Periodic striped ground states
in Ising models with competing interactions

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Based on joint works with J. Lebowitz, E. Lieb and R. Seiringer

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Outline

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Periodic patterns

The spontaneous emergence of periodic states is an ubiquitous phenomenon in nature.

Nevertheless, a fundamental understanding of why crystals, or ordered patterns, form is still missing.

In this talk I would like to focus on the phenomenon of formation of periodic arrays of stripes or slabs, which are observed in a variety of systems, ranging from magnetic films to superconductors, polymer suspensions, twinned martensites, etc.

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Stripes

Fig. 3. Magnetic image of the 308 nm Py film (6 μm scan). The half period is 220 ± 30 nm, slightly smaller than the film thickness.

Stripes
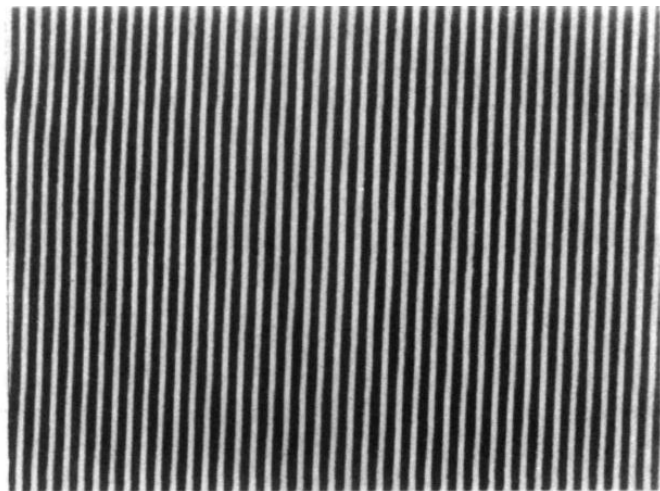
Magnetic garnet film $(YGdTm)_3(FeGa)_6O_2$ at $H = 0$

Competing interactions

The basic mechanism behind stripe formation seems to be the competition between a short-range attractive and a long-range repulsive interaction.

The resulting frustration induces the system to form mesoscopic islands of a uniform phase, which alternate regularly on the scale of the whole sample.

Striped and slabbed patterns

Theoretically, the understanding of these regular patterns is based on a variational computation of the "best energy" among a selected class of periodic states.

Remarkably, in many different situations, the "best state" seems to be striped or slabbed. But why?

Outline

A standard model for competing interactions

In many situations, the following Hamiltonian is the simplest realistic model for pattern formation in systems with competing interactions:

$$H = -J \sum_{\langle \mathbf{x}, \mathbf{y} \rangle} (\sigma_{\mathbf{x}} \sigma_{\mathbf{y}} - 1) + \frac{1}{2} \sum_{\mathbf{x} \neq \mathbf{y}} \frac{(\sigma_{\mathbf{x}} \sigma_{\mathbf{y}} - 1)}{|\mathbf{x} - \mathbf{y}|^p}$$

The long range interaction can model:

- a Coulomb potential ($p = 1$),

- a dipolar potential ($p = 3$).

More general values of p describe a "generic" antiferromagnetic power law potential.

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Known facts about the ground state

What are the ground states of the system?

Limiting cases:

- $J = +\infty$: homogeneous FM state
- $J = 0$: AF state (by RP: Fröhlich-Israel-Lieb-Simon 1978)

The FM state is stable for J large, for all $p > d + 1$

(Ginibre-Grossmann-Ruelle 1966)

The AF state is stable for $J \gtrsim 0$ for all d, p.

If $p > d + 1$, the FM transition line can be

computed exactly (G-Lebowitz-Lieb 2011):

$$J = J_c(p) = \frac{1}{2} \sum_{\mathbf{y} \in \mathbb{Z}^d} \frac{|y_1|}{|\mathbf{y}|^p}$$

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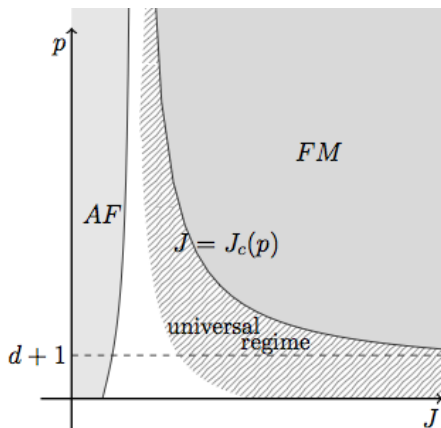- $J = 0$:   AF state (by RP: Fröhlich-Israel-Lieb-Simon 1978)

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The expected ground state phase diagram

What happens in the intermediate region?

Expected: periodic patterns!

Which periodic patterns?

Standard procedure: minimize the energy among a class of periodic states, say stripes and checkerboard. If $d = 2$, in the universal regime:

$$\boxed{2 < p < 3} \quad e_s \sim -J^{-\frac{p-2}{3-p}} , \qquad\qquad e_c \sim -J^{-\frac{p-2}{3-p}}$$

$$\boxed{p = 3} \quad e_s \sim -e^{-J/2} , \qquad\qquad e_c \sim -e^{-J/2}$$

$$\boxed{3 < p \leq 4} \quad e_s \sim -(J_c - J)^{\frac{p-2}{p-3}} , \quad e_c \sim -(J_c - J)^{\frac{p-2}{p-3}}$$

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Rigorous results

1. If the ground state is striped, then it is periodic. In particular, in $d = 1$ it is always periodic (G-Lebowitz-Lieb 2006)

2. Upper and lower bounds on the g.s. energy, matching at the dominant order (G-Lebowitz-Lieb 2006, 2007, 2011)

3. $p > 2d$: upper lower bounds on the g.s. energy, matching at the 1st subdominant order (G-Lieb-Seiringer 2013)

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Other results, I

Our methods have been generalized and applied to other models:

1. **1D continuum functionals with magnetic field** (G-Lebowitz-Lieb): RP + *convexity*;

2. **1D models where the FM interaction is not n.n.** (Buttà-Esposito-G-Marra): RP + *coarse graining*;

3. **anisotropic 2D system for martensitic phase transitions** (G-Müller): RP + *localization bounds*;

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Other results, II

More results on related models (in the continuum):

In $d = 1$:

- Periodicity of 1D minimizers

(Chen-Oshita, Hubbard, Müller, Kohn-Müller, ...)

In $d \geq 2$:

- Computat'n of the ground state energy at dominant order
- Proof of the self-similarity of the finite volume free energy
- Derivation of effective functionals for meso-patterns
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Periodicity in $d \geq 2$

There are just a few very special cases for which one can prove the existence of non-trivial periodic structures in $d \geq 2$. Remarkable: existence of triangular lattice by Heilmann-Radin, Theil, Süto.

Open problem: prove (or disprove) the periodicity of the minimizers for our model, or related models in the continuum, in $d \geq 2$ and $p \geq 1$.

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Main results

Theorem [G-Seiringer]. *Let $d \geq 2$, $p > 2d$ and $J_c = J_c(p)$ be the location of the FM transition line.*

There exists $\epsilon > 0$ such that, if $J_c - \epsilon < J < J_c$, then, in any finite box, the ground state with "optimal striped boundary conditions" is unique.

Such ground state is periodic and striped, with stripes all of width $h^*(J)$.

Remarks

- Our theorem is based on quantitative lower bounds on the excess energy of generic spin states with optimal striped boundary conditions.

- Our result can be restated by saying that all the infinite periodic striped configurations are (strictly stable) infinite volume ground states.

- We also prove that any infinite volume g.s., invariant under translations by $d - 1$ independent integer vectors, consists of two periodic striped halves separated by a finite interface.

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Main steps

Restrict to $d = 2$ and $p > 4$, for simplicity.
The main steps of the proof are:

1. Representation of the energy in terms of droplet self-energies and droplet-droplet interactions.

2. Localization of the droplets' energy functional in bad tiles of side ℓ containing at least one corner and good striped regions.

3. Key fact: the localized energy of the bad tiles has an excess energy $\propto N_c$.

4. The localized energy of the good regions can be bounded from below via reflection positivity and a (subtle) control of the boundary error.

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The droplet representation

In the presence of $+$ boundary conditions, we define the droplets δ_i to be the maximal connected regions of negative spins. Their boundaries Γ_i are the usual low-temperature contours of the Ising model.
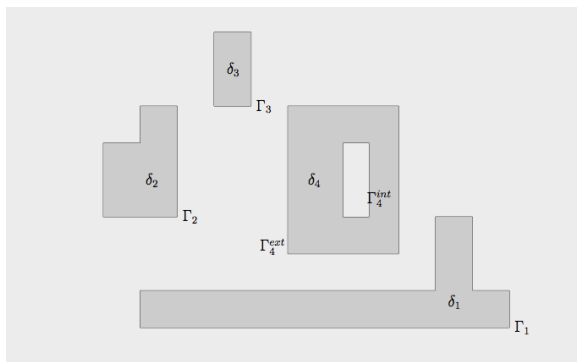
The droplet representation

The total energy can be rewritten as:

$$H = \sum_i \left[2J|\Gamma_i| + U(\delta_i) \right] + \sum_{i<j} W(\delta_i, \delta_j)$$

where:

$$U(\delta) = -2 \sum_{\mathbf{x} \in \delta} \sum_{\mathbf{y} \in \mathbb{Z}^2 \setminus \delta} \frac{1}{|\mathbf{x} - \mathbf{y}|^p}$$

$$W(\delta, \delta') = 4 \sum_{\mathbf{x} \in \delta} \sum_{\mathbf{y} \in \delta'} \frac{1}{|\mathbf{x} - \mathbf{y}|^p}$$

Note: the droplet-droplet interaction is positive.

Energy of a droplet ($p > 4$)
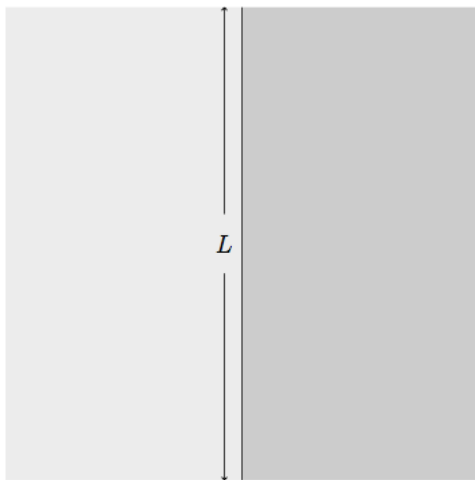
$$E = (2J - 2J_c)L$$

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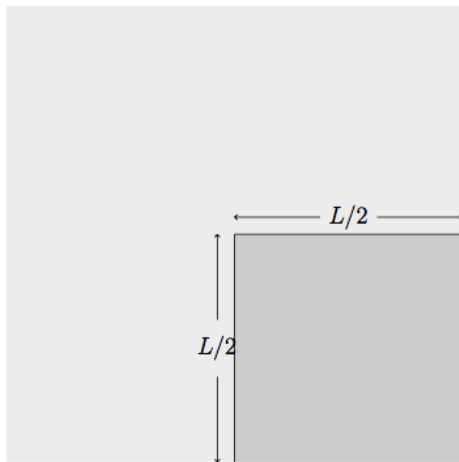
$$E = (2J - 2J_c)L + \kappa_c$$

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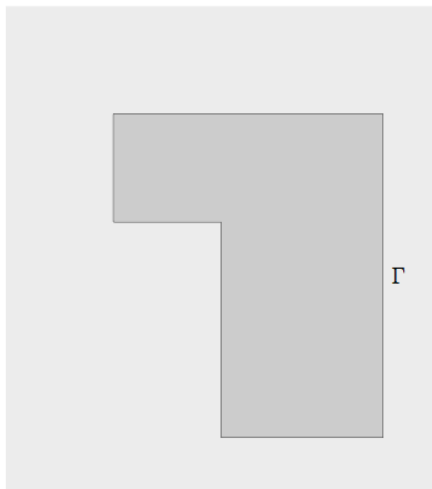
$$E \geq 2(J - J_c)|\Gamma| + \kappa_c N_c$$

Corners

Take home message: corners cost a finite energy!
They look like elementary excitations.

However: how do we eliminate corners by local
moves? How do we exclude that their presence does
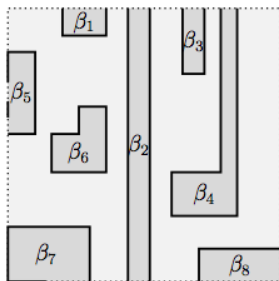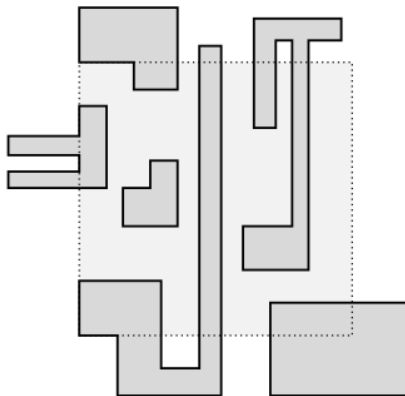not decrease the interaction energy substantially?

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Localization

We localize in bad tiles T_i of side $h^* \ll \ell \ll (J_c - J)^{-1}$ and good regions G_i.

Lower bounds on the local energies

As far as $W(\delta_i, \delta_j)$ is concerned, we just neglect the terms with δ_i, δ_j belonging to different tiles T_i or good regions G_i.

The local energy of a bad tile is $\geq e_s(h^*)\ell^2 + n_c(T)$.

Non trivial part: lower bound on the energy of the good regions, of the form

$$E_G \geq e_s(h^*)|G| - (J_c - J)|\partial G| + \sum_{h \neq h^*}(e_s(h) - e_s(h^*))A_h(G)$$

with $A_h(G)$ the area covered by stripes of width h.

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Lower bounds on the local energy of the good regions, I

A good region G can have a complicated shape.
We deform and slice G, thus reducing it to a union
of rectangles, keeping track of boundary errors.
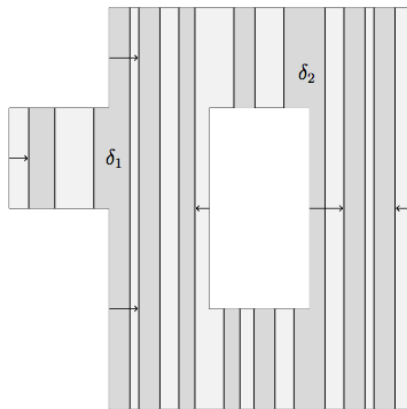
Lower bounds on the local energy of the good regions, II

We first move the vertical boundaries to the interior, so that they coincide with boundaries of the rectangular droplets.
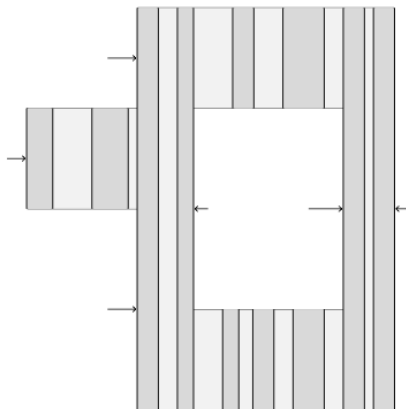
Lower bounds on the local energy of the good regions, III

Next we slice in horizontal slices, and show that we only pay a boundary error from the external boundary, of the order $\sim (J_c - J)|\partial G|$.
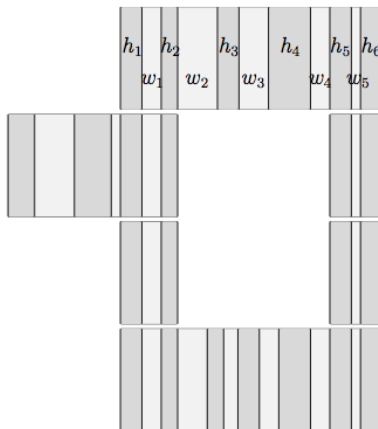
Reflection positivity

Finally, in order to get an optimal bond on the energy of a slice, we use block reflection positivity, as developed by G-Lebowitz-Lieb in the study of the 1D version of our model.

Summary

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- The proof combines localization bounds, based on a convenient droplet representation of the energy, with block reflection positivity.

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Open problems

1. Generalize to $d < p \leq 2d$. In particular, do stripes form in $d = 2$ for $p = 3$ at large J?

2. Prove stability of stripes (LRO) at positive temperatures in $d = 3$.

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Thank you!

Reflection positivity

Once we reduced to a local striped configuration, the model is effectively one dimensional:

$$H_{1D}(\underline{\sigma}) = -J \sum_x \sigma_x \sigma_{x+1} + \sum_{x<y} \sigma_x \, v(x-y) \, \sigma_y$$

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Let us temporarily focus on the long range part:

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Block reflection positivity

Summarizing: the long range energy decreases by reflecting left onto right, or viceversa. If $J > 0$, then we can reflect only at the "broken bonds":

After repeated reflections we end up with a lower bound involving periodic arrays of blocks all of the same size $\ldots, h_i, h_i, h_i, h_i, \ldots$

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