East model: Rigorous results for the low temperature non-equilibrium dynamics

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Overview

1. East model: definition
2. High density non-equilibrium regime is described via a Hierarchical Coalescence Process (HCP)
3. Previous talk: Universal limiting behavior for HCP
4. $\longrightarrow$ staircase for one-time quantities (e.g. density);
   $\longrightarrow$ aging for two-time quantities (e.g. density-density autocorrelation);
   $\longrightarrow$ scaling limit for the distribution.

The East model
[Jackle, Eisinger '91]

One dimensional KCM with Glauber dynamics

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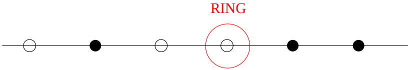
Kinetic constraint: the right neighbor should be empty



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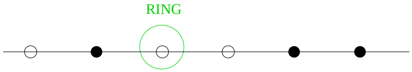

$\Downarrow$

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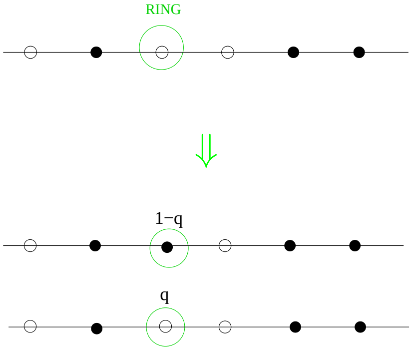


A little bit of notation

Configurations : $\sigma \in \{0,1\}^{\mathbb{Z}}, \quad \forall x \in \mathbb{Z} : \quad \sigma(x) \in \{0,1\}$

σ_t:=process at time t with initial config. σ

$\mathbb{E}_\sigma$:= mean over the process started at σ

Reversible w.r.t. Bernoulli product measure:
$\mu(\sigma) = \prod_{x\in\mathbb{Z}} \mu_x(\sigma(x)), \quad \mu_x(0) = q \quad \rightarrow$ μ is invariant

Old results

Exponential convergence to equilibrium

Theorem (Cancrini, Martinelli, Schonmann, C.T. '09)

Let $\nu := \prod_{x\in\mathbb{Z}} \nu_x(\sigma(x))$ *with* $\nu_x(0) = q_0 \in [0,1)$, $\quad q_0 \neq q$

$$\left| \int d\nu(\sigma)\mathbb{E}_\sigma(f(\sigma_t)) - \mu(f) \right| \leq C_f \exp(-t \;\; gap/2)$$

Fast divergence of relaxation time $T = 1/\text{gap}$

Theorem (Aldous, Diaconis '02 + Cancrini, Martinelli, Roberto, C.T. '08)

- *gap*> 0 *for* $q \in (0,1)$
- *gap*$(q) \simeq \exp\left(-\frac{|\log q|^2}{2\log 2}\right) \qquad\qquad q \downarrow 0$

High density non-equilibrium dynamics

Setting:

- Start at time zero from a non-equilibrium measure ν;
- run dynamics with $q \ll 1$

We focus on the pre-asymptotic regime:

- large time-scales τ, $\lim_{q\downarrow 0} \tau(q) = \infty$
- $\tau(q) \ll T(q) = 1/\mathrm{gap(q)}$

Key observations I

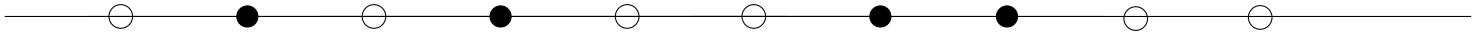
$q \downarrow 0$:
$\longrightarrow$ evolution dominated by killing excess vacancies
= coarsening of intervals separated by consecutive vacancies

kinetic constraints:
$\longrightarrow$ to kill a vacancy there must be a vacancy on the right n.n., otherwise it should be created starting from the next vacancy

$\rightarrow$ cooperative relaxation / energy barriers.

Which energy barrier should I overcome to kill a vacancy which sees the next vacancy on the right at distance ℓ?

Key observations II



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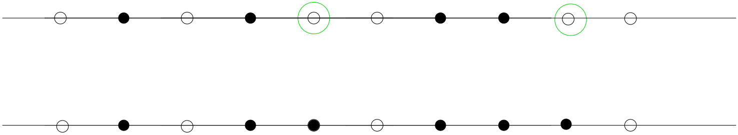

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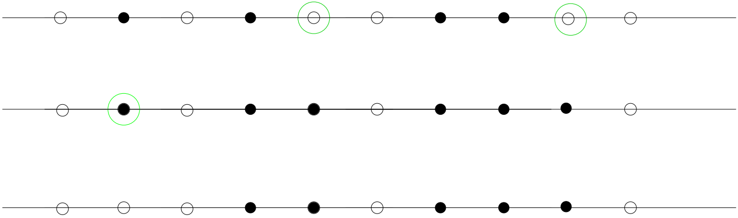
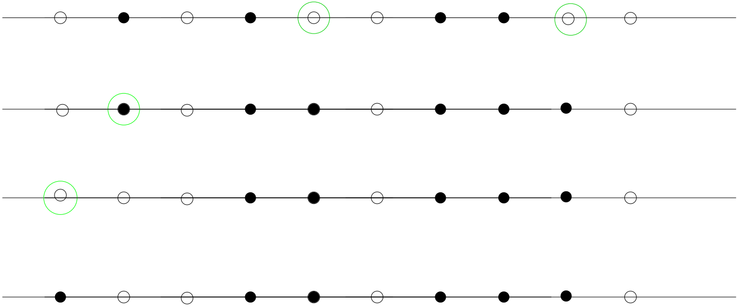


One extra vacancy → probability q, typical time $1/q$

Key observations II



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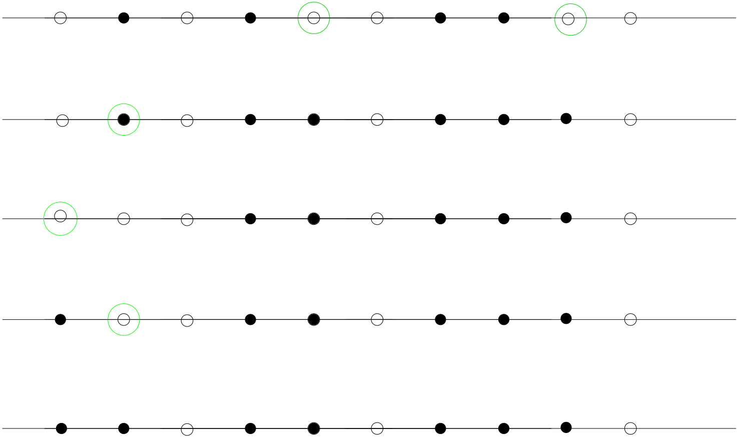
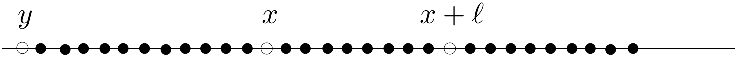


One extra vacancy → probability q, typical time $1/q$

Key observations II

[Evans, Sollich '99]

$$\ell \in [2^{n-1}+1, 2^n], \qquad n \geq 1$$



To kill vacancy at x we should create n simultaneous vacancies

$\rightarrow$ probability q^n= energy barrier n= activation time

$$t_n = (1/q)^n$$

Domain = interval separating two consecutive vacancies
Killing vacancy at x = coalescing domain $[y, x]$ with $[x, x+\ell]$

Active and stalling periods

Domain $[x, x+\ell]$ is of class n if $\ell \in [2^{n-1}+1, 2^n]$, $n \geq 1$
Vacancy at x is of class n if it is left border of domain of class n

$$\lim_{q \downarrow 0} t_n / t_{n-1} = \infty, \quad t_n^+ := t_n^{1+\epsilon} = \left(\frac{1}{q^n}\right)^{1+\epsilon}, \quad t_n^- := t_n^{1-\epsilon} = \left(\frac{1}{q^n}\right)^{1-\epsilon}$$

Up to time t_n^- vacancies of class n cannot disappear;
at time t_n^+ all vacancies of class n have disappeared.

$[t_n^-, t_n^+] = n$-th active period:
all vacancies of class n are killed or become of larger class.

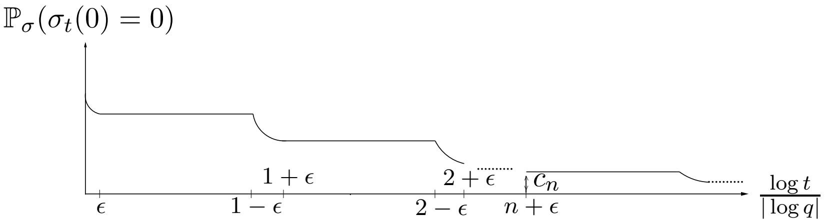
$[t_n^+, t_{n+1}^-] = n$-th stalling period:
nothing happens.

Staircase behavior

Theorem (I)

$\forall \sigma \in \{0,1\}^{\mathbb{Z}}$, $\forall n \in \mathbb{N}$ *there exists* $c_n(\sigma)$ *s.t.*

$$\lim_{q \downarrow 0} \sup_{t \in [t_n^+, t_{n+1}^-]} |\mathbb{P}_\sigma(\sigma_t(0) = 0) - c_n(\sigma)| = 0$$



Heigth of plateaux

x_k = position of the k-th vacancy ($x_0 \leq 0 < x_1$ and $x_k < x_{k+1}$).
ν= measure on $[1, 2, \dots)$
Ren(ν)= stationary renewal distribution with interval law ν:
- invariant under translations on $\mathbb{Z}$;
- w.r.t. Ren(ν)($\cdot|\sigma(0) = 0$) the domain lengths $\{x_k - x_{k-1}\}_{k=-\infty}^{\infty}$ are i.i.d. with law ν.

Theorem (II)

If the initial distribution is $Q = Ren(\nu)$ then

$$\lim_{n\to\infty} \lim_{q\downarrow 0} \mathbb{E}_Q(c_n)(2^n + 1) = 1$$

Coarsening

Let $\tilde{X}$ be a random variable with

$$\mathbb{E}(\exp(-s\tilde{X})) = 1 - \exp\left\{ -c_0 \int_1^\infty \frac{e^{-sx}}{x} dx \right\}$$

Theorem (III)

If the initial distribution is $Q = Ren(\nu)$ then for any bounded function f and any $k \in \mathbb{Z}$

$$\lim_{n\uparrow\infty} \lim_{q\downarrow 0} \sup_{t\in[t_n^+, t_{n+1}^-]} \left| \mathbb{E}_Q\big(f(X_k^{(n+1)}(t))\big) - E\big(f(\tilde{X})\big) \right| = 0$$

where $X_k^{(n)}(t) := (x_{k+1}(t) - x_k(t))/(2^{n-1}+1)$

$\Longrightarrow$ Mean domain length grows as $t^{\log 2/|\log q|}$

Aging

Two-time quantities age

$\mathrm{Cov}_\sigma(\sigma_t(x);\sigma_s(x)) := \mathbb{E}_\sigma(\sigma_t(x)\sigma_s(x)) - \mathbb{E}_\sigma(\sigma_t(x))\mathbb{E}_\sigma(\sigma_s(x))$

Theorem (IV)

Fix $\sigma \in \{0,1\}^{\mathbb{Z}}$, $\forall n,m \in \mathbb{N}$ *there exists* $c_{n,m,x}(\sigma)$ *s.t.*

$$\lim_{q\downarrow 0} \sup_{t\in[t_n^+,t_{n+1}^-]} \sup_{s\in[t_m^+,t_{m+1}^-]} \left|Cov_\sigma(\sigma_t(x);\sigma_s(x)) - c_{n,m,x}(\sigma)\right| = 0$$

If $Q = Ren(\nu)$ *set* $\rho_x := Q(\sigma(x)=0)$, *then*

$$\lim_{n\to\infty}\lim_{m\to\infty}\lim_{q\downarrow 0}\left|\mathbb{E}_Q(c_{n,m,x}) - \frac{\rho_x}{(2^n+1)}\left(1-\frac{\rho_x}{(2^m+1)}\right)\right| = 0$$

Key ingredient: Persistence of zeros = density of zeros for $q\downarrow 0$:
$\lim_{q\downarrow 0}\left|\mathbb{P}_\sigma(\sigma_t(x)=0) - \mathbb{P}_\sigma(\sigma_s(x)=0,\ \ \forall s\in[0,t])\right| = 0$

East vs HCP I

Look at evolution in $[t_n^-, t_n^+]$ (active + stalling period n), then:

1. initially domains are of class at least n ($\ell \in [2^{n-1}+1, 2^n]$);
2. domains of class n coalesce with their left neighbour;
3. the newly created domain is of class at least $n+1$;
4. metastability: typical time t_n to kill a vacancy but time needed to perform killing is t_{n-1}

→ Mapping with the evolution inside one epoch of HCP with:

1. active domains $d \in [d^{(n)}, d^{(n+1)})$, $d^{(n)} = 2^{n-1}+1$;
2. $\lambda_r^{(n)}(d) = 0$;
3. $\lambda_l^{(n)}(d) > 0$ is expressed in terms of a large deviation probability for East

East vs HCP II

Mapping + our results for HCP (cfr. previous talk)
$\rightarrow$ Theorem II + III

Heuristic picture + connection to HCP : Evans, Sollich '99.

Our main novelties:

- Rigorous proof (for the approximation via HCP, for the scaling limit of HCP);
- scaling limit of the domain distribution when the initial distribution is not Bernoulli (Theorem II: different scaling if ν is in the domain of attraction of α-stable law);
- scaling for the position of the first vacancy.

Out of equilibrium for other KCMs

The sad or interesting state of the art

From the mathematical point of view:

- convergence to equilibrium is expected if q, q_0 are in the ergodicity regime (where $gap > 0$).
  Only for FA1f it can be proven and only in a limited regime $q \in [\bar{q}, 1]$ (instead of $q \in (0, 1]$) work in progress
- ...

From the physical point of view:

- KCM with ergodicity breaking transition at $q_c \in [0, 1]$ (e.g. North-East, spiral): asymptotics when $q_0 > q_c$ and $q < q_c$?
  numerical results for spiral Corberi, Cugliandolo '09: coarsening process with analogy to quenched ferromagnets
- asymptotic behavior when $q \downarrow q_c$?
  only other known results are for FA1f, Sollich, Mayer '07

Bibliography

A.Faggionato, F.Martinelli, C.Roberto, C.T. arXiv:1012.4912

A.Faggionato, F.Martinelli, C.Roberto, C.T. arXiv:1007.0109

P.Sollich, M.Evans, Phys.Rev.Lett, 83 (1999), p.3238–3241

P.Sollich, M.Evans, Phys.Rev. E (2003), 031504