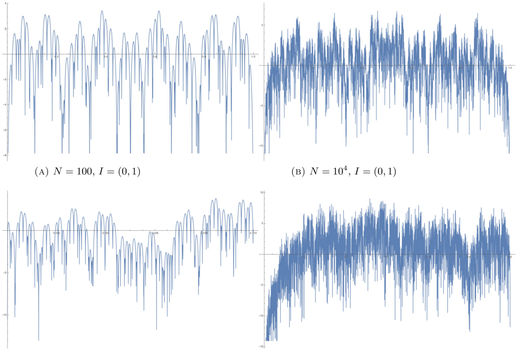
Theorem (Cook-Z '17)

$M_N^*/\log N \to x_0 \sim 0.65$

Max asymptotics not determined simply by tail of (1), which would give $x_0 = 1$. In fact, expect Gaussian fluctuations of M_N^* due to fluctuations in total number of cycles, so any hope for restoring log-cor story is by conditioning on it.

Random permutation char poly



(A) $N = 100$, $I = (0,1)$ (B) $N = 10^4$, $I = (0,1)$

(C) $N = 10^4$, $I = (0.1, 0.11)$ (D) $N = 10^9$, $I = (0, 0.6)$

FIGURE 1. Simulations of the field X_N on subintervals $I \subset \mathbb{T}$, computed from the cycle structures for the permutations P_N using the formula (2.2)). The cycle structures are random partitions of $[N]$ generated using the Chinese restaurant process. The respective partitions are:

(A): $\{56, 22, 9, 9, 4\}$,
(B/C): $\{6310, 1914, 909, 668, 79, 47, 33, 19, 12, 5, 3, 1\}$,
(D): $\{892060223, 78087020, 19479718, 9152317, 630684, 352623, 114502, 104059, 8973, 8193, 1641, 33, 5, 3, 2, 2, 1, 1\}$.

In (D) there are noticeable dips in the field near the rationals 0, 1/2, and 1/3.

(From Cook, Zeitouni ’17)

Towards the wild: GβE

We take $X_N \sim$ GβE, ie joint distribution of eigenvalues on $\mathbb{R}^N$

$$\prod_{i<j} |\lambda_i - \lambda_j|^\beta e^{-\beta \frac{N}{4} \sum \lambda_i^2}$$

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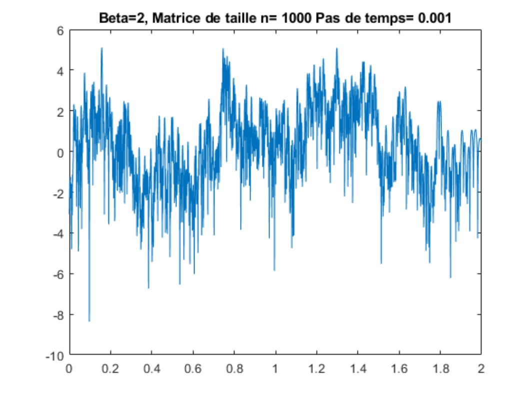
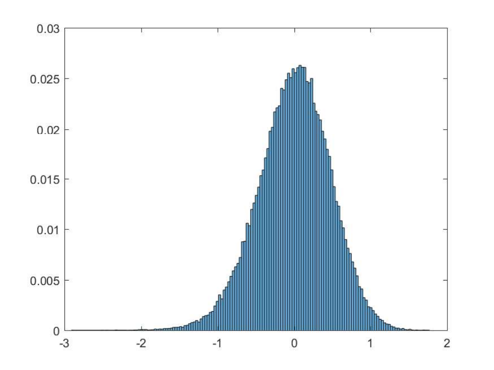
For general β: even CLT of log-det not clear!

log-det trajectory



Empirical facts



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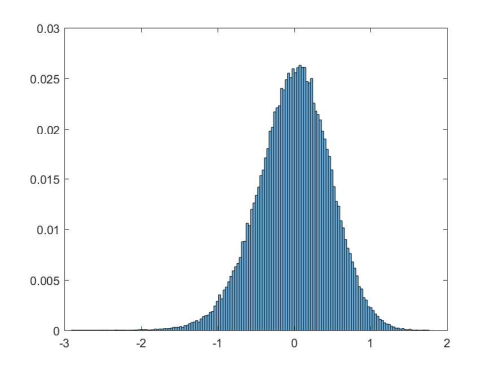
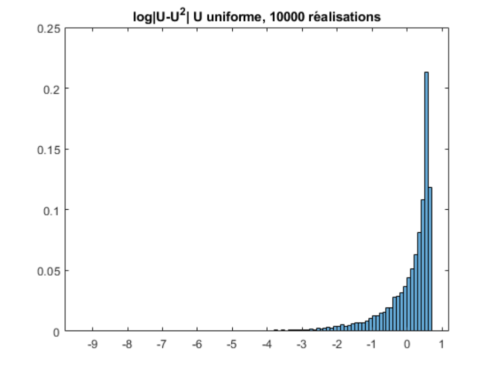


Skewed?

Reason for skewness in simulations



CLT for log determinant GβE

The case $z = 0$ is special.

Theorem (Tao-Vu '11)

$(M_N(0) - N \int \log|z-x|\sigma(dx) - a_\beta \log N)/\sqrt{\log N}$ converges (for Wigner matrices, 4 matching moments) to standard Gaussian.

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Bourgade-Mody '19: extends w/out matching 4 moments.
By replacement principle, the key step in the TV proof is the result for GβE, $\beta = 1, 2$. Their proof extends to general $\beta > 0$, and is based on recursions.

The Dumitriu-Edelman representation

Theorem (Dumitriu-Edelman '05)

X_N from $G\beta E$ is unitarily equivalent to the following 3-diagonal Jacobi matrix

$$\frac{1}{\sqrt{N}} X_N = \frac{1}{\sqrt{N}} \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \mathbf{0} & a_{N-1} & b_N \end{pmatrix}$$

where $b_i \sim N(0, \sqrt{2/\beta})$, $a_i \sim \chi_{i\beta}/\sqrt{\beta}$.

Here $a_i \sim \chi_{i\beta}/\sqrt{\beta}$ means a_i^2 has chi-square distribution with $i\beta$ degrees of freedom, ie $\chi_{i\beta} \sim \sqrt{i} + \sqrt{1/2\beta} G + O(1/i)$.

Recursions

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We set

$$\Psi_k(z) = \phi_k(z\sqrt{N})\frac{e^{-k\beta U(z\sqrt{N/k})+ck}}{\sqrt{k!}}$$

and then

$$\Psi_k(z) = (z\sqrt{N} - b_k)\Delta_k\Psi_{k-1}(z) - a_{k-1}^2\Delta_k\Delta_{k-1}\Psi_{k-2}(z).$$

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Here, $\Delta_k = 1/\sqrt{k}\alpha(t_k)$, $t_k = z\sqrt{N/k}$, and $\alpha(t) = 1$ for $t < 2$ and $\alpha(t) = \sqrt{t^2/4 - 1} + t/2$ for $t \geq 2$.

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Set k_0 so that $t_{k_0} = 2$ (if $z = 0$ then $k_0 = 1$). In matrix form, for $k \geq k_0$,

$$\begin{pmatrix} \Psi_{k+1}(z) \\ \Psi_k(z) \end{pmatrix}$$

$$\sim \begin{pmatrix} \omega_k & -1 + 1/2k \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Psi_k(z) \\ \Psi_{k-1}(z) \end{pmatrix} + \begin{pmatrix} a_k/\sqrt{k} & g_k/\sqrt{k} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Psi_k(z) \\ \Psi_{k-1}(z) \end{pmatrix}$$

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In the Tao-Vu $z = 0$ case, $\omega_k = 0$, and except for perturbation, we have a pure rotation.
Tao-Vu show that $\Psi_{k-1}(z)^2 + \Psi_{k-1}(z)^2$ (essentially) forms a martingale with quadratic variation process of increment $\sim 1/k$. This gives the CLT.

Recursions - general z

The following is joint work with Fanny Augeri and Raphael Butez, in progress. We have a CLT for log-characteristic polynomial, and work on the log-correlated structure.

Recursions - general z

There are several regimes to consider. Fix $\epsilon > 0$, recall that $k_0 = z^2 N/4k$.

- $k < \epsilon k_0$: one easily checks that $\Psi_k(z) \sim 1$.
- $k \in [\epsilon k_0, k_0]$: write

$$X_k = \Psi_k/\Psi_{k-1} = 1 + \delta_k, \quad X_k = A_k + B_k/X_{k-1}$$

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for appropriate A_k, B_k. In this regime, $\delta_k \sim 0$ and one obtains a recursion

$$\delta_k \sim u_k + v_k \delta_{k-1}$$

where $u_k \sim b_k/\sqrt{k\alpha_k^2} + 1/2k\alpha_k^2 - g_k/\sqrt{k\alpha_k^4}$,
$v_k = (1 - 1/2k + g_k/\sqrt{k})/\alpha_k^2$, which one solves.

- $k > k_0$: Oscillatory regime, most interesting.

Recursions - general z - the scalar regime

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is a martingale, and small. We need to compute $\sum \delta_k$, and δ_k are correlated!.

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In fact, such analysis was just posted (January 24, arXiv:2001.09042) by Lambert-Paquette (hyperbolic regime).

Recursions - general z - the oscilatory regime

$$X_k = \begin{pmatrix} \Psi_{k+1} \\ \Psi_k \end{pmatrix}, k > k_0.$$

We have

$$X_{k+1} = (A_k + W_k)X_k,$$

where,

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Eigenvalues of A_k for $k > k_0$ are complex of (essentially) unit norm. Change basis to eigenvector basis, get

$$\hat{X}_k = Q_k \prod_{i=k_0}^{k-1} Q_{i+1}^{-1} Q_i \left(R_i + \hat{W}_i\right) Q_{k_0}^{-1} \hat{X}_{k_0},$$

where R_i are rotation matrices of angle $\theta_k \sim \sqrt{k/k_0 - 1}$.

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First order approximation: divide to blocks of length $\ell_i = (k_0/i)^{1/3}$, linearize in each block, and get contribution to variance of order $1/i$.

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Solution: along block we have $\prod R_i = I$, but the vector $(1, 0)^T$ is not mapped to $\rho_i(1, 0)$ due to the noise. So instead, stop (at random time) where

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Computing correlation between different zs is complicated in the regime $|z - z'| < N^{-2/3}$ because of block structure.