Interacting Persistent Random Walkers

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References:

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Plan

Plan

- **I** Persistent random walkers
- **II** Active matter and run and tumble bacteria
- **III** Single persistent random walker: exceptional point
- **IV** Exact results for two interacting persistent random walkers

Persistent Random Walker

Persistent random walker in continuous time

$$\begin{array}{ccccc} +\,0 & \rightarrow & 0\,+ & \text{rate} & \gamma \\ 0\,- & \rightarrow & -\,0 & \text{rate} & \gamma \\ + & \leftrightarrow & - & \text{rate} & \omega \end{array}$$

Also known as

- Random walk with memory
- Run and tumble particle
- Telegraphic noise process

History

- G. I. Taylor 1921 *Diffusion by continous movement*; particle in turbulent motion
- S. Goldstein 1951 *On diffusion by discontinuous movements, and on the telegraph equation*
- M. Kacs 1974 *A stochastic model related to the telegrapher's equation*

Continuum limit: telegrapher's equation

Lattice spacing a, limit $\gamma \to \infty$, $a \to 0$ with $\gamma a = v_0$

$$\begin{aligned} \partial_t P_+(x,t) &= -v_0 \partial_x P_+(x,t) + \omega[P_-(x,t) - P_+(x,t)], \\ \partial_t P_-(x,t) &= +v_0 \partial_x P_-(x,t) + \omega[P_+(x,t) - P_-(x,t)] \end{aligned}$$

$P(x,t) = P_+(x,t) + P_-(x,t)$ obeys

telegrapher's equation

$$\partial_t^2 P(x,t) + 2\omega \partial_t P(x,t) = v_0^2 \partial_x^2 P(x,t)$$

Interpolates between:

ballistic limit $\omega \to 0$ and

diffusive limit $\displaystyle \lim_{v_0,\omega\to\infty} \frac{v_0^2}{2\omega} = D$

Active matter: self-propelled constitutents

The constituents continuously expend energy so are not in thermal equilibrium

Examples

- birds flocking
- self-phoretic synthetic colloids
- run and tumble bacteria

Generally we have non-equilibrium states exhibiting collective behaviour

Fluctuation-dissipation theorem and other equilibrium/near equilibrium concepts do not apply

Bacteria for physicists

- Unicellular prokaryotic organism (no nucleus or organelles)
- Active unit - spherocylinder often with flagella acting as propellor
- Lots of them!
  "There are approximately ten times as many bacterial cells in the human flora as there are human cells in the body, with the largest number of the human flora being in the gut flora, and a large number on the skin".- wikipedia
- Form dense aggregates such as biofilms
- From statistical physics perspective ideal candidate for 'active matter' microconstituent: driven, noisy, profuse

Run and Tumble Bacteria

- Bacteria run then with rate α go into tumble state duration τ and choose new direction of motion stochastically
- Phenomenological 'coarse-grained' description of this yields

$$\dot{\rho}(\underline{x}, t) = -\underline{\nabla} \cdot [\underline{v}(\rho, \alpha, ...)\rho - D(v, \alpha, ...)\underline{\nabla}\rho]$$

  which can generate Motility-induced phase separation
- see e.g.

Schimansky-Geier et al (1995) Phys. Lett. A,
Toner, Tu, Ramaswamy (2005) Ann. Phys.
Tailleur and Cates (2008) Phys. Rev. Lett
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Or if you're bored with biology just consider . . .

Persistent Random Walkers as microscopic dynamical model for active matter
c.f. random walker as microscopic model for passive matter

Microscopic Model: Persistent Random Walkers

- instantaneous tumble rate ω on a one-dimensional lattice

Microscopic dynamics

$$\begin{array}{cccll} +\,0 & \to & 0\,+ & \text{rate} & \gamma \\ 0\,- & \to & -\,0 & \text{rate} & \gamma \\ + & \leftrightarrow & - & \text{rate} & \omega \quad \text{instantaneous tumble} \end{array}$$

Master equation (with $\gamma = 1$ and p.b.c. $P_\sigma(n+L) = P_\sigma(n)$).

$$\begin{aligned} \partial_t P_+(n,t) &= P_+(n-1,t) - P_+(n,t) + \omega[P_-(n,t) - P_+(n,t)], \\ \partial_t P_-(n,t) &= P_-(n+1,t) - P_-(n,t) + \omega[P_+(n,t) - P_-(n,t)], \end{aligned}$$

is of the form

$$\partial_t \begin{pmatrix} P_+(n,t) \\ P_-(n,t) \end{pmatrix} = M \begin{pmatrix} P_+(n,t) \\ P_-(n,t) \end{pmatrix}$$

M is Markov matrix.

Spectral solution of single persistent walker

Simple eigenstates of Markov matrix

$$\begin{pmatrix} u_+(n) \\ u_-(n) \end{pmatrix} = \frac{1}{2L} \begin{pmatrix} \mathbb{I} \\ \mathbb{I} \end{pmatrix}$$ Stationary state eigenvalue 0

$$\begin{pmatrix} u_+(n) \\ u_-(n) \end{pmatrix} = \begin{pmatrix} \mathbb{I} \\ -\mathbb{I} \end{pmatrix}$$ Tumble mode eigenvalue -2ω

Solve for all eigenvalues by Fourier transform: eigenvalues $\lambda(k)$ $k = 1, \ldots, L$

$$\boxed{\lambda^{\pm}(k) = -2\sin^2\left(\frac{\pi k}{L}\right) - \omega \pm \sqrt{\omega^2 - \sin^2\left(\frac{2\pi k}{L}\right)}} \tag{1}$$

$\omega \to \infty$ Symmetric random walker $\lambda(k) \to -2\sin^2\left(\frac{\pi k}{L}\right)$ real

$\omega \to 0$ Asymmetric walker $\lambda(k) \to -2\sin^2\left(\frac{\pi k}{L}\right) \pm i\left|\sin\left(\frac{2\pi k}{L}\right)\right|$ complex

Spectral solution of single persistent walker: Exceptional Point

$\omega = 1$ Exceptional Point: transition from real to complex spectrum

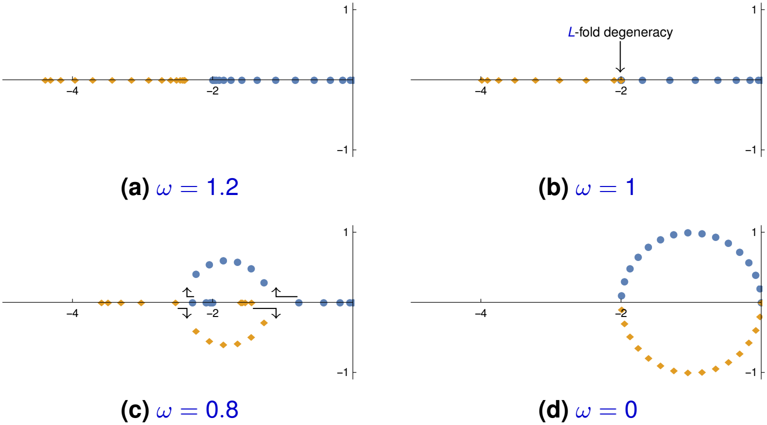


Figure: One-particle spectrum $\lambda^{\pm}(k)$ in the complex plane for $L = 31$.

Well known exceptional point: critical damping

Damped SHM

$$\ddot{x} + 2\gamma\dot{x} + w_0^2 x = 0$$

Trial Solution $x = \mathrm{e}^{\lambda t}$

$$\lambda_\pm = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

$$x = A\mathrm{e}^{\lambda_+ t} + B\mathrm{e}^{\lambda_- t}$$

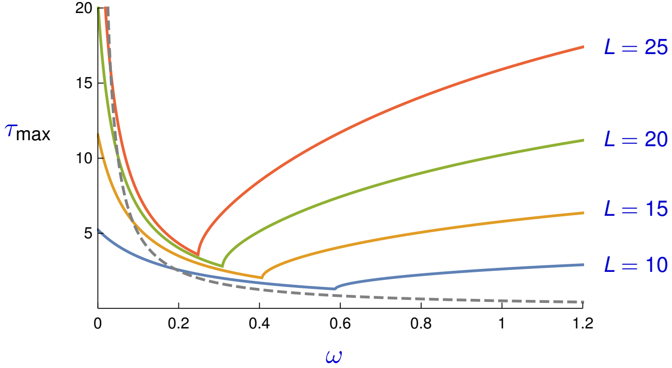
Critical damping $\gamma = \omega_0$ is Exceptional Point

$$x = (A + Bt)\mathrm{e}^{-\omega_0 t}$$

Critical damping gives fastest relaxation.

Persistent Random Walker: Relaxation Time

Slowest relaxation time is given by $\tau_{\max} = 1/\text{Re}\lambda^+(1)$



Cusp at $\omega^* = \sin(2\pi/L)$ Exceptional Point and $\boxed{\tau_{\max}|_{\omega=\omega^*} \approx \dfrac{1}{\omega^*} \sim L}$.

Dashed line is tumble mode $\lambda = -2\omega$

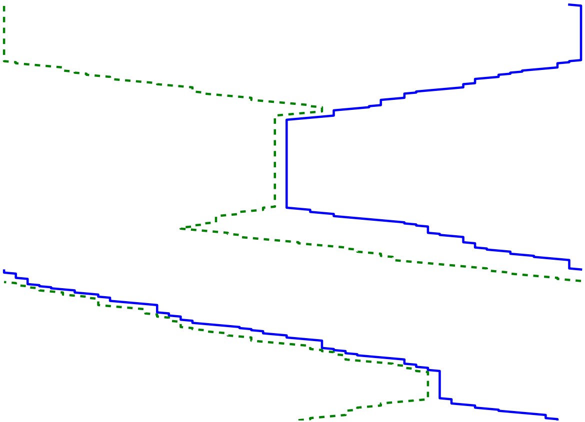
Microscopic Model: Interacting Persistent Random Walkers

- hard-core repulsion
- instantaneous tumble rate ω
- on a one-dimensional lattice

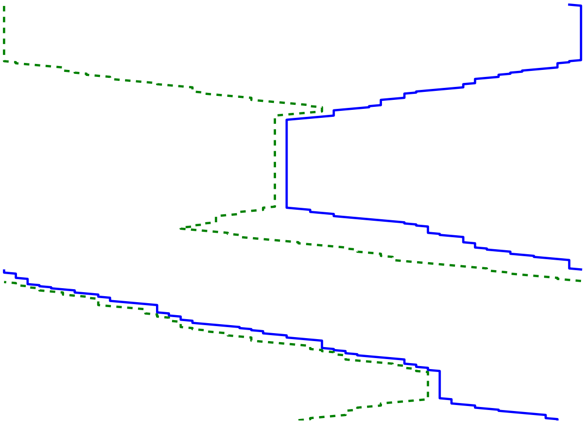
Microscopic dynamics

$$
\begin{array}{llll}
+\,0 & \to & 0\,+ & \text{rate} \quad \gamma \\
0\,- & \to & -\,0 & \text{rate} \quad \gamma \\
+\,- & & & \text{jammed} \\
+ & \leftrightarrow & - & \text{rate} \quad \omega
\end{array}
$$

Two R n T Random Walkers



Two R n T Random Walkers



direction of time must be up - time irreversibility.

Master equation for two R n T Random Walkers

Time evolution of the probabilities of separation n in the four velocity sectors $++$, $+-$, $-+$, $--$ (take $\gamma = 1$):

$$\begin{aligned}\dot{P}_{++}(n) = {} & P_{++}(n-1)I_{n>1} + P_{++}(n+1)I_{L-n>1} \\ & + \omega[P_{+-}(n) + P_{-+}(n)] \\ & - P_{++}(n)[2\omega + I_{n>1} + I_{L-n>1}]\end{aligned}$$

$$\begin{aligned}\dot{P}_{+-}(n) = {} & 2P_{+-}(n+1)I_{L-n>1} + \omega[P_{++}(n) + P_{--}(n)] \\ & - P_{+-}(n)[2\omega + 2I_{n>1}]\end{aligned}$$

Plus symmetries $P_{--}(n) = P_{++}(n)$ and $P_{-+}(n) = P_{+-}(L-n)$.

Indicator $I_{k>1} = 1$ if $k > 1$ and is zero otherwise.

Circulation of probability between different sectors - no detailed balance

Exact Steady State for two R n T Random Walkers

Generating function approach yields

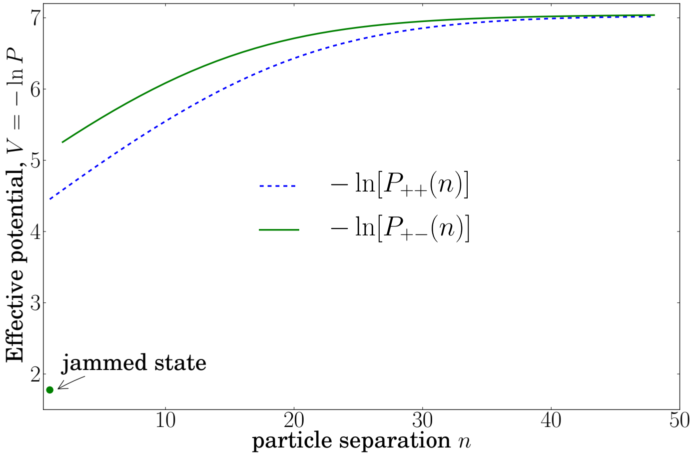
$$P_{++}(n) = \frac{1}{Z}\left[(1-z^2)(z^n + z^{L-n}) + q(z)\right]$$
$$P_{+-}(n) = \frac{1}{Z}\left[(1-z)^2(z^n - z^{L-n}) + q(z) + \delta_{n,1}\Delta(z)\right]$$

where

$$z = 1 + \omega - \sqrt{\omega(2+\omega)} \qquad 0 < z < 1$$
$$q(z) = (1-z)^2(1-z^L)$$
$$\Delta(z) = 2(1+z)(z-z^L)$$
$$Z = 4[\Delta(z) + (L-1)q(z)]\,.$$

Three pieces: attractive, extended and jammed

Effective Pair Potential



Limiting cases

$\omega/\gamma \to \infty$ High tumble rate: recovers symmetric exclusion process

$$P(n) \sim \frac{1}{L-1}\left[1 + \frac{1}{2\omega}\left(\delta_{n,1} + \delta_{n,L-1} - \tfrac{2}{L-1}\right) + O\left(\tfrac{1}{\omega^2}\right)\right]$$

$\omega/\gamma \to 0$ Scaling limit: $L \to \infty$, $\gamma = L$; lattice spacing $a = \ell/L$;

physical velocity $\gamma a = \ell$ is fixed

$$P_{++}(x) = \frac{\omega + 4\ell\delta(x) + 4\ell\delta(\ell - x)}{4(\omega+8)\ell}$$

$$P_{+-}(x) = \frac{\omega + 8\ell\delta(x)}{4(\omega+8)\ell}$$

Jammed state persists in both sectors

Spectral Solution for two R n T Random Walkers

Eigenvalues of Markov matrix

$$\lambda = z_1 + \frac{1}{z_1} - 2(1+\omega)$$

where z_1 and z_2 are solutions of polynomial equations

$$(z_1 + \bar{z}_1)\left[2(z_2 + \bar{z}_2) - (z_1 + \bar{z}_1)\right] = 4(1-\omega^2),$$

$$z_2^{L-1} = \frac{2z_2 - (z_1 + \bar{z}_1)}{2\bar{z}_2 - (z_1 + \bar{z}_1)} \cdot \frac{(\bar{z}_1 - \bar{z}_2) + (z_1 - \bar{z}_2)z_1^{L-1}}{(\bar{z}_1 - z_2) + (z_1 - z_2)z_1^{L-1}},$$

with $\bar{z}_i = 1/z_i$.

Eigenfunctions

$$\begin{aligned}
u_{++}(n) &= A_L(z_1^n + z_1^{L-n}) + B_L(z_2^n + z_2^{L-n}), \\
u_{+-}(n) &= A'_L(z_1^n - z_1^{L-n}) + B'_L(z_2^n + z_2^{L-n}) + C\delta_{n,1} \\
u_{-+}(n) &= \pm u_{+-}(L-n), \\
u_{--}(n) &= \pm u_{++}(n),
\end{aligned}$$

Spectral Solution for two R n T Random Walkers

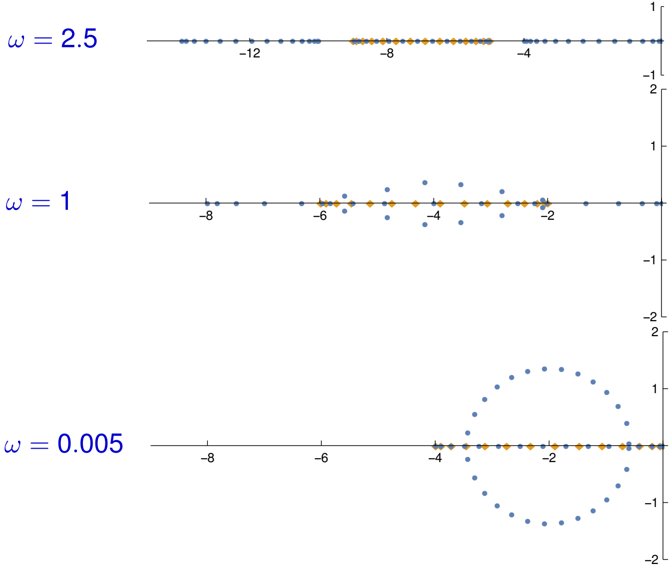


Figure: Plot of two-particle spectrum in the complex plane for $L = 30$.

Two RnT Random Walkers: Slowest Relaxation Time

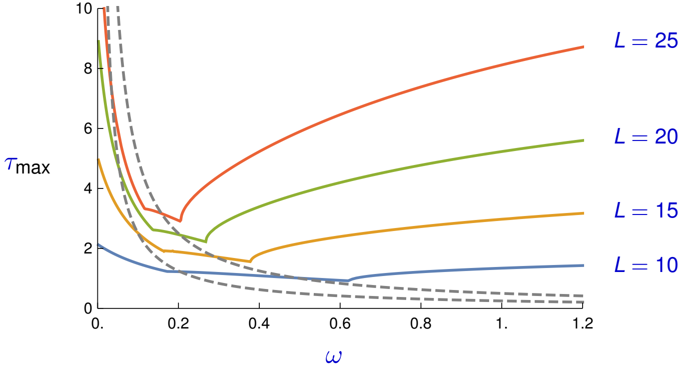


Figure: Coloured lines: longest system-size dependent relaxation time. Dashed lines: tumble relaxation times $1/2\omega$ and $1/4\omega$. Each coloured curve has two minima and cusps which signify dynamical transitions

Finite tumbling times

Tumbling dynamics: Introduce tumbling state $*$

$$\begin{array}{ccclll} \pm & \to & * & \text{rate} & \alpha & \alpha \text{ is rate of entering tumble} \\ * & \to & + & \text{rate} & \beta/2 & \\ * & \to & - & \text{rate} & \beta/2 & \beta \text{ is rate of exiting tumble} \end{array}$$

Stationary state scaling limit: $L \to \infty$, lattice spacing ℓ/L

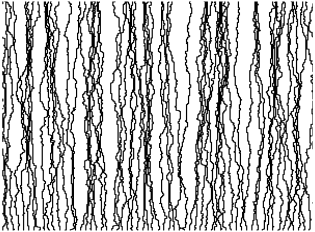
$$P_{++}(x) \propto b_{++} + a'_{++}\left[\delta(x) + \delta(\ell - x)\right] + a_{++}\left[\mathrm{e}^{-\mu x} + \mathrm{e}^{-\mu(\ell - x)}\right]$$

where

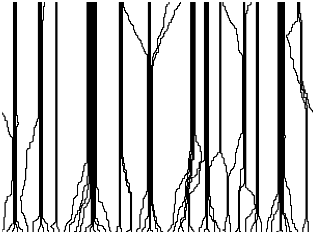
$$\mu = \left(\frac{(\theta+\phi)(\theta+2\phi)}{2}\right)^{1/2} \qquad \text{new inverse length}$$

$$\frac{\alpha}{\gamma} = \frac{\phi}{L} \qquad \frac{\beta}{\gamma} = \frac{\theta}{L}$$

Open Problem: Many Body Problem



Symmetric exclusion ($\omega \to \infty$)
(Exactly solvable equilibrium system)



RnT random walkers

Can one solve the many body problem to demonstrate clustering?

Spectral Solution for two particle problem has some similarity to Bethe ansatz (which solves symmetric exclusion process) .....

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