Diffusion in Hamiltonian Systems

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based on work with O. Ajanki, A. Kupiainen, and earlier work with J. Fröhlich.

Main goal

Prove that a Hamiltonian (in particular: deterministic) system exhibits diffusion for long times.
$\rightarrow$ Emergence of irreversibility from deterministic dynamics
common wisdom phyiscally, but hard to make rigorous

Outline

- I discuss the strategy following the Rayleigh gas (for which we have no results whatsoever).
- At the end, I give the real (quantum) model (with result). Differences with the Rayleigh model are irrelevant except for those that will be highlighted

Previous results (scarce)

Diffusion proven in

- *Sinai and Bunimovich, 81,* Billiards with finite horizon (Lorentz gas)

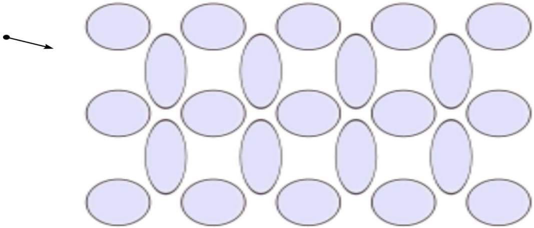


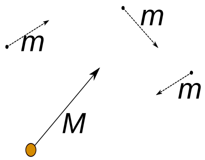
Figure: Tracer particle bounces elastically off periodic objects. No 'free corridors'.

- *Knauf, 90,* A lattice of $-1/r$ attractive potentials in 2D $\Rightarrow$, smooth+uniformly hyperbolic.
- *Harris and Spitzer, 69,* 1D gas of point particles, all masses equal: exactly solvable

Rayleigh gas: particle in ideal gas

- Heavy particle with mass M in a gas of point particles with mass m.
- Elastic collisions $M - m$ and free flow between collisions.
- Initial positions $\sim$ point process with density ρ.
- Maxwellian initial velocities $\rho_{\mathrm{E}}^{\beta}(\mathrm{d}v) \sim \mathrm{e}^{-\frac{\beta}{2}mv^2}\mathrm{d}v$
- Formally;

$$\rho_{\mathrm{E}}^{\beta}(\mathrm{d}\underline{x}\mathrm{d}\underline{v}) \sim \prod_i \rho \mathrm{e}^{-\frac{\beta}{2}mv_i^2}\,\mathrm{d}x_i\mathrm{d}v_i$$



elastic collisions

Figure: Particle M has volume, particles m are points: No collisions between M and m.

Rayleigh gas: Diffusion?

The heavy particle is with position $X(t)$ is originally placed at the origin. $X(0) = 0$. Does the particle diffuse?

$$\frac{|X(t)|^2}{t} \underset{t \nearrow \infty}{\overset{\text{in expectation}}{\longrightarrow}} D$$

....and in probability? CLT? invariance principle?

If you assume each gas particle collides only once, then $(X(t), \frac{\mathrm{d}X(t)}{\mathrm{d}t})$ is just a Markov process (fully stochastic). Diffusion follows easily upon analyzing this Markov process (Linear Boltzmann Equation)

However, what about recollisions?
$\Rightarrow$ Markov property breaks down

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Rayleigh gas: Scaling limit

- Make recollisions infinitely unlikely $\Rightarrow$ easier problem. This is the idea of scaling limits. For example, let density of the gas particles be small and observe the system for long times

$$\text{density} \sim \epsilon, \qquad \text{time} \sim 1/\epsilon, \qquad \epsilon \searrow 0.$$

In the limit, $\epsilon \searrow 0$, the probability of one collision, respectively a recollision is

$$(1/\epsilon) \times \epsilon, \qquad (1/\epsilon) \times \epsilon^2$$

One expects that $(\epsilon X^\epsilon(\tau/\epsilon), V^\epsilon(\tau/\epsilon)$ converges to a Markov process (Linear Boltzmann equation) in τ as $\epsilon \searrow 0$.

- This was done (essentially) by Durr-Goldstein-Lebowitz in 1981. However, without scaling limit (i.e. ϵ fixed), no results available!

Scaling limits are not the end of the story

The dynamical system gets adjusted as time grows.

$\Rightarrow$ Does not give information on the long-time limit of the fixed ϵ dynamical system.

Example

2D Anderson model is well-described by LBE for short times, but localized for large times: With probability $\exp^{-\epsilon^{-1}}$, the particle is sent back to its starting place. The scaling limit is lying!

Results (NOT exhaustive) on scaling limits

- *Yau, Erdös, '99, Yau, Erdös, Salmhofer, '05, Lukkarinen, Spohn, '08,* quantum or wave models
- *Toth, Holley, Dürr-Goldstein-Lebowitz, '81,* Rayleigh gas
- *Komorowski, Ryzhik, '04,* particle in random force field
- *Dolgopyat, Liverani, '10,* coupled Anosov systems.

Why is it not obvious?

- Problem is formulated as perturbation of ballistic system
- Strategy: first establish stochasticity on short time-scales. View the system as a perturbation of the stochastic system.
- Time for stochasticity to set in (E.g. gap of of the velocity process) is of the order of $1/\epsilon$ (at least one collision)
- However, in time $1/\epsilon$, the particle travels a distance $1/\epsilon$. The probability that two particles cross the swept-out space region simultaneously is $O(1)$. **No small parameter**

Solution: creeping particle

Make mass M ϵ-dependent, such that velocity $v \to 0$ as $\epsilon \to 0$ and swept-out space cylinder shrinks. Then the recollisions are manifestly subleading!

But Let p, q be pre-collision S and E respectively, and p', q' post-collision.

$$\begin{cases} p+q = p'+q' \\ \frac{p^2}{2M} + \frac{q^2}{2m} = \frac{(p')^2}{2M} + \frac{(q')^2}{2m} \end{cases}$$

Indeed, let $M \nearrow \infty$ and $\frac{p^2}{2M} = O(\beta^{-1}) = O(1)$ (Maxwell distribution), then

$$\frac{|p'-p|}{|p|} = \frac{|q'-q|}{|p|} \sim \frac{O(1)}{O(\sqrt{M})}$$

Hence gap of velocity process vanishes, we lose effective stochasticity.

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Solution: creeping particle with internal spring

- Assume the particle has some internal dgf. s with energy $E_{int}(s)$, e.g. it is a molecule with a vibrational mode.
- Replace $\frac{p^2}{2M} \to \frac{1}{M}E_{kin}(p)$, with $E_{kin}(\cdot)$ bounded (realistic for quantum particle on the lattice). Instead of collisions, consider absorption, and emission.

$$\text{emit } q' \Rightarrow \begin{cases} p &= p' + q', \\ \frac{1}{M}E_{kin}(p) + E_{int}(s) &= \frac{1}{M}E_{kin}(p') + \frac{(q')^2}{2m} + E'_{int}(s') \end{cases}$$

If $M \nearrow \infty$ and $E_{int} = E'_{int}$, then $p = p'$, $\Rightarrow$ No momentum transfer.
If $M \nearrow \infty$ and $E_{int} - E'_{int} = O(1)$, then $p - p' = O(1)$, $\Rightarrow$ No problem, momentum randomized after $O(1)$ collisions

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Quantum dynamics and diffusion

- Ingredients
  - Hilbert space $\mathcal{H}$ and a positive-definite density matrix $\rho \in \mathcal{B}_1(\mathcal{H})$ with $\mathrm{Tr}\,\rho = 1$.
  - A unitary time-evolution U_t: $U_t U_{t'} = U_{t+t'}$, given by $U_t = \mathrm{e}^{\mathrm{i}tH}$.
  - A position observable $X = X^* \in \mathcal{B}(\mathcal{H})$
- Evolved density matrix $U_t \rho U_{-t}$ describes system at time t:

$$\mathbb{E}_0[F(X)] \sim \mathrm{Tr}[\rho F(X)], \qquad \mathbb{E}_t[F(X)] \sim \mathrm{Tr}[U_t \rho U_{-t} F(X)]$$

E.g. $F = 1_x$, then $\mathrm{Tr}[\rho 1_x(X)]$ prob. to measure $X = x$.

- Diffusion: for ρ such that originally, $\mathrm{Tr}[\rho F(X)] = F(0)$

$$\mathrm{Tr}[U_t \rho U_{-t} \mathrm{e}^{\mathrm{i}\frac{\gamma}{\sqrt{t}} X}] \underset{t \nearrow \infty}{\to} \mathrm{e}^{-\frac{1}{2}(\gamma, D\gamma)}$$

for some *diffusion tensor* D.

Setup: Quantum particle coupled to a field I.

- The Hilbert space is of the form (S=system, E=reservoir)

$$\mathcal{H} := \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{E}}$$

  where

$$\mathcal{H}_{\mathrm{E}} = \Gamma_{symm}(\mathfrak{h}) = \Omega \oplus \mathfrak{h} \oplus (\mathfrak{h} \otimes_{symm} \mathfrak{h}) \oplus \dots$$

  The space $\mathfrak{h} = L^2(\mathbb{T}^d, \mathrm{d}q)$ or $\mathfrak{h} = L^2(\mathbb{R}^d, \mathrm{d}q)$ is the Hilbert space of one field quantum, a phonon or photon in our case.
- The Hamiltonian is

$$H_\lambda := H_{\mathrm{S}} + H_{\mathrm{E}} + \lambda H_{\mathrm{SE}}$$

  where $H_{\mathrm{S}} = H_{\mathrm{S}} \otimes 1$ acts on $\mathcal{H}_{\mathrm{S}}$ and $H_{\mathrm{E}} = 1 \otimes H_{\mathrm{E}}$ on $\mathcal{H}_{\mathrm{E}}$.
  The coupling strength λ will be assumed to be small.
  For $\lambda = 0$, particle and field are decoupled $\Rightarrow$ particle is ballistic.

Setup II: The field

- The free dynamics $t \to U_t^0$ of one field quantum (one gas particle):

$$U_t^0 \phi(q) = \mathrm{e}^{\mathrm{i}\omega_{\mathrm{E}}(q)t} \phi(q), \qquad \text{Hamiltonian} \omega(q)$$

Examples: $\omega_{\mathrm{E}}(q) = |q|$ (photons) or $\omega_{\mathrm{E}}(q) = \sqrt{m^2 + q^2}$ (optical phonons)

- Free dynamics on $\Gamma_{symm}(\mathfrak{h}) = \mathbb{C} \oplus \mathfrak{h} \oplus (\mathfrak{h} \otimes_{symm} \mathfrak{h}) \oplus \ldots$

$$H_{\mathrm{E}} = \mathrm{d}\Gamma(\omega_{\mathrm{E}}) = 0 \oplus \omega \oplus (1 \otimes \omega_{\mathrm{E}} + \omega_{\mathrm{E}} \otimes 1) \oplus \ldots$$

- Initial (field) density matrix

$$\rho_{\mathrm{E}} = \frac{1}{Norm} \mathrm{e}^{-\beta H_{\mathrm{E}}}$$

(Restrict to finite volume $\Lambda \subset \mathbb{Z}^d$)

Setup III: Particle

- Hilbert space: translation dgf. and internal dgf. (IDF)

$$\mathcal{H}_{\mathrm{S}} = l^2(\mathbb{Z}^d) \otimes \mathbb{C}^N$$

- Hamiltonian

$$H_{\mathrm{S}} = \lambda^2 \Delta \otimes 1 + 1 \otimes H_{IDF}$$

  with Δ the discrete Laplacian and H_{IDF} some Hermitian matrix.

- The free time-evolution is ballistic! Let X be position operator on $l^2(\mathbb{Z}^d)$, then

$$\mathrm{Tr}[(X \otimes 1)\left(\mathrm{e}^{-\mathrm{i}tH_{\mathrm{S}}} \rho_{\mathrm{S}} \mathrm{e}^{\mathrm{i}tH_{\mathrm{S}}}\right)] \propto t$$

  for generic initial density matrices, e.g.

$$\rho_{\mathrm{S}} = \rho_{\mathrm{S}}(x, y, IDF) = \delta_{x,0}\delta_{y,0} \otimes \rho_{\mathrm{S}}(IDF)$$

Setup IV: Interaction

The interaction

$$H_{\mathrm{SE}} := \sum_x \{1_x \otimes W \otimes a(\phi_x) + 1_x \otimes W \otimes a^*(\phi_x)\}$$

- W is a Hermitian matrix acting on $\mathbb{C}^N$
- $a^*(\phi_x)/a(\phi_x)$ creates/annihilates a field quantum with wavefunction $\phi_x \in \mathfrak{h} = L^2(\mathbb{T}^d)$.

  $$a^*(\psi)\mathrm{Sym}[\psi_1 \otimes \ldots \otimes \psi_k] := \sqrt{k+1}\mathrm{Sym}[\psi \otimes \psi_1 \otimes \ldots \otimes \psi_k]$$

  with Sym projection on symmetric subspace, and $a(\phi) = (a^*(\phi))^*$. Satisfy CCR:

  $$[a(\psi'), a^*(\psi)] = \langle \psi', \psi \rangle$$

- $\phi_x(q) = \mathrm{e}^{iqx}\phi(q)$. The function $\phi(q)$ is the *form factor*, containing appropriate IR and UV cutoffs: it determines the *form* of the particle.

Setup: Summary

The Hamiltonian

$$H_\lambda := \lambda^2 \Delta \otimes 1 \otimes 1 + 1 \otimes H_{IDF} \otimes 1 + 1 \otimes 1 \otimes \mathrm{d}\Gamma(\omega)$$
$$+ \lambda \sum_x \{1_x \otimes W \otimes a(\phi_x) + 1_x \otimes W \otimes a^*(\phi_x)\}$$

where H_{IDF} and W are $N \times N$-matrices.

Initial density matrix

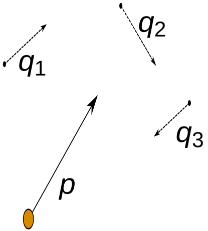
$$\rho_0 = \delta_{x,0}\delta_{y,0} \otimes \rho(IDF) \otimes \frac{1}{Norm} \mathrm{e}^{-\beta H_{\mathrm{E}}}$$

Quanitity of interest

$$\mathbb{E}_t[\mathrm{e}^{\mathrm{i}\frac{\gamma}{\sqrt{t}}X}] = \mathrm{Tr}[\mathrm{e}^{\mathrm{i}\frac{\gamma}{\sqrt{t}}X} \mathrm{e}^{-\mathrm{i}tH_\lambda} \rho_0 \mathrm{e}^{\mathrm{i}tH_\lambda}]$$

with $X = X \otimes 1_{IDF} \otimes 1_{\mathrm{E}}$ position operator on $l^2(\mathbb{Z}^d)$.

Particle in contact with reservoir



Particle (p) in

a bath of phonons (q)

at Temp > 0

coupling strength: λ

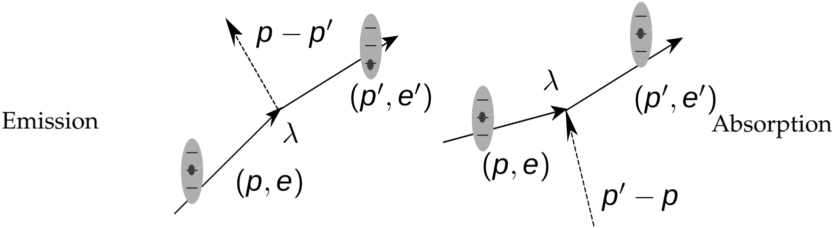


Figure: p, p' are the particle momenta, e, e' are the internal dgf. ($\sim$ vibrational levels)

Dispersive properties of the free bosons

- The bosons enter only via the 'free' correlation function

$$\zeta(x,t) := \mathrm{Tr}\left[\rho_{\mathrm{E}}^{\beta}\Phi(x,t)\Phi(0,0)\right]$$

with the time-evolved, space-translated interaction terms

$$\Phi(x,t) := \int_{\mathbb{T}^d} \mathrm{d}q \left\{ \mathrm{e}^{\mathrm{i}(qx-\omega t)}\phi(q)a_q) + \mathrm{h.c.} \right\}$$

- If $\omega(q) = \sqrt{m^2+|q|^2}$ and ϕ smooth, then

$$\sup_x |\zeta(x,t)| = O(|t|^{-d/2}) \qquad \text{diffusion eq.}$$

- If $\omega(q) = |q|$ and ϕ smooth, then

$$\sup_x |\zeta(x,t)| = O(|t|^{-(d-1)/2}) \qquad \text{lin. wave eq.}$$

Result (D.R., Kupiainen)

Assume that the particle is sufficiently coupled *(Fermi Golden Rule*, later) and that

$$\sup_x |\zeta(x,t)| \leq O(|t|^{-(1+\alpha)}), \qquad \begin{cases} \alpha > 1/2 & \textit{if noneq.} (\beta_1 \neq \beta_2) \\ \alpha > 1/4 & \textit{if eq.} \end{cases}$$

(noneq. setup: replace the field by two fields)
Then, for λ small enough but not zero, the particle motion is diffusive: for any $\kappa \in \mathbb{R}^d$;

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \mathrm{Tr}[\rho_t \mathrm{e}^{\mathrm{i}\gamma \frac{X}{\sqrt{t}}}] \quad \to \quad \mathrm{e}^{-(\gamma, D_\lambda \gamma)}, \qquad \rho_t = \mathrm{e}^{-\mathrm{i}tH} \rho_0 \mathrm{e}^{\mathrm{i}tH}$$

as $t \to \infty$, with

$$D_\lambda = \lambda^2 (D_m + o(|\lambda|^0))$$

and D_m corresponds to Markov approximation (later).

Comments on results

- D_m is the diffusion constant of a Markovian approximation (to be defined). The strict positivity of D_m is the omitted assumption about 'sufficient coupling'.
- **Noneq**: No reasonable $3D$ model satisfies the time-decay assumption, yes in $4D$.
- **Eq**: There are $3D$ models where $|\zeta(x,t)| \sim O(t^{-3/2})$.
- Earlier 4D model (*D.R., J. Fröhlich)* has additional assumption:

$$|\zeta(x,t)| \leq \mathrm{e}^{-ct} \qquad \text{for small 'speed'}: |x| \leq v^*|t|$$

(satisfied if field consists of photons ($\omega(q) = |q|$) in the continuum+ ultrastrong infrared regularity)

The Van Hove limit (weak coupling limit)

Convergence of the reduced density matrix [*Davies, 74*]

Assume that $\int \sup_x |\zeta(x,t)| \,\mathrm{d}t < \infty$. Then

$$\mathrm{e}^{\mathrm{i}t[H_{IDF},\cdot]}\rho_{\mathrm{S},t} \quad \underset{t=\lambda^{-2}\tau}{\overset{\lambda\searrow 0}{\longrightarrow}} \quad \mathrm{e}^{\tau\mathcal{M}}\rho_{\mathrm{S},0}$$

where $\rho_{\mathrm{S},t} \equiv \mathrm{Tr}_{\mathrm{E}}(\mathrm{e}^{-\mathrm{i}tH}\rho_0\mathrm{e}^{\mathrm{i}tH})$ and $\mathcal{M}$ is a “Lindblad operator”

- Partial trace Tr_{E} (*cfr. marginal distribution)* defined by

$$\mathrm{Tr}[(A_{\mathrm{S}}\otimes 1)\rho_{\mathrm{S+E}}] = \mathrm{Tr}_{\mathrm{S}}[A_{\mathrm{S}}\,\mathrm{Tr}_{\mathrm{E}}[\rho_{\mathrm{S+E}}]], \qquad \forall A_{\mathrm{S}} \in \mathcal{B}(\mathcal{H}_{\mathrm{S}})$$

- $\lambda^{-2}(=\epsilon^{-1})$ is timescale where effects of field become visible. The ’fast’ evolution $\mathrm{e}^{-\mathrm{i}t[H_{IDF},\cdot]}$ is subtracted.
- $\mathrm{e}^{\tau\mathcal{M}}$ is a ’Quantum Markov semigroup’; a semigroup of positivity-preserving, trace-preserving maps.

The Van Hove limit: properties of the generator

First, turn off hopping: $\mathcal{H}_S = \mathbb{C}^N$, e.g. $M = \infty$.

- Let e, e' be non-degenerate eigenvalues of H_{IDF},

$$\nu(e) = \langle e|\rho_S|e\rangle, \qquad \text{occupation prob.}$$

with $\langle e|$ the corresponding eigenvector. Then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\nu(e) = \sum_{e'} \left[r(e' \to e)\nu(e') - r(e \to e')\nu(e) \right]$$

with detailed balance $r(e' \to e) = \mathrm{e}^{-\beta(e-e')} r(e \to e')$.
Autonomous behaviour: diagonal independent of off-diagonal: Pauli master equation

- Decoherence

$$\langle e|\rho_S|e'\rangle \sim \mathrm{e}^{-\tau/\tau_{dc}}, \qquad e \neq e'$$

Off-diagonal elements vanish (no Schrodinger cat states!)

Bloch-Boltzmann equation (with hopping)

- Let e, e' eigenvalues of H_{IDF}, and p, p' (quasi)momenta, and

  $$\nu(e,p) := \langle e,p|\rho_S|e,p\rangle$$

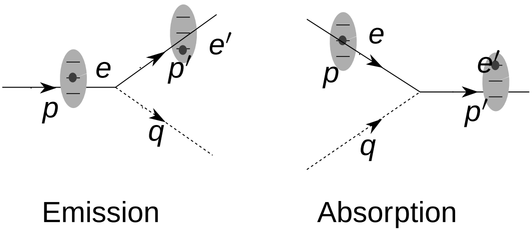
  Diagonal elements evolve as

  $$\frac{\mathrm{d}}{\mathrm{d}\tau}\nu(e,p) = \sum_{e',p'} \left[r(e',p' \to e,p)\nu(e',p') - r(e,p \to e',p')\nu(e,p)\right]$$

  with detailed balance
  $r(e',p' \to e,p) = \mathrm{e}^{-\beta(e-e')} r(e,p \to e',p')$.
- Transport: At momentum p, the particle moves with (group) velocity $v_g(p) = \mathcal{O}(1)$. (position ”slaved” by p, like in LBE)
- Center of mass decoherence (off diagonal position elements)

  $$\langle e,x|\rho_S|e',x'\rangle \sim \mathrm{e}^{-|x-x'|/\ell_{dc}} + \mathrm{e}^{-\tau/\tau_{dc}}$$

The support of the rates $r(e, p \to e', p')$ is determined by 'collision rules'.



$$\text{emission} \begin{cases} p = p' + q \\ e = e' + \omega(q) \end{cases} \qquad \text{absorption} \begin{cases} p + q = p' \\ e + \omega(q) = e' \end{cases}$$

Kinetic energy of particle ($\mathcal{O}(\lambda^2)$) vanishes in these collision rules!

Why do we need the internal degrees of freedom

$$\text{emission} \begin{cases} p = p' + q \\ e = e' + \omega(q) \end{cases} \qquad \text{absorption} \begin{cases} p + q = p' \\ e + \omega(q) = e' \end{cases}$$

Why IDF

If $e = e'$, then necessarily $p = p'$. Hence no change of direction.

Diffusion in the Markovian approximation

By the statement that the system is 'sufficiently well-coupled', we mean that the Bloch-Boltzmann equation exhibits diffusion. This can be checked within the theory of stochastic processes.

Dealing with longer time scales

- We know that on time scales $t \approx \lambda^{-2}$, the particle looks like a random walk. It takes a few steps in this time.
- The corrections to this behaviour are manifestly non-Markovian and long-range in time. The range is determined by the correlation function $\zeta(x,t)$
- This looks like the problem of proving an annealed central limit theorem for a random walk in a time-dependent random environment, with long-range memory
- More generally, this looks like doing perturbation theory around a stochastic system, rather than around the unperturbed Hamiltonian system.

Conclusion

- Realistic Hamiltonian models for diffusion. 3D case included. Only mild assumptions on details of the model.
- Only soft mathematics required: Van Hove scaling limit and perturbation of stochastic systems. Thanks to the introduction of a new time-scale.
- **Phenomenology is beautiful**: Diffusion, decoherence, thermalization, transport, fluctuation-dissipation, quantum ratchets.

Strategy: RWRE I (random walk in random environment)

Let $\tilde{U}_{\tau\in\mathbb{N}}$ be *random* transition kernels on $\mathbb{Z}^d$

- law of $\tilde{U}_\tau$ invariant under rotations and space, time-translations.
- $\mathbb{E}(\tilde{U}_\tau) = \tilde{T}$ is transition kernel of simple random walk, say $\tilde{T} \in \mathcal{B}$ with $\mathcal{B} = \mathcal{B}(l^1(\mathbb{Z}^d) \to l^1(\mathbb{Z}^d))$
- Hence, $\tilde{U}_\tau = \tilde{T} + \tilde{B}_\tau$ with $\tilde{B}_\tau$ 'dynamical disorder'.

$$
\begin{aligned}
\mathbb{E}(\tilde{U}_N \ldots \tilde{U}_1) &= \tilde{T}^N + \sum_{A\neq\emptyset} \mathbb{E}\left(\underbrace{\tilde{B}_{\tau_i} \ldots \tilde{T} \ldots \tilde{B}_{\tau_j}}_{\tilde{B} \text{ at times } \tau_1,\ldots,\tau_m} \right) \\
&= \tilde{T}^N + \sum_{A\neq\emptyset} \mathcal{T}[\otimes_{\tau\in A^c} \tilde{T}_\tau \otimes \mathbb{E}\left(\tilde{B}_{\tau_m} \ldots \tilde{B}_{\tau_1}\right)]
\end{aligned}
$$

where $\mathcal{T}$ time-orders operators: $\mathcal{T}[V_3 \otimes V_2 \otimes V_1] = V_3 V_2 V_1$ and the correlation function $\mathbb{E}(\tilde{B}_{\tau_m} \ldots \tilde{B}_{\tau_1})$ takes values in $\mathcal{B}^{\otimes m}$.

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Let $\tilde{U}_{\tau\in\mathbb{N}}$ be *random* transition kernels on $\mathbb{Z}^d$

- law of $\tilde{U}_\tau$ invariant under rotations and space, time-translations.
- $\mathbb{E}(\tilde{U}_\tau) = \tilde{T}$ is transition kernel of simple random walk, say $\tilde{T} \in \mathcal{B}$ with $\mathcal{B} = \mathcal{B}(l^1(\mathbb{Z}^d) \to l^1(\mathbb{Z}^d))$
- Hence, $\tilde{U}_\tau = \tilde{T} + \tilde{B}_\tau$ with $\tilde{B}_\tau$ 'dynamical disorder'.

$$
\begin{aligned}
\mathbb{E}(\tilde{U}_N \dots \tilde{U}_1) &= \tilde{T}^N + \sum_{A\neq\emptyset} \mathbb{E}\left(\underbrace{\tilde{B}_{\tau_i} \dots \tilde{T} \dots \tilde{B}_{\tau_j}}_{\tilde{B} \text{ at times } \tau_1,\dots,\tau_m}\right) \\
&= \tilde{T}^N + \sum_{A\neq\emptyset} \mathcal{T}[\otimes_{\tau\in A^c} \tilde{T}_\tau \otimes \mathbb{E}\left(\tilde{B}_{\tau_m} \dots \tilde{B}_{\tau_1}\right)]
\end{aligned}
$$

where $\mathcal{T}$ time-orders operators: $\mathcal{T}[V_3 \otimes V_2 \otimes V_1] = V_3 V_2 V_1$ and the correlation function $\mathbb{E}(\tilde{B}_{\tau_m} \dots \tilde{B}_{\tau_1})$ takes values in $\mathcal{B}^{\otimes m}$.

Strategy: RWRE II

Result (Ajanki, D.R., Kupiainen, in progress)

Let $b_\tau(x) = \sum_{x'} |\tilde{B}_\tau(x,x')| \mathrm{e}^{c|x'-x|}$ and assume (for all m)

$$\sum_{1=\tau_1<\ldots<\tau_m} \prod_{j=2}^{m} (|\tau_j - \tau_{j-1}|^{\alpha}) \sup_{x_1,\ldots,x_m} \left| \mathbb{E}^c \left(\prod_{j=1}^{m} b_{\tau_j}(x_j) \right)^T \right| < \delta^m,$$

(Here $\mathbb{E}^c$ stands for the connected correlation function) Then, if $\delta < \delta_0$ and $\alpha > 0$, there is annealed CLT

$$\left[\mathbb{E}(\tilde{U}_N \ldots \tilde{U}_1)\right](0, \sqrt{N}\,\cdot) \quad \underset{N\nearrow\infty}{\longrightarrow} \quad \mathrm{Gauss}_\sigma(\cdot)$$

- Similar framework for RWRE was pioneered in $'91$ by Bricmont-Kupiainen. Here: much easier because integrable correlations.
- Proof: RG + cluster expansion.

Strategy: reduced dynamics

Let U_t be time-evolution acting on probability measures ρ_{SE} of system (S) dgf. (X, V) and environment (E) dgf. $\underline{x}, \underline{v}$. Set

$$T\rho_{\mathrm{S}} := \mathrm{Tr}_{\mathrm{E}}\, U_{\epsilon^{-1}}(\rho_{\mathrm{S}} \times \rho^{\beta}_{\mathrm{E}}) = \int \mathrm{d}\underline{x}\mathrm{d}\underline{v} U_{\epsilon^{-1}}(\rho_{\mathrm{S}} \times \rho^{\beta}_{\mathrm{E}})$$

Then, one expects (modulo space rescaling)

$$T = T_{\epsilon} \quad \underset{\epsilon \searrow 0}{\longrightarrow} \quad \mathrm{e}^{\mathcal{M}}, \qquad \text{as operators on } \rho_{\mathrm{S}}$$

with $\mathrm{e}^{\mathcal{M}}$ time 1 transition kernel of the linear Boltzmann equation.

Good properties of T for small ϵ

- T is a translation-invariant transition kernel, acting on $\rho_{\mathrm{S}}(X, V)$.
- If $\mathrm{e}^{\mathcal{M}}$ has a gap of $O(1)$ (when restricted to functions of V), then so does T. (by spectral perturbation theory)
- discrete-time Markov process defined by $T^N, N \in \mathbb{N}$ has diffusive behaviour of the position (cfr. CLT for LBE)

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Strategy: excitations

Let (U_t^{E} is free E-evolution)

$$U_{\epsilon^{-1}} = T \times U^{\mathrm{E}}_{\epsilon^{-1}} + B, \qquad \text{(this defines } B\text{)}$$

We are really interested in

$$\rho_{\mathrm{S},N\epsilon^{-1}} := \mathrm{Tr}_{\mathrm{E}}(U_{\epsilon^{-1}})^N(\rho_{\mathrm{S}} \times \rho^{\beta}_{\mathrm{E}})$$

can be expanded in sets $A = \{\tau_1, \tau_2, \ldots, \tau_m\} \subset \{1, \ldots, N\}$:

$$\rho_{\mathrm{S},N\epsilon^{-1}} := T^N \rho_{\mathrm{S}} + \sum_{A \neq \emptyset} \mathrm{Tr}_{\mathrm{E}} \underbrace{\ldots B \ldots (T \times U^{\mathrm{E}}_{\epsilon^{-1}}) \ldots B \ldots}_{B \text{ at times } \tau_{1,\ldots,\tau_m}} (\rho_{\mathrm{S}} \times \rho^{\beta}_{\mathrm{E}})$$

$$:= T^N \rho_{\mathrm{S}} + \sum_{A \neq \emptyset} \mathcal{T}[\otimes_{\tau \in A^c} T_\tau \otimes \mathbb{E}\,(B_{\tau_m} \ldots B_{\tau_1})]$$

last line *defines* $\mathbb{E}\,(B_{\tau_m} \ldots B_{\tau_1})$ formally. Unlike T, the B depend on environment dgf.

Good properties of B for small ϵ?

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Good properties of B for small ϵ?

Strategy: excitations

- If T satisfy CLT and $\mathbb{E}^c(B_{\tau_m} \dots B_{\tau_1})$ satisfy the same condition as for RWRE, then we have diffusion, but we need here $\alpha > 1/2$ (power of time decay) because one loses a Ward identity in the RG compared to RWRE.
- Originally, decay of $B - B$ correlations comes from properties of the ideal gas, roughly $\langle b_\tau(x) b_0(0) \rangle$ should originate from

$$\mathbb{E}_E^\beta(\delta(x - x(t))\delta(x(0) - 0)) \sim t^{-d}, \qquad t = \tau\epsilon^{-1}$$

- Smallness δ of B comes from ϵ.
- Controlling all cumulants (as required in our approach) seems out of reach for models like Rayleigh gas with unit mass. But realistic with huge mass

Remarks

- Quenched CLT requires some additional assumption.
- B can not be viewed as enviromnent RV's because they are influenced by particle.
- Much simpler Kipnis-Varadhan approach for symmetric disorder $\tilde{U}(x,x') = \tilde{U}(x',x)$ (then reversible markov process). The above theorem does not exploit this. However, our Hamiltonian model is reversible, so perhaps there is a shortcut possible.