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EINDHOVEN
UNIVERSITY OF
TECHNOLOGY

Critical percolation on scale-free random graphs

Remco van der Hofstad

Interacting Random Systems 2021

"Structure and function of complex networks: epidemics and optimization"

IHP Paris, January 26-27, 2021

Joint work with:
▷ Shankar Bhamidi (UNC)
▷ Johan van Leeuwaarden (TU/e)
▷ Souvik Dhara (MIT/MSR)
▷ Sanchayan Sen (ICTS Bangalore)

EURANDOM NETWORKS

Structure and function of complex networks: epidemics and optimization
Moderator: **Remco van der Hofstad** (Eindhoven)

9h00 – 9h10:	Opening
9h10 – 10h00:	**Remco van der Hofstad** (Eindhoven): Critical percolation on scale-free random graphs
10h00 – 10h45:	**Jean-Stéphane Dhersin** (Paris): Spatial evolution of an epidemic and “social" networks
10h45 – 10h55:	“Coffee" Break
10h55 – 11h40:	**Christina Goldschmidt** (Oxford): The scaling limit of a critical random directed graph
11h40 – 12h25:	**Nicolas Broutin** (Paris): The Brownian parabolic tree
12h25 – 13h30:	“Lunch" Break
13h30 – 14h15:	**Lenka Zdeborová** (Lausanne): Epidemic mitigation by statistical inference from contact tracing data
14h15 – 15h00:	**Amin Coja-Oghlan** (Frankfurt): Group testing
15h00 – 15h10:	“Coffee" Break
15h10 – 15h55:	**Laurent Massoulié** (Paris): Partial alignment of sparse random graphs
15h55 – 16h40:	**Pieter Trapman** (Stockholm): Herd immunity, population structure and the second wave of an epidemic

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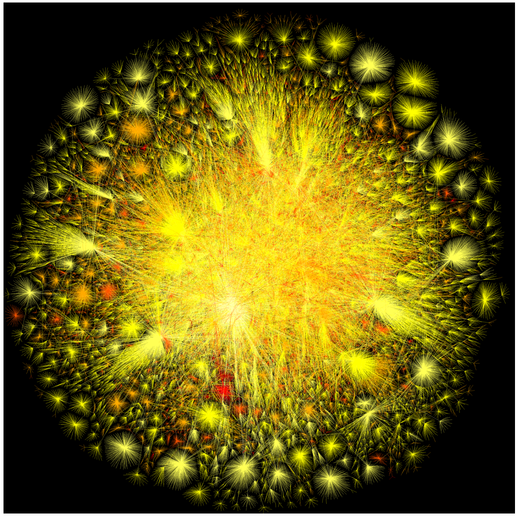
EURANDOM NETWORKS

Part 1: Complex networks and percolation on them

Complex networks

Yeast protein interaction network[a]



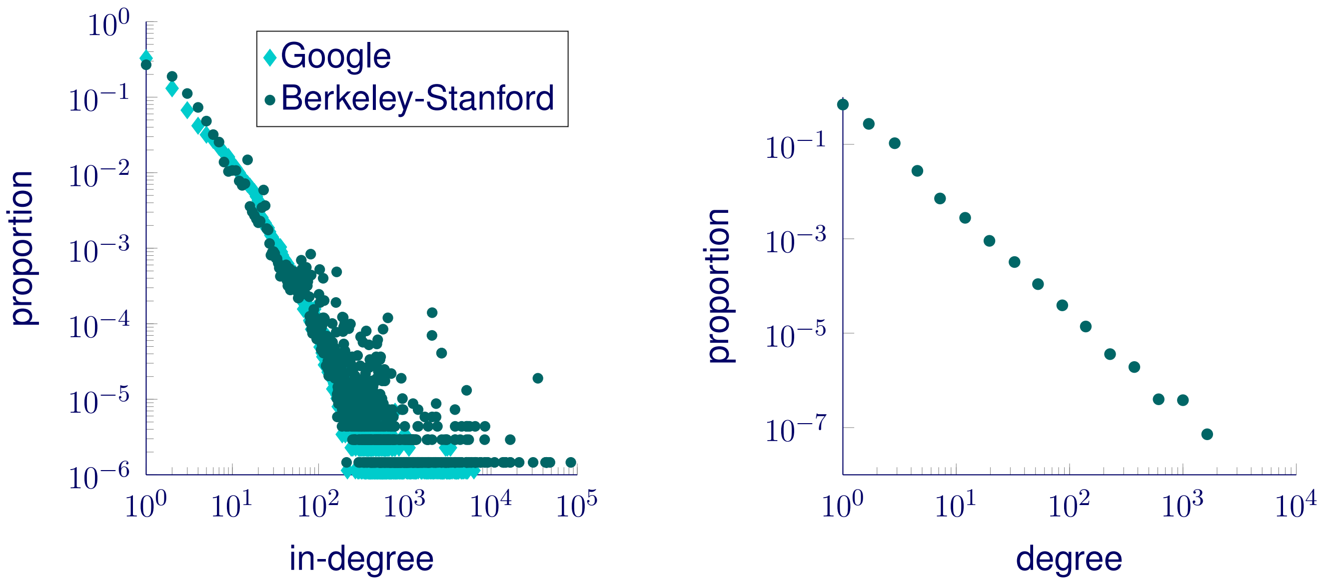
Internet 2010[b]

Attention focussing on unexpected commonality.

[a] Barabási & Óltvai 2004

[b] Opte project http://www.opte.org/the-internet

Scale-free paradigm



Loglog plot degree sequences WWW in-degree and Internet

▷ Straight line: proportion p_k of vertices of degree k satisfies $p_k = ck^{-\tau}$.

▷ Empirical evidence: Often $\tau \in (2,3)$ reported.

Percolation

▷ Percolation: edges are retained independently with fixed probability $p \in [0, 1]$.

▷ Infinite graphs: Phase transition at some $p \in [0, 1]$:

$\star$ $p < p_c$: No infinite connected component;

$\star$ $p > p_c$: Infinite connected component(s) exist.

▷ Most infinite graphs have $p_c \in (0, 1)$:

Non-trivial phase transition.

Most interesting behavior occurs when p is close to p_c :

Self-similar fractal critical (finite?) components.

Percolation finite graphs

▷ We will be mainly interested in

percolation on finite graphs,

where even definition critical value is non-trivial and not unique.

[Borgs-Chayes-vdH-Slade-Spencer 05-06, Nachmias-Peres07, Janson-Warnke17]

▷ Often, exists (increasing) one-parameter family of critical values $\lambda \mapsto p_c(n;\lambda)$ s.t.

$\star\ \lambda_n \to -\infty$: All components are dust;

$\star\ \lambda_n \to \infty$: Unique largest component of almost deterministic size.

$\lambda \mapsto p_c(n;\lambda)$ is critical window.

Most interesting behavior occurs for p in critical window:

Fractal random components of mesocopic size.

Here, study percolation on **random graphs.**

Part 2: Network models and their giants

Configuration model

▷ n number of vertices;
▷ $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ sequence of degrees.

[Bender-Canfield (78), Bollobás (80), Molloy-Reed (95), Newman-Strogatz-Watts (01)]

▷ Assign d_j half-edges to vertex j. Assume total degree even, i.e.,

$$\ell_n = \sum_{i \in [n]} d_i \qquad \text{even.}$$

▷ Pair half-edges to create edges as follows:
Number half-edges from 1 to ℓ_n in any order.
First pair first half-edge at random to one of other $\ell_n - 1$ half-edges.

▷ Continue with second half-edge (when not connected to first) and so on, until all half-edges are connected.

Choice degrees

$\triangleright$ Aim: Proportion of vertices i with $d_i = k$ is close to

$$F(k) - F(k-1) = p_k = \mathbb{P}(D = k),$$

where D has distribution function F.

$\star$ Power law degrees: precise structure of large degrees crucial.

$\star$ Take $\boldsymbol{d} = (d_1, \ldots, d_n)$ as i.i.d. rvs with distribution function F.

$\star$ Take $d_i = [1-F]^{-1}(i/n)$, with F distribution function on $\mathbb{N}$.

Power-law degrees:

$$[1-F](k) \approx ck^{-(\tau-1)}, \quad \text{so that} \quad d_j \approx (cn/j)^{1/(\tau-1)}.$$

Phase transition

Let $D \sim F$ and $\nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D] > 1$. Then

[Cohen et al. (00, 02), Callaway et al. (00), Newman et al. (01), Dorogovtsev-Goltsev-Mendes (07)]

[Molloy-Reed (95), Janson-Luczak (09), Bollobś-Riordan (09), Fountoulakis (07), Janson (09)]

$\triangleright$ largest component $\sim \rho n$ with $\rho \in (0,1)$ for $p > 1/\nu$;
$\triangleright$ largest component $o(n)$ for $p \leq 1/\nu$.

Identifies percolation critical value CM as

$$p_c = 1/\nu, \qquad \text{where} \qquad \nu = \mathbb{E}[D(D-1)]/\mathbb{E}[D] > 1,$$

ν is expected number forward neighbors of vertex in uniform edge.

$\triangleright$ When $\mathbb{E}[D(D-1)] = \infty$, graph always supercritical $p_c = 0$:

Robustness of the giant.

Poissonian RGs

Vertex $i \in [n]$ has vertex weight $w_i \geq 0$, make edge between vertices i, j independently occupied with probability

$$p_{ij} = 1 - \mathrm{e}^{-w_i w_j/\ell_n}, \qquad \text{where} \qquad \ell_n = \sum_{i \in [n]} w_i \qquad \text{is total weight.}$$

Motivation is that w_i is close to expected degree vertex i

▷ **Percolation can be thought of as replacing w_i by pw_i.**

⋆ CM and PRG both "uncorrelated networks", strongly related by conditioning on simplicity and degrees.

⋆ Again take $w_j = [1-F]^{-1}(j/n)$ with $[1-F](k) \approx ck^{-(\tau-1)}$, so that

$$w_j \approx (cn/j)^{1/(\tau-1)}.$$

Phase transition

Let $W \sim F$ and $\nu = \mathbb{E}[W^2]/\mathbb{E}[W] > 1$. Then

[Bollobás-Janson-Riordan (07), Chung-Lu (02), Soderberg (02)]

▷ largest component $\sim \rho n$ with $\rho \in (0,1)$ for $p > 1/\nu$;
▷ largest component $o(n)$ for $p \leq 1/\nu$.

Identifies percolation critical value GRG as

$$p_c = 1/\nu, \qquad \text{with} \qquad \nu = \mathbb{E}[W^2]/\mathbb{E}[W].$$

ν is expected number forward neighbors of vertex in uniform edge.

▷ When $\mathbb{E}[W^2] = \infty$, graph always supercritical:

Robustness of the giant.

Part 3: Critical percolation on Poisson random graphs

Finite third moments

Let $\mu = \mathbb{E}[W], \sigma^2 = \mathbb{E}[W^3]/\mathbb{E}[W]$. Let B be standard Brownian motion, and

$$B_s^\lambda = \sigma B_s + s\lambda - s^2\sigma^2/(2\mu), \qquad R_s^\lambda = B_s^\lambda - \min_{0\leq u\leq s} B_s^\lambda.$$

Aldous (97): Can order excursions of R^λ as $\gamma_1(\lambda) > \gamma_2(\lambda) > \ldots$.

Let $|\mathcal{C}_{(1)}(\lambda)| \geq |\mathcal{C}_{(2)}(\lambda)| \geq |\mathcal{C}_{(3)}(\lambda)| \ldots$ be ordered cluster sizes of percolated Poisson random graph with $p = p_n(\lambda) = (1 + \lambda n^{-1/3})/\nu$.

Theorem 1. (Aldous 97, BvdHvL10, Turova 13) Assume that $\mathbb{E}[W^3] < \infty$. Then, for every $\lambda \in \mathbb{R}$,

$$\big(n^{-2/3}|\mathcal{C}_{(i)}(\lambda)|\big)_{i\geq 1} \xrightarrow{d} \big(\gamma_i(\lambda)\big)_{i\geq 1}.$$

$\star$ Homogeneous case $w_i = 1$ of Erdős-Rényi RG has long history:

[Erdős-Rényi (60), Bollobás (84), Łuczak (90), Janson et al. (93), Aldous (97),...]

Infinite third moments

Let $|\mathcal{C}_{(1)}(\lambda)| \geq |\mathcal{C}_{(2)}(\lambda)| \geq |\mathcal{C}_{(3)}(\lambda)| \ldots$ denote ordered cluster sizes of percolated Poisson random graph with

$$p = p_n(\lambda) = (1 + \lambda n^{-(\tau-3)/(\tau-1)})/\nu.$$

Theorem 2. (BvdHvL12) In power-law case with $\tau \in (3,4)$, for every $\lambda \in \mathbb{R}$,

$$\big(n^{-(\tau-2)/(\tau-1)}|\mathcal{C}_{(i)}(\lambda)|\big)_{i\geq 1} \xrightarrow{d} \big(H_i(\lambda)\big)_{i\geq 1}.$$

Moreover, for every pair of hubs i, j fixed,

$$\mathbb{P}(i \longleftrightarrow j) \to q_{ij}(\lambda) \in (0,1).$$

$\triangleright$ Limits $H_i(\lambda)$ correspond to ordered hitting times of 0 of certain fascinating 'thinned' Lévy process.

$\star$ Powers of n predicted in physics community

[Kalisky-Cohen (06), Dorogovtsev, Goltsev, Mendes (07),...]

Configuration model

Theorem 3. (SB-SD-vdH-vL, Riordan 10) Theorems 1-2 hold for critical configuration model with deterministic degrees, under suitable conditions on degrees very alike those on weights in Theorems 1-2.

Theorem 4. (Adrien Joseph 11) Theorems 1-2 hold for critical configuration model with i.i.d. degrees, under suitable conditions.

$\star$ Remarkably, scaling limit is notably different for i.i.d. degrees by extreme value statistics degrees.

▷ Extensive follow-up work for metric structure

[Broutin et al. (10,12), Bhamidi-Broutin et al. (14), Conchon-Kerjan-Goldschmidt (19+), BvdHS (17), BDvdHS (18)]

Proof: weak convergence

Proof relies on three main ideas:

(1) Subsequent exploration of clusters;

(2) Removal of vertices found: **depletion-of-points effect;**

(3) In critical window, exploration process converges weakly;

$\mathbb{E}[W^3] < \infty$: steps exploration process have finite variance, and Brownian motion appears in limit:

'power to the masses!'

$\tau \in (3, 4)$: high-weight vertices dominate exploration:

'power to the wealthy!'

Part 4: Critical percolation on scale-free random graphs

⋆ In scale-free regime, degrees have infinite variance, largest degrees are $\gg \sqrt{n}$, and giant is robust:

critical value tends to zero with network size.

What is appropriate critical value?

Critical scale-free CM

Let $|\mathcal{C}_{(1)}(\lambda)| \geq |\mathcal{C}_{(2)}(\lambda)| \geq |\mathcal{C}_{(3)}(\lambda)| \ldots$ be ordered cluster sizes, where

$$p = p_n(\lambda) = \frac{\lambda}{n^{(3-\tau)/(\tau-1)}}.$$

Theorem 5. (DvdHvL18+) Assume that $\tau \in (2,3)$. Then, for every $\lambda > 0$, there exists random vector $\big(\gamma_i(\lambda)\big)_{i\geq 1}$ s.t.

$$\big(n^{-(\tau-2)/(\tau-1)}|\mathcal{C}_{(i)}(\lambda)|\big)_{i\geq 1} \xrightarrow{d} \big(\gamma_i(\lambda)\big)_{i\geq 1}.$$

$\star$ Power of n in $p_n(\lambda)$ satisfies $(3-\tau)/(\tau-1) \in (0,1)$.

[Cohen et al. (00), Braunstein et al. (07)]

$\star$ Power of n in $|\mathcal{C}_{(i)}(\lambda)|$ is that of $p_n(\lambda)n^{1/(\tau-1)}$ and satisfies

$$(\tau-2)/(\tau-1) \in (0,1/2).$$

Fractal scaling!

Ingredients proof CM

▷ Largest degree vertices (=hubs) have degree $\Theta(n^{1/(\tau-1)})$;

▷ Number of edges between two hubs is

$$\Theta(d_i d_j/\ell_n) = \Theta(n^{2/(\tau-1)-1}) = \Theta(n^{(3-\tau)/(\tau-1)}).$$

▷ Number of edges between hubs surviving percolation close to Poisson parameter $p_n(\lambda) d_i d_j/\ell_n$:

surviving edges random for this choice of $p_n(\lambda)$!

▷ Barely super and sub-critical regimes also studied to show that

this is right critical value.

Critical scale-free PRG

Let $|\mathcal{C}_{(1)}(\lambda)| \geq |\mathcal{C}_{(2)}(\lambda)| \geq |\mathcal{C}_{(3)}(\lambda)| \ldots$ be ordered cluster sizes, where

$$p = p_n(\lambda) = \frac{\lambda}{n^{(3-\tau)/2}} \gg \frac{\lambda}{n^{(3-\tau)/(\tau-1)}}.$$

Theorem 6. (BDvdHvLb21+) Assume that $\tau \in (2,3)$. Then, there exist $\lambda_c > 0$ and random vector $\big(\gamma_i(\lambda)\big)_{i\geq 1}$ s.t., for every $\lambda \in (0, \lambda_c)$,

$$\Big(n^{-(\tau^2-4\tau+5)/[2(\tau-1)]}|\mathcal{C}_{(i)}(\lambda)|\Big)_{i\geq 1} \xrightarrow{d} \big(\gamma_i(\lambda)\big)_{i\geq 1};$$

Theorem 7. (BDvdHvL21b+) For $\lambda > \lambda_c$, there exists $\zeta = \zeta_\lambda > 0$ s.t.

$$|\mathcal{C}_{(1)}(\lambda)| \geq \zeta\sqrt{n}.$$

Phase transition!

- ▷ Note that $\frac{\tau^2-4\tau+5}{2(\tau-1)} < 1/2$ for $\tau \in (2,3)$;
- ▷ $\frac{\tau^2-4\tau+5}{2(\tau-1)} = \frac{1}{\tau-1} + \frac{\tau-3}{2}$ equals exponent of n in $p_n(\lambda)w_i$ for hubs.

Single-edge constraint

▷ Hubs have weight $\Theta(n^{\alpha})$, with $\alpha = 1/(\tau-1)$;

▷ Number of two-paths between two hubs is of order

$$f_n(i,j) \equiv \sum_{v\in[n]} [1-\mathrm{e}^{w_i w_v/\ell_n}][1-\mathrm{e}^{w_v w_j/\ell_n}] = (n^{-\alpha}w_i)(n^{-\alpha}w_j)\Theta(n^{3-\tau}).$$

▷ Number of two-paths surviving percolation close to Poisson

$$p_n(\lambda)^2 f_n(i,j).$$

Surviving two-paths random for our choice of $p_n(\lambda)$!

Single-edge constraint

▷ Calculus:

$$p_n(\lambda)^2 f_n(i,j) \to c\lambda^2 h(i,j),$$

where $h(i,j) \sim (i \wedge j)^{-\alpha}(i \vee j)^{-(1-\alpha)}$.

▷ $h(i,j)$ is approximately homogeneous of degree -1, i.e.,

$$h(t(i,j)) \approx t^{-1}h(i,j).$$

▷ Such processes investigated by Durrett-Kesten (90) on $\mathbb{Z}_+$.
Prove that limiting process has phase transition at specific value λ_c.

▷ Below λ_c, limiting graph on $\mathbb{Z}_+$ disconnected, above connected.

▷ Connected regime corresponds to existence tiny giant.

Conclusions

▷ Scaling limits of cluster sizes inhomogeneous random graphs depend sensitively on

number of finite moments of degrees;

Scaling limits described by excursions of limiting exploration process. These processes are rather different when degrees have

finite versus infinite-third moments.

▷ Extension to metric convergence, where scaling limit has fractal structure.

Conclusions scale-free

▷ Scale-free regime highly sensitive to

single-edge constraint.

Even order of critical value changes.

▷ Critical value determined by

hub connectivity.

▷ One-paths vs two-paths depending on single-edge constraint.

▷ Single-edge constraint:

finite critical value.

▷ Proof: inhomogeneous random graph comparisons.

▷ Similar results for uniform scale-free RG, erased CM?

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