Geometry of Quantum Transport

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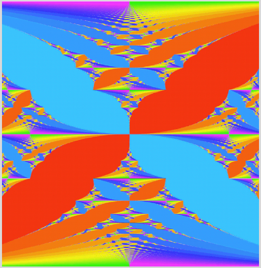

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2. Open systems
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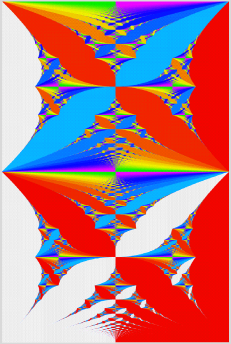
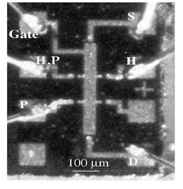
3. Quantum response
   - Dephasing Lindbladians
4. Compatibility

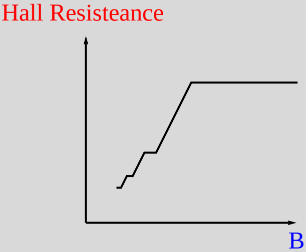


Landau-TKNN butterfly

Motivation: Quantum Hall effect

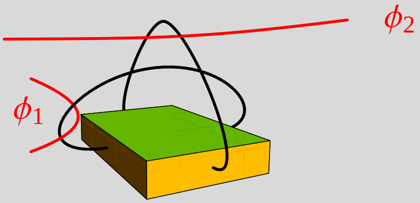
- Ill characterized
- Quantized resistivity $\frac{h}{e^2}\frac{1}{\mathbb{Z}}$
- Great accuracy
- Quantum response=Adiabatic curvature
- Hall conductance= Chern number
- Geometry of response in open q-system
- Chern numbers in open q-system





Quantum Control

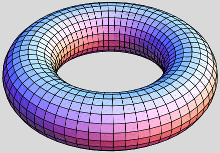
- Controls: Magnetic flux tubes
- $H(\phi)$: *Control space* $\mapsto$ *Hamiltonians*
- Aharonov-Bohm periodicity: $H(\phi) \equiv H(\phi + 2\pi)$
- Control space: $\mathbb{T}^2$

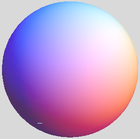


Topology of QHE in physical space

Controlled Hamiltonians

- Controls: ϕ; Space of controls $\mathcal{M}$;
  Two dimensional good enough
- Controlled Hamiltonian: $H(\phi)$;
  2×2 matrices good enough
- $\mathcal{M}$: a-priori topology, no a-priori metric
  e.g. $\mathbb{T}^2, \mathbb{S}^2, \mathbb{R}^2$

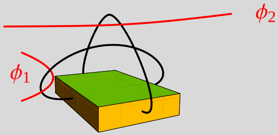




$\mathcal{M}$: Control spaces

Quantum Response

- Controls $= \phi_\mu$; e.g. magnetic fluxes
- Response: $\frac{\partial H}{\partial \phi_\nu}$ e.g. Loop currents
- Driving= control rates $= \dot{\phi}_\mu$ e.g. emf
- Quantum evolution: $\dot{\rho}_t = \mathcal{L}(\rho)$



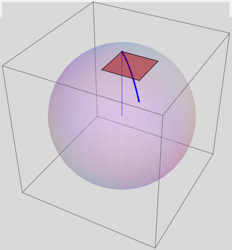
Loop currents, emf

Definition (Response matrix)

$$\underbrace{\mathrm{Tr}(\rho_t \partial_\mu H)}_{response} = \underbrace{f_{\mu\nu}(\phi)}_{response\ \ matrix} \times \underbrace{\dot{\phi}_\nu}_{driving} + \dots$$

- $f = \underbrace{f^S}_{dissipative} + \underbrace{f^A}_{reactive}$

Geometry in Hilbert space



Metric and symplectic structures

- $P(\phi)$ smooth family of projections
- Example: qubit $P(\phi) = \frac{1+\hat{\phi}\cdot\sigma}{2}$

Definition (Fubini-Study; adiabatic curvature)

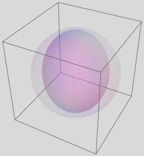
$$g_{\mu\nu} + i\omega_{\mu\nu} = 2\,\mathrm{Tr}(A_\mu A_\nu^*), \quad A_\mu = P_\perp \partial_\mu P, \quad P_\perp = 1 - P$$

- $g \geq 0$; symmetric
- ω anti-symmetric, defines a symplectic structure
- Endows control space with geometry

Open systems

- Self-adjoints (Hamiltonians): Generate unitary
- Lindbladians: Generate (completely) positive maps
- Kraus:

$$\rho \to \sum_{j=1}^{n} A_j \rho A_j^*, \quad \sum A_j^* A_j = 1$$



Contraction of Bloch sphere

$n = 1$ unitary

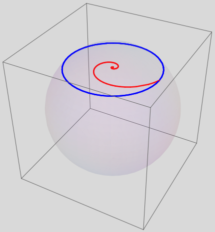
- Interpretation: Measurement, Coupling to a Markovian bath; Stochastic evolution

Lindbladians

Definition (Lindbladian)

$$\dot{\rho} = \mathcal{L}(\rho) = -i[H, \rho] + \sum [\Gamma_a, \rho\Gamma_a^*] + [\Gamma_a\rho, \Gamma_a^*]$$

- $H = H^*$; Γ_a anything.
- Unitary: $\Gamma_a = 0$
- Dephasing: $\Gamma = \Gamma(H)$
- Interpretation: Measurement of H
- P_j, spectral projections of H, stationary states $\mathcal{L}(P_j) = 0$

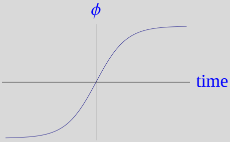


Unitary vs dephasing orbits

Adiabatic evolutions

- Controls $\phi(s)$ and $H(\phi)$ change adiabatically,

  $\dot\phi = O(\epsilon)$



Adiabatic switching of controls

- $H(\phi)$ determines $\mathcal{L}_\phi$
- Initial data: $P(\phi)$, instantaneous stationary state
- Adiabatic evolutions $\epsilon\dot\rho = \mathcal{L}(\rho)$ (re-scaled time)

Theorem (Adiabatic)

If $P(\phi)$ smooth, $\rho_t = P(\phi) + O(\epsilon)$

- Clings to instantaneous stationary for long time, $t = O(1/\epsilon)$

Main result

Theorem (Geometric transport)

Response matrix of dephasing Lindbladians with $\Gamma(H) = \sqrt{\gamma H}$

$$f = \frac{\gamma}{1+\gamma^2} + \frac{1}{1+\gamma^2}\,\omega$$

- Good news: response is geometric
- Bad (?) news: Hall conductance $= \frac{Chern}{1+\gamma^2}$
- Metric compatible with symplectic if

$$g\omega^{-1} + \omega g^{-1} = 0$$

Theorem (Immunity)

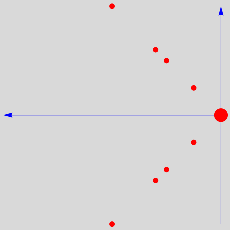
Compatibility implies: $(f^{-1}) = \gamma\, g^{-1} + \omega^{-1}$

Spectral properties

- Instantaneous: $H\,|j\rangle = e_j\,|j\rangle$
- $\mathcal{L}(|j\rangle\,\langle k|) = \lambda_{jk}\;\; |j\rangle\,\langle k|\,, \quad Re\,\lambda_{jk} \le 0$
- Dephasing Lindblad $\lambda_{jj} = 0$; multiply degenerate
- Suppose H simple

$$Ker\ \mathcal{L} = \{\,|j\rangle\,\langle j|\,, j = 1, \dots\,\}$$

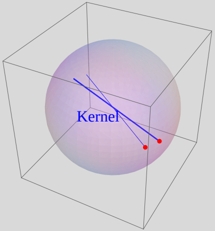
$$Range\ \mathcal{L} = \{\,|i\rangle\,\langle j|\,, i \ne j = 1, \dots\,\}$$



Spectrum Lindblad

Adiabatic evolution

- $Ker\ \mathcal{L}_\phi : \mathcal{M} \mapsto \mathcal{H} = Ker\ \mathcal{L} \oplus Range\ \mathcal{L}$
- Basic identity $P^2 = P \Longrightarrow P\dot{P}P = 0$
- It follows that $Ker\ \mathcal{L}_\phi$ evolves like a rigid body
- Characterize motion in ker and in *range*;



Motion of Kernel

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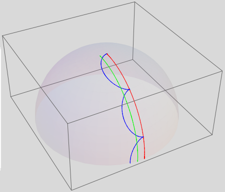


Qubit adiabatic orbit

Theorem (Adiabatic)

$$\rho_t = \underbrace{P(\phi) + O(\epsilon)}_{in\ \ kernel} + \underbrace{\mathcal{L}^{-1}(\partial_\mu P)\dot{\phi}^\mu + O(\epsilon^2)}_{in\ \ range} + \dots$$

Linear response

- First order in adiabaticity:

$$f_{\mu\nu} = \mathrm{Tr}\,\Big(\underbrace{\partial_\mu H}_{observable}\;\underbrace{\mathcal{L}^{-1}(\partial_\nu P)}_{\delta\rho}\Big)$$

- Geometric Hamiltonians:

$$H(\phi) = \sum_j \underbrace{e_j}_{fixed}\underbrace{P_j(\phi)}_{moving}$$

- Kubo type formula:

$$f_{\mu\nu} = \sum_j e_j\, tr\big((\partial_\mu P_j)\; \mathcal{L}^{-1}(\partial_\nu P)\big)$$

Geometric linear response

- Kubo type formula:

$$f_{\mu\nu} = \sum_j e_j tr\big((\partial_\mu P_j)\ \mathcal{L}^{-1}(\partial_\nu P)\big)$$

- Special: Two level system $P_1 = P_\perp$:

$$f_{\mu\nu} = \frac{1}{i+\gamma} tr\big(\partial_\mu P\ P_\perp \partial_\nu P\big), \quad \gamma = -\frac{Re\ \lambda_{01}}{e_1 - e_0}$$

- Less special: $\Gamma(h) = \sqrt{\gamma H}$:

$$f_{\mu\nu} = \frac{1}{i+\gamma} tr\big(\partial_\mu P\ P_\perp \partial_\nu P\big)$$

- In either case: transport geometric

$$f = \frac{\gamma}{1+\gamma^2}\, g + \frac{1}{1+\gamma^2}\omega$$

Compatibility: Immunity

- g allows to compute length; ω allows to compute area
- Compatibility in 2-D: $\det g = \det \omega$
- Compatibility in n-D: $g\omega^{-1} + \omega g^{-1} = 0$
- Possible only if time reversal is broken: $\omega = 0$ if time reversal holds
- Immunity from dephasing

$$f = \frac{\gamma}{1+\gamma^2}\, g + \frac{1}{1+\gamma^2}\, \omega \Longleftrightarrow f^{-1} = \gamma g^{-1} + \omega^{-1}$$

- Chern number

$$2\pi \underbrace{Chern}_{integer} = \int \omega_{12}(\phi)\ d\phi_1 \wedge d\phi_2 = \int \underbrace{\omega_{12}^{-1}(\phi)}_{Hall\ \ resist.}\ \underbrace{\det g(\phi)\ d\phi_1 \wedge d\phi_2}_{averaging}$$

Compatibility: Tests

- Test: is there $\tau \in \mathbb{C}$ such that

$$P_\perp \bar{\partial} P = 0, \quad \bar{\partial} = \tau \partial_1 - \partial_2$$

plugging in

$$g = \mathrm{Tr}(P_\perp \{\partial_\mu P, \partial_\nu P\}), \quad \omega = i \, \mathrm{Tr}(P_\perp [\partial_\mu P, \partial_\nu P])$$

gives

$$g_{22} = |\tau|^2 g_{11}, \quad g_{12} = \tau_1 g_{11}, \quad \omega_{12} = \tau_2 g_{11}.$$

compatible

Compatibility: Tests and complex structure

Theorem (Complex structure)

$J = g\omega^{-1}$ with g and ω compatible, endows space of controls with complex structure: $J^2 = -1$.

Theorem (Holomorphy test)

Suppose $P = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$ with $\bar\partial\,|\psi\rangle = 0$ (for some τ) then g and ω are compatible

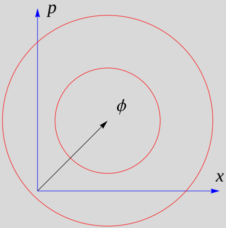
$$P_\perp \bar\partial P = P_\perp \bar\partial \left(\frac{|\psi\rangle\,\langle\psi|}{\langle\psi|\psi\rangle} \right) = P_\perp \underbrace{(\bar\partial\,|\psi\rangle)}_{0} \left(\frac{\langle\psi|}{\langle\psi|\psi\rangle} \right) + \underbrace{P_\perp\,|\psi\rangle}_{0}\ \bar\partial \left(\frac{\langle\psi|}{\langle\psi|\psi\rangle} \right)$$

Qubit , Coherent states and QHE

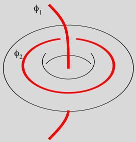
Theorem

Compatible metric and curvature:

- *Qubit:* $H(\phi) = \hat{\phi} \cdot \sigma$
- *Coherent states:* $H(\phi) = (p - \phi_1)^2 + (x - \phi_2)^2$
- *QHE:* $H(\phi) = D^* D$



- $D = i\partial + 2\pi\phi - 2\pi\bar{\tau}By$
- $\partial = \bar{\tau}\partial_x - \partial_y, \quad \phi = \bar{\tau}\phi_1 + \phi_2$
- Coherent states:



$$|\phi_1, \phi_2\rangle = e^{i(\phi_1 x - \phi_2 p)}\,|0\rangle = \underbrace{N(\phi, \bar{\phi})}_{\textit{normalization}}\;\underbrace{e^{\phi a^*}\,|0\rangle}_{\textit{holomorphic}}$$

Summary

- Transport in certain open quantum systems geometric
- Hilbert space projections induce geometry on control space
- Dissipation $\propto$ Fubini-Study metric
- Non-dissipative transport $\propto$ adiabatic curvature
- Kähler structure $\Longrightarrow$ immunity to dephasing of certain transport coefficints
- Chern numbers still relevant