Universal distributions in one dimensional coarsening models

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REFERENCES

A.J. Bray, B. Derrida, C. Godrèche
Non-trivial exponents in the zero temperature dynamics of the 1d Ising and Potts model J. Phys. A27, L357-L361 (1994)

Non-trivial algebraic decay in a soluble model of coarsening Europhys. Lett. 27, 175-180 (1994)

B. Derrida, V. Hakim, V. Pasquier
Exact first passage exponents in 1d domain growth: relation to a reaction-diffusion model Phys. Rev. Lett. 75, 751-754 (1995)

B. Derrida, R. Zeitak
Distribution of domain sizes in the zero temperature Glauber dynamics of the 1d Potts model Phys. Rev. E 54, 2513-2525 (1996)

B. Derrida
Non-trivial exponents in coarsening phenomena Physica D 103, 466-477 (1997)

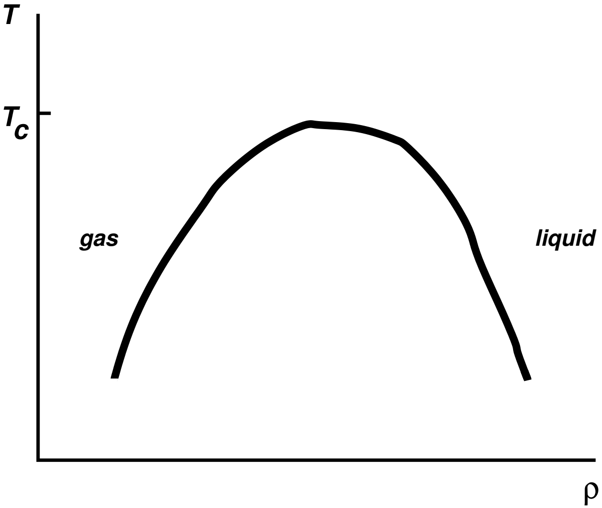
OUTLINE

Introduction

Stochastic Ising Model

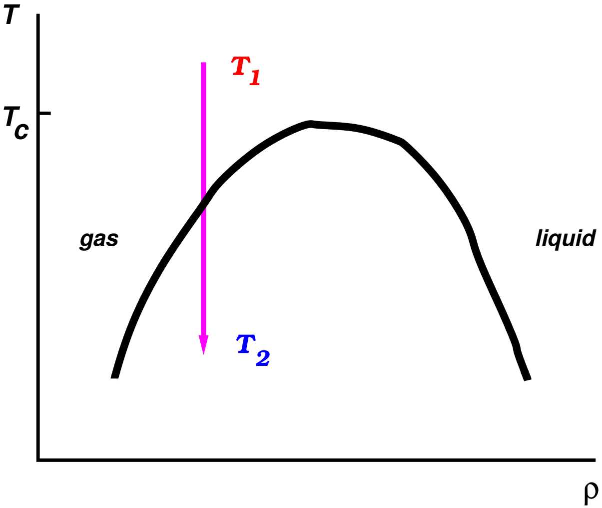
Landau Ginzburg equation

Quench



Fluid

Quench



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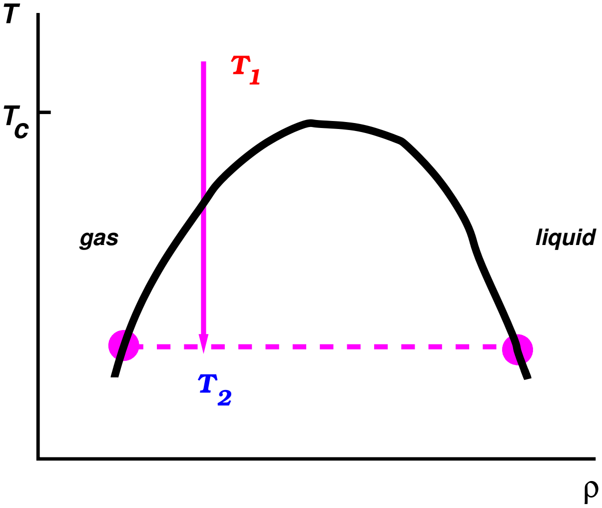

Fluid

Quench of a two dimensional Ising model

Zero temperature Glauber dynamics:

Start with a random $\{S_i(0)\}$

At each time step $t \longrightarrow t + \frac{1}{N}$

- Choose a site i at random
- $S_i \left(t + \frac{1}{N}\right) = \mathrm{Sign}\left[\sum_j S_j(t)\right]$
- If $\sum_j S_j(t) = 0$, then

$$S_i \left(t + \frac{1}{N}\right) = \pm 1$$

with equal probability

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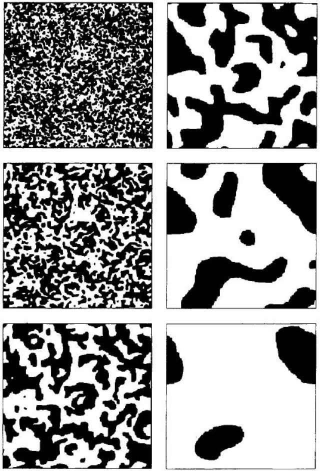

400×400

$t = 8, 32, ..8192$

Scaling limit $t \to \infty$ (Glauber dynamics)

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- Size of domains $\sim t^s$ $\qquad s = 1/2$ Bray 1994
- Energy $E(t) - E(\infty) \sim [\mathrm{Size}]^{-1}$
- Autocorrelation $\langle S_i(t) S_i(0) \rangle \sim [\mathrm{Size}]^{-a}$

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- Geometry and **distribution of domain sizes**

Zero temperature of the q state Potts model

$S_i(t) = 1, 2, ..q$ $\qquad$ $E = \sum_i \delta(S_i(t), S_{i+1}(t))$

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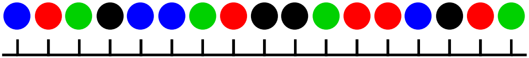

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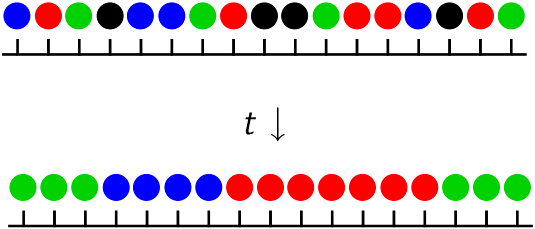

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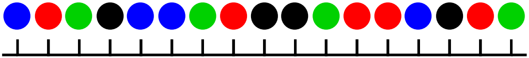
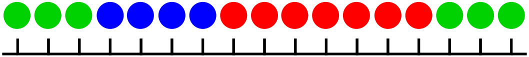


$$t \downarrow$$



$$\boxed{\text{Zero Temperature Glauber dynamics} \equiv \text{Voter model}}$$

Normalized distribution $p(x)$ of domain sizes

$q = \infty$

- $p(x) = \frac{\pi}{2} x e^{-x^2 \pi/4}$
- Universality classes
- Correlations $\frac{\langle x_i x_{i+1} \rangle}{\langle x \rangle^2} = \frac{3}{\pi} \neq 1$

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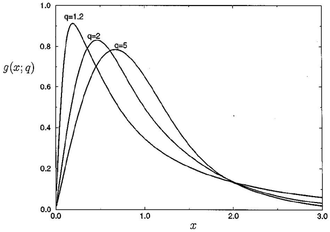
Arbitrary q

$$x = \begin{array}{cl} l_1 & \text{with prob. } 1 - q^{-1} \\ l_1 + l_2 & \text{with prob. } q^{-1} - q^{-2} \\ \dots & \\ l_1 + l_2 + \dots l_n & \text{with prob. } q^{-n+1} - q^{-n} \end{array}$$



Results
Arbitrary q

D Zeitak 1996

- Exact expression of $p(x)$

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For $q > 2$, $A = \frac{q}{4(q-1)} \sum_{n=1}^{\infty} \frac{\mu^n}{n^{3/2}}$ with $\mu = \frac{4(q-1)}{q^2}$

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$$\text{For } q < 2, \quad = \frac{q}{4(q-1)} \left[\sum_{n=1}^{\infty} \frac{\mu^n}{n^{3/2}} + 4\sqrt{-\pi \log(\mu)} \right]$$

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- Small x expansion

$$p(x) = \frac{\pi}{2}\frac{q}{q-1}x - \frac{\pi^2}{8}\frac{q^3}{(q-1)^3}x^3 + \frac{\pi^2}{24}\frac{q^3}{(q-1)^4}x^4 + \cdots$$
$$+ \frac{\pi^6}{1857945600}\frac{q^{10}(64 - 64q + 315\pi q^3)}{(q-1)^{13}}x^{13} + O(x^{14}) .$$

Derivation

- Free Fermions

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- Non intersecting random walks for $0 < \tau < t$

$$Q_n \left\{x_1(0) \to x_1(t), .. < x_n(0) \to x_n(t))\right\} = \mathrm{Det}\left[Q_1 \left\{x_i(0) \to x_j(t)\right\}\right]$$

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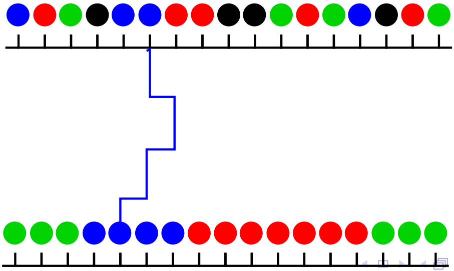

Derivation

- Free Fermions

- Non intersecting random walks for $0 < \tau < t$

$$\mathrm{Pro}_n \{x_1(\tau) < x_2(\tau).. < x_n(\tau)\} = \mathrm{Det}\left[\mathrm{Pro}_2 \{x_i(\tau) < x_j(\tau)\}\right]$$

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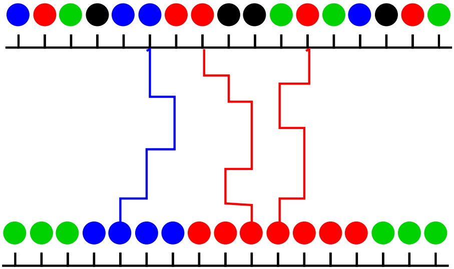


Universality (at $q = \infty$)

If $\langle x \rangle$ **finite at time** $t = 0$

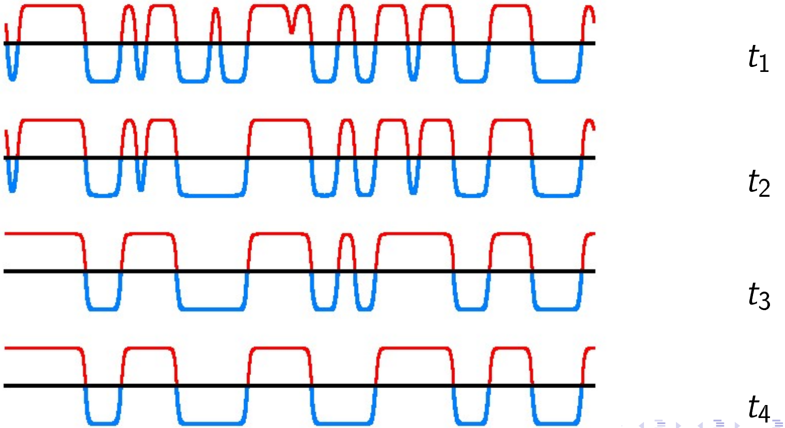
$$p(x) = Cxe^{-x^2/2}$$

If $\langle x \rangle = \infty$ **and** $p(x) \sim x^{-1-\alpha}$ **at** $t = 0$

$$p(x) = Ce^{-x^2/2} \int_0^\infty u^{-1-\alpha} \sinh(xu)\, e^{-u^2/2}\, du$$

Landau Guinzburg equation

$$\boxed{\frac{d\phi}{dt} = \frac{d^2\phi}{dx^2} + \phi - \phi^3}$$



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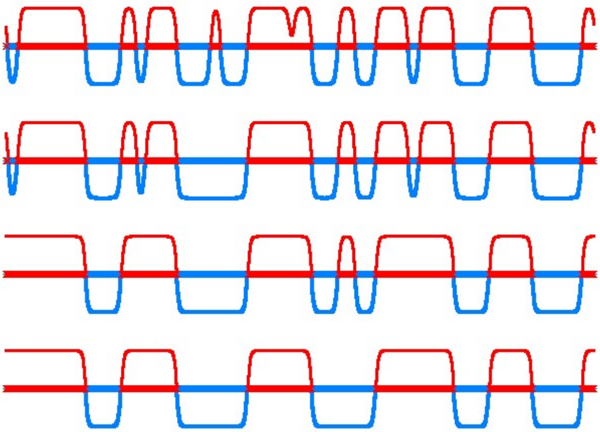

t_1

t_2

t_3

t_4

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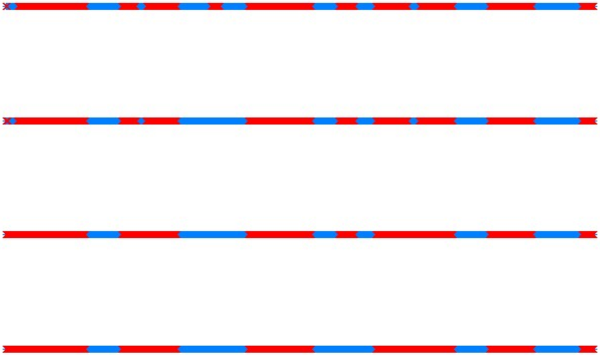

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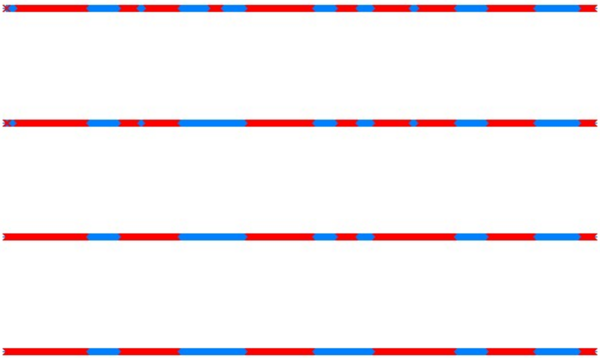
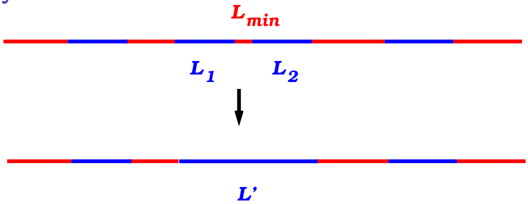
Long time dynamics



$$\boxed{L' = L_1 + L_{\min} + L_2}$$

Long time dynamics



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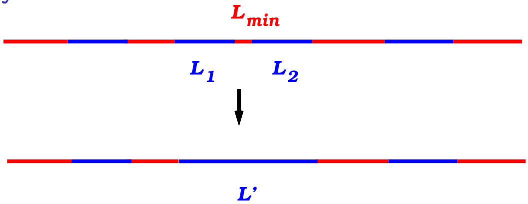

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N total number of intervals

n_L number of intervals of length L

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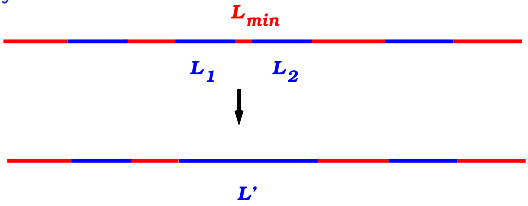

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Under the elimination of all intervals of length $L_{min} \equiv L_0$

$$
\begin{aligned}
N' &= N - 2n_{L_0} \\
n'_L &= n_L \left(1 - \frac{2n_{L_0}}{N}\right) + n_{L_0} \sum_{i=L_0}^{L-2L_0} \frac{n_i}{N} \, \frac{n_{L-i-L_0}}{N}
\end{aligned}
$$

Scaling limit

$L_0 \longrightarrow L_0 + 1$

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$$\boxed{n_L = \frac{N}{L_0} f\left(\frac{L}{L_0}\right)}$$

$$f(x)+xf'(x)+\theta(x-3)f(1)\int_1^{x-2} dy\, f(y)f(x-y-1) = 0$$

Scaling limit

Nagai Kawasaki 1986

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One parameter family of solutions indexed by $f(1)$

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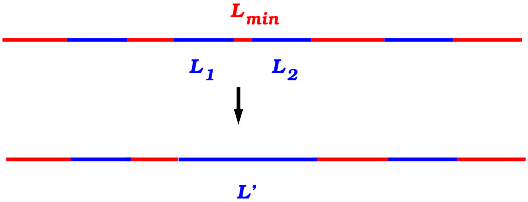
$$\phi(p) = 1 - Ap^{2f(1)}(1+O(p)) \quad \Rightarrow \quad \boxed{f(1) = \frac{1}{2}}$$

Persistence



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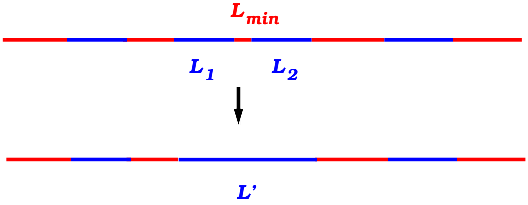

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n'_L d'_L &= n_L d_L \left(1 - \frac{2n_{L_0}}{N}\right) + n_{L_0} \sum_{i=L_0}^{L-2L_0} \frac{n_i}{N} \, \frac{n_{L-i-L_0}}{N} \, (d_i + d_{L-i-L_0}) \; .
\end{aligned}$$

Scaling limit

Bray D Godrèche 1994

$$\boxed{n_L = \frac{N}{L_0} f\left(\frac{L}{L_0}\right) \; , \qquad n_L d_L = N\, L_0^{\beta-1}\, g\left(\frac{L}{L_0}\right)}$$

$$(1-\beta)g(x) + xg'(x) + 2\theta(x-3)f(1)\int_1^{x-2} dy\, g(y)f(x-y-1) = 0$$

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Small p: $\psi(p) = A(\beta) + B(\beta)p^{1-\beta} + O(p) \quad \Rightarrow$

$$B(\beta) = 0 \qquad \Rightarrow \qquad \boxed{\beta = 0.82492412\ldots}$$

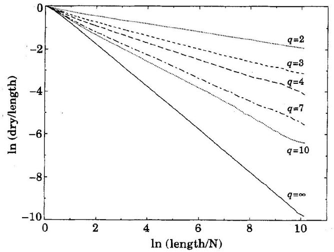
$$B(\beta) = \int_0^\infty dq\, q^{\beta-2}\, e^{-q} \left[(1-q-e^{-q})e^{r(q)} + 2(1-\beta)q + (1-\beta)q^2 e^{-r(q)}\right]$$

with

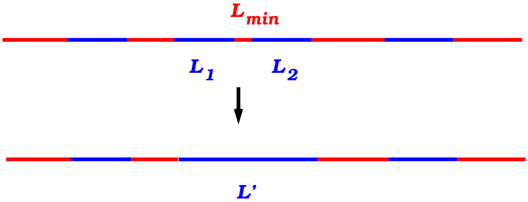
$$r(q) = -\gamma - \sum_{n=1}^{\infty} \frac{(-q)^n}{n\ n!}$$

$$B(\beta) = 0 \quad \Rightarrow \quad \boxed{\beta = 0.82492412\ldots}$$

Persistence $\quad \mathrm{Pro}(S_i(t') = S_i(0)), 0 < t' < t) \sim [\mathrm{Size}]^{\beta-1}$

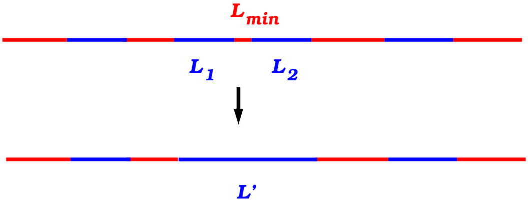


Autocorrelation



$$\boxed{L' = L_1 + L_{\mathrm{min}} + L_2 \qquad ; \qquad A' = A_1 - A_{\mathrm{min}} + A_2}$$

Autocorrelation



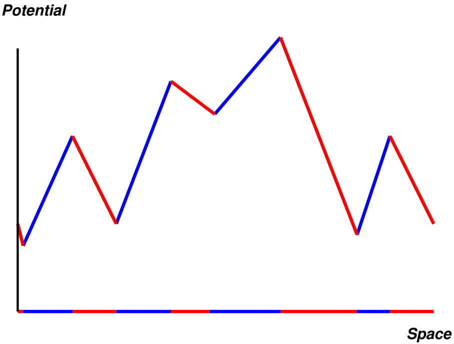
$$\boxed{L' = L_1 + L_{\min} + L_2 \qquad ; \qquad A' = A_1 - A_{\min} + A_2}$$

Under the elimination of all intervals of length $L_{min} \equiv L_0$

$$
\begin{aligned}
N' &= N - 2n_{L_0} \\
n'_L &= n_L \left(1 - \frac{2n_{L_0}}{N}\right) + n_{L_0} \sum_{i=L_0}^{L-2L_0} \frac{n_i}{N} \; \frac{n_{L-i-L_0}}{N} \\
n'_L a'_L &= n_L a_L \left(1 - \frac{2n_{L_0}}{N}\right) + n_{L_0} \sum_{i=L_0}^{L-2L_0} \frac{n_i}{N} \; \frac{n_{L-i-L_0}}{N} \; (a_i - a_{L_0} + a_{L-i-L_0})
\end{aligned}
$$

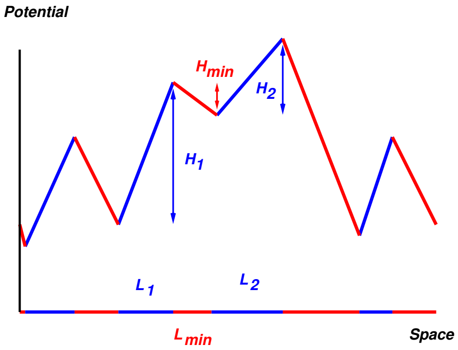
Sinai Diffusion

Le Doussal Monthus Fisher 1999
Igloi Monthus 2005



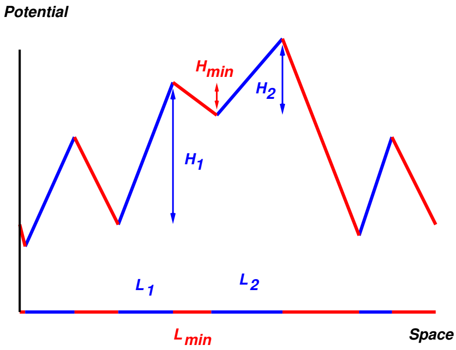
Sinai Diffusion

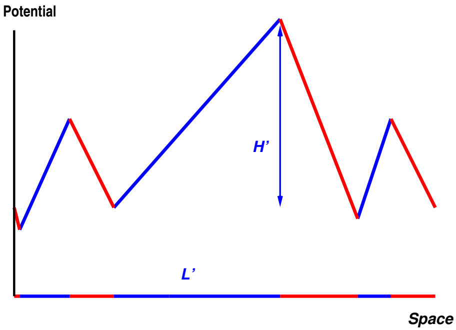
Le Doussal Monthus Fisher 1999
Igloi Monthus 2005



Sinai Diffusion

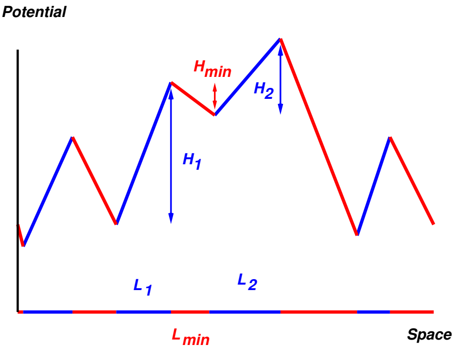
Le Doussal Monthus Fisher 1999
Igloi Monthus 2005

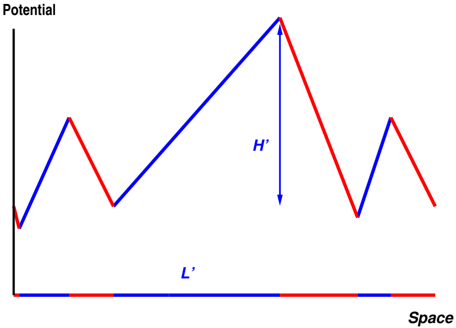




Sinai Diffusion

Le Doussal Monthus Fisher 1999
Igloi Monthus 2005

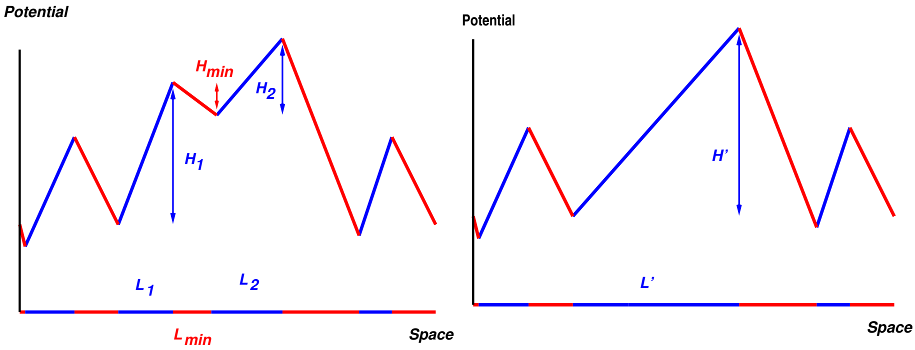




$$\boxed{L' = L_1 + L_{\min} + L_2 \qquad ; \qquad H' = H_1 - H_{\min} + H_2}$$

Sinai Diffusion

Le Doussal Monthus Fisher 1999
Igloi Monthus 2005



$$\boxed{L' = L_1 + L_{\rm min} + L_2 \qquad ; \qquad H' = H_1 - H_{\rm min} + H_2}$$

Under the elimination of all intervals of height $H_{min} \equiv H_0$

Conclusion

- Universality
- Exact real space renormalization