Scaling limits in one dimensional hierarchical coalescence processes

Alessandra Faggionato

Department of Mathematics
University La Sapienza

Joint work with F. Martinelli, C. Roberto, C. Toninelli

Motivations

Evolution of 1d systems dominated by coalescence of domains, e.g.

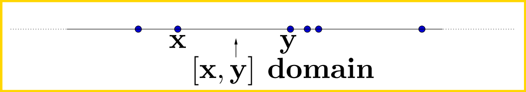
- large vapor droplets in breath figures,
- interacting particle systems at low temperature.

Physics literature: appearance of a scale–invariant morphology for large times

- supported by simulations,
- exact calculation of the limit state (physically relevant), assuming convergence,
- different models have the same or similar limit states.

Hierarchical coalescence processes

State: $\xi \subset \mathbb{R}$ locally finite



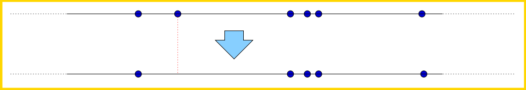
- $[\mathbf{x}, \mathbf{y}]$ **domain**
- $\mathbf{y} - \mathbf{x}$ **domain length**

Hierarchical time:

- **infinite epochs indexed by** $\mathbf{n} \geq 1$,
- $\mathbf{t} \in [0, \infty]$ **time inside** $\mathbf{n}^{\mathbf{th}}$**–epoch.**

Evolution (jump dynamics):

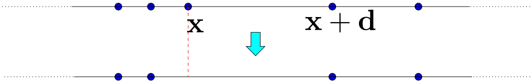
- **Sequence of trajectories** $\left(\xi^{(\mathbf{n})}(\mathbf{t}) : \mathbf{t} \in [\mathbf{0}, \infty]\right)_{\mathbf{n} \geq \mathbf{1}}$,
- $\xi^{(\mathbf{n})}(\infty) = \xi^{(\mathbf{n+1})}(\mathbf{0})$,
- **Jump: coalescence of two neighboring domains.**



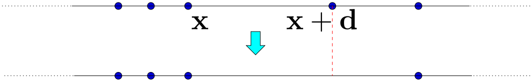
Jump dynamics at epoch n

Jump rates:

- $\lambda_\ell^{(n)}(d)$: rate a domain $[x, x+d]$ of length d merges with its left neighbor.



- $\lambda_r^{(n)}(d)$: similar



Dynamics

Dynamical parameters:

- $\mathbf{0 < d^{(1)} < d^{(2)} < \cdots \to \infty}$
- **Jump rates:** $\lambda_\ell^{(\mathbf{n})}(\mathbf{d})$, $\lambda_\mathbf{r}^{(\mathbf{n})}(\mathbf{d})$

$$\lambda^{(\mathbf{n})}(\mathbf{d}) = \lambda_\ell^{(\mathbf{n})}(\mathbf{d}) + \lambda_\mathbf{r}^{(\mathbf{n})}(\mathbf{d})$$

$\lambda^{(\mathbf{n})}(\mathbf{d})$ **rate a domain of length d incorporates one of its neighbors**

Assumptions

$\lambda^{(n)}(d)$ rate a domain of length d incorporates one of its neighbors

Assumptions:

- **(A1)** $\lambda^{(n)}(d) > 0$ iff $d \in [d^{(n)}, d^{(n+1)})$
- **(A2)** $d, d' \geq d^{(n)} \Rightarrow d + d' \geq d^{(n+1)}$

i.e. $2d^{(n)} \geq d^{(n+1)}$

Consequences:

- **(A1)** $\Rightarrow$ a domain can incorporate one of its neighbors iff $d \in [d^{(n)}, d^{(n+1)})$
- **(A2)** $\Rightarrow$ a domain resulting from a coalescence cannot incorporate other domains if at the beginning of the epoch n domain lenghts are at least $d^{(n)}$

Consequences

$\mathcal{N}(\mathbf{d}) \quad : \xi \subset \mathbb{R}$ **locally finite, domains of length** $\geq \mathbf{d}$

Suppose $\xi^{(\mathbf{1})}(\mathbf{0}) \in \mathcal{N}(\mathbf{d}^{(\mathbf{1})})$**. Then:**

- $\xi^{(\mathbf{1})}(\mathbf{t})$, $\mathbf{t} \in [\mathbf{0}, \infty)$, **is constant on finite intervals eventually**
- $\xi^{(\mathbf{1})}(\infty) := \lim_{\mathbf{t}\uparrow\infty} \xi^{(\mathbf{1})}(\mathbf{t}) \in \mathcal{N}(\mathbf{d}^{(\mathbf{2})})$
- **Setting** $\xi^{(\mathbf{2})}(\mathbf{0}) := \xi^{(\mathbf{1})}(\infty)$, **same properties hold for epoch 2 and so on.**

The dynamics is well defined for all epochs

Example: paste–all model

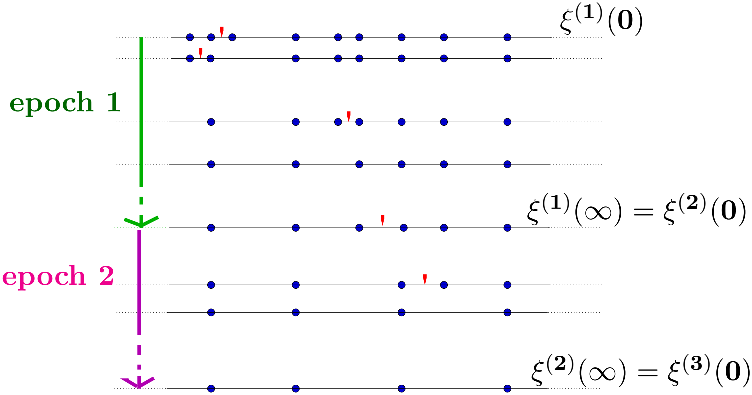
Integer domain lengths

At each step one searches for the shortest domain which is pasted as a whole to either one of its neighbors, with equal probability

Model:

- $\xi^{(\mathbf{1})}(\mathbf{0}) \subset \mathbb{Z}$
- $\mathbf{d^{(n)} := n}$,
- $\lambda_\ell^{(\mathbf{n})}(\mathbf{d}) = \lambda_{\mathbf{r}}^{(\mathbf{n})}(\mathbf{d}) = \frac{\mathbf{1}}{\mathbf{2}}\mathbb{I}(\mathbf{n} \leq \mathbf{d} < \mathbf{n} + \mathbf{1})$

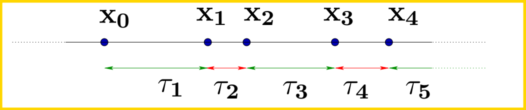
Paste–all model



Renewal Simple Point Processes

Def. A random set $\xi \subset \mathbb{R}$ is called **right renewal SPP** with **first point law** ν, **interval law** μ, shortly $\xi \sim \mathrm{Ren}(\nu, \mu)$ if:

- $\xi = \{x_k : k \geq 0\}$, $x_k < x_{k+1}$
- x_0 has law ν
- $\tau_k = x_k - x_{k-1}$ $(k \geq 1)$ has law μ
- x_0, τ_k $(k \geq 1)$ are independent



($\mathbb{Z}$–)Stationary renewal SPP

- $\xi = \{\mathbf{x_k} : \mathbf{k} \in \mathbb{Z}\}$, $\mathbf{x_k} < \mathbf{x_{k+1}}$
- $\xi \sim \mathbf{Ren}_{\mathbb{Z}}(\mu)$ **law invariant by integer translations**
- $\xi \sim \mathbf{Ren}(\mu)$ **law invariant by real translations**
- $\int \mu(\mathbf{x})\mathbf{x} < \infty$ **!!!**

Renewal initial distribution

Let $\xi^{(1)}(0) \sim \mathrm{Ren}(\nu, \mu)$

- ν first point law, probability on $\mathbb{R}$
- μ interval law, probability on $[d^{(1)}, \infty)$

Fact:

- For all $n \geq 1$ and $t \in [0, \infty]$, $\xi^{(n)}(t) \sim \mathrm{Ren}(\nu_t^{(n)}, \mu_t^{(n)})$;
- $\nu_t^{(n)} \Rightarrow \nu_\infty^{(n)}$, $\mu_t^{(n)} \Rightarrow \mu_\infty^{(n)}$.

Question: what about $\nu_\infty^{(n)}$ and $\mu_\infty^{(n)}$, $n \to \infty$?

Similar situation with $\xi^{(1)}(0) \sim \mathrm{Ren}_{\mathbb{Z}}(\mu), \mathrm{Ren}(\mu)$.
Then $\xi^{(n)}(t) \sim \mathrm{Ren}_{\mathbb{Z}}(\mu_t^{(n)}), \mathrm{Ren}(\mu_t^{(n)})$

Scaling limit for $\mu_0^{(n)}$

Let X_n be a random variable with law $\mu_0^{(n)}$

Recall $\mu_0^{(1)} = \mu$

Theorem. Let $g(s) := \int_{[d^{(1)},\infty)} e^{-sx}\mu(dx)$. Suppose that

$$\lim_{s\downarrow 0} -\frac{sg'(s)}{1-g(s)} = c_0,$$

then

(i) $c_0 \in [0,1]$,

(ii) $X_n/d^{(n)} \Rightarrow X_\infty$,

(iii) X_∞ has Laplace transform

$$g_\infty^{(c_0)}(s) = 1 - \exp\Big\{ - c_0 \int_1^\infty \frac{e^{-sx}}{x} dx\Big\}.$$

Remarks

- **Universality: The asymptotics depends only on $\mathbf{c_0}$, not on the dynamics (apart $\mathbf{d^{(n)}}$–rescaling)**
- **Limit theorems for $\nu_{\mathbf{0}}^{(\mathbf{n})}$, after same rescaling. Classes of universality, e.g.**
  (i) $\lambda_{\mathbf{r}}^{(\mathbf{n})} \equiv \mathbf{0}$, **(ii)** $\lambda_{\ell}^{(\mathbf{n})} \equiv \mathbf{0}$,
  (iii) $\lambda_{\ell}^{(\mathbf{n})} = \gamma \lambda_{\mathbf{r}}^{(\mathbf{n})}$ **with** $\gamma \in (\mathbf{0}, \infty)$

Remarks

- **Scaling limits for 1d hierarchical coalescence processes with triple coalescences, i.e. a domain can merge also with both of its neighbors**
- **Scaling limits for initial exchangeable simple point processes**

Sufficient conditions

- $\mathbf{g(s)} = \int_{[\mathbf{d}^{(1)},\infty)} \mathbf{e}^{-\mathbf{sx}}\mu(\mathbf{dx})$,
- $(\star)$ $\lim_{\mathbf{s}\downarrow 0} -\frac{\mathbf{sg}'(\mathbf{s})}{1-\mathbf{g}(\mathbf{s})} = \mathbf{c_0}$

Fact.

(i) μ **has finite mean** $\Rightarrow$ $(\star)$ **holds with** $\mathbf{c_0} = 1$

In particular, same scaling limit when starting with ($\mathbb{Z}$–)stationary renewal SPPs.

(ii) **If** $\mu([\mathbf{x},\infty)) = \mathbf{x}^{-\alpha}\mathbf{L}(\mathbf{x})$,
$\mathbf{L}(\mathbf{x})$ **slowly varying as** $\mathbf{x} \to \infty$ **and** $\alpha \in [0,1]$,
then $(\star)$ **holds with** $\mathbf{c_0} = \alpha$.

Remarks

- $\mathbf{L(x)}$ **slowly varying as** $\mathbf{x \to \infty}$:

$$\lim_{\mathbf{x}\to\infty} \mathbf{L(cx)/L(x) = 1}\,, \quad \forall \mathbf{c > 0}\,.$$

- **Fixed** $\alpha \in (\mathbf{0}, \mathbf{1})$,

$$\mu\big([\mathbf{x}, \infty)\big) = \mathbf{x}^{-\alpha}\mathbf{L(x)}$$

with $\mathbf{L(x)}$ **slowly varying as** $\mathbf{x} \to \infty$ **iff** μ **belongs to domain of attraction of the** α**–stable law.**

- **Suppose** μ **law of** $\mathbf{e^Z}$, $\mathbf{Z}$ **geometric r.v. of parameter** $\mathbf{p = 1 - e^{-\lambda}}$, $\lambda \in (\mathbf{0}, \mathbf{1})$. **Then**

$$\nexists \lim_{\mathbf{s}\downarrow\mathbf{0}} -\frac{\mathbf{sg'(s)}}{\mathbf{1 - g(s)}}\,.$$

Recursive identities

- $\mathbf{X_n}$ **random variable with law** $\mu_{\mathbf{0}}^{(\mathbf{n})}$
- $\mathbf{g_n(s)} := \mathbb{E}\left(\exp\{-\mathbf{sX_n}/\mathbf{d^{(n)}}\}\right)$
- $\mathbf{h_n(s)} := \mathbb{E}\left(\exp\{-\mathbf{sX_n}/\mathbf{d^{(n)}}\}\,;\, \mathbf{d^{(n)}} \le \mathbf{X_n} \le \mathbf{d^{(n+1)}}\right)$
- $\mathbf{a_n} := \mathbf{d^{(n+1)}}/\mathbf{d^{(n)}}$

$$(*) \qquad \mathbf{1 - g_{n+1}(a_n s) \;=\; (1 - g_n(s))\, e^{h_n(s)}}\,, \qquad \mathbf{n \ge 1}$$

Hard non–linear problem !

Key transformation

Fact:

$\exists \mathbf{t_n}$ **non–negative measure on** $[\mathbf{0}, \infty)$ **such that**

$$\mathbf{g_n(s) = 1 - \exp\left\{ - \int_0^\infty \frac{e^{-s(1+x)}}{1+x} t_n(dx) \right\}}$$

Limit points $\mathbf{t_n(dx) = c_0\, dx}$

Benefit of key transformation

- $\Phi_n(x) := a_n(1 + x) - 1$
- $t_n \circ \Phi_n(A) := t_n\,(\,\Phi_n(A)\,)\,, \qquad A \subset [0, \infty)$

Then the recursive system $(*)$ becomes

$$t_{n+1} = (1/a_n)t_n \circ \Phi_n\,, \qquad n \geq 1$$

Treatable non–linear problem !

Bibliography

- **F. M. R. T. Universality in one dimensional hierarchical coalescence processes. Preprint (2010). arXiv:1007.0109**
- **F. M. R. T. Universality in one dimensional hierarchical coalescence processes with double and triple coalescences. Forthcoming.**