Universal laws for hulls in large planar maps

Emmanuel Guitter (IPhT, CEA Saclay)

IHP January 24, 2017

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arXiv:1511.01773 The distance-dependent two-point function of triangulations: a new derivation from old results

arXiv:1512.00179 The distance-dependent two-point function of quadrangulations: a new derivation by direct recursion

arXiv:1602.07433 Some results on the statistics of hull perimeters in large planar triangulations and quadrangulations

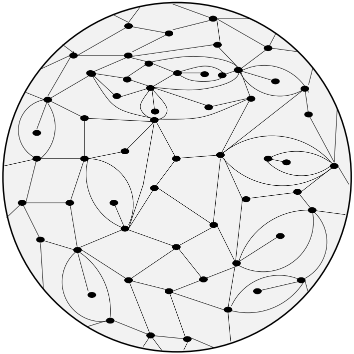
arXiv:1606.06532 Eulerian triangulations: two-point function and hull perimeter statistics

arXiv:1611.02871 Refined universal laws for hull volumes and perimeters in large planar maps

The hull: a heuristic presentation

Consider:

- a planar map, say a quadrangulation, i.e. a connected graph embedded on the sphere whose all faces have degree 4

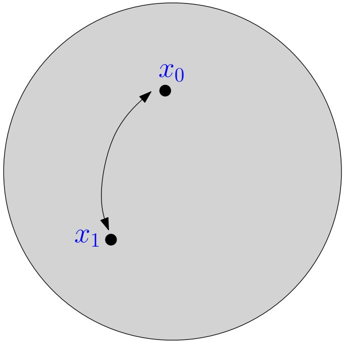


The hull: a heuristic presentation

Consider:

- a planar map, say a quadrangulation, i.e. a connected graph embedded on the sphere whose all faces have degree 4
- with two marked vertices x_0 et x_1 at graph distance

$$d(x_0, x_1) = k$$



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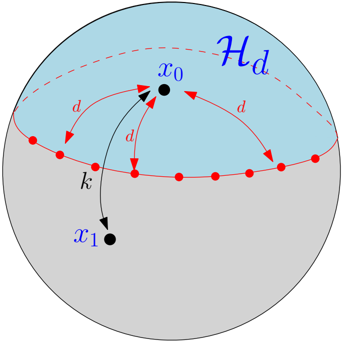
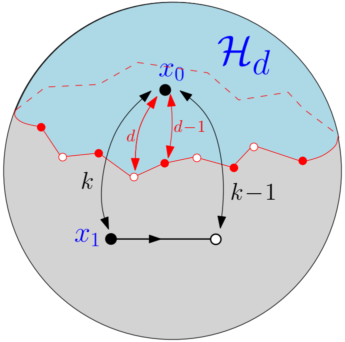


Pour $d < k$:

$\exists$ a closed line "at distance" d from x_0 and separating x_0 from x_1

The hull at distance d is the part of the map lying on the same side as x_0 from this separating line (here denoted by $\mathcal{H}_d$)

More precisely, we shall work in the ensemble $\mathcal{Q}_{k,N}$ of quadrangulations with N faces, with two marked vertices x_0 et x_1 at graph distance $d(x_0, x_1) = k$ and (this simplifies the combinatorics) with a marked edge from x_1 to a vertex at distance $k-1$ from x_0 (such an edge always exists)



The dividing line will be chosen as a simple closed curve following edges of the map and visiting alternately vertices at distance d and $d-1$ from x_0

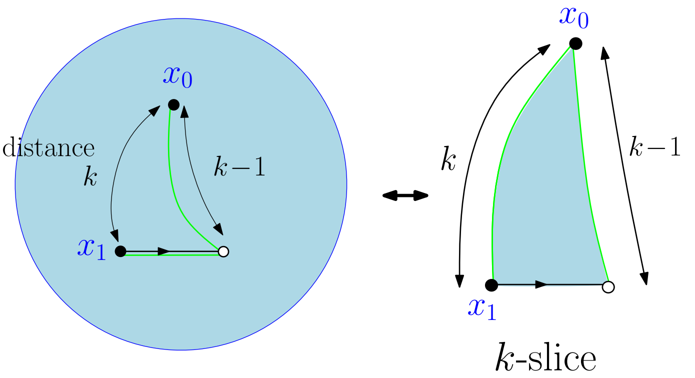
Quantities of interest

Hull perimeter: $\mathcal{L}(d)$ = length (number of edges) of the separating line
Hull volume: $\mathcal{V}(d)$ = area (number of faces) of the hull

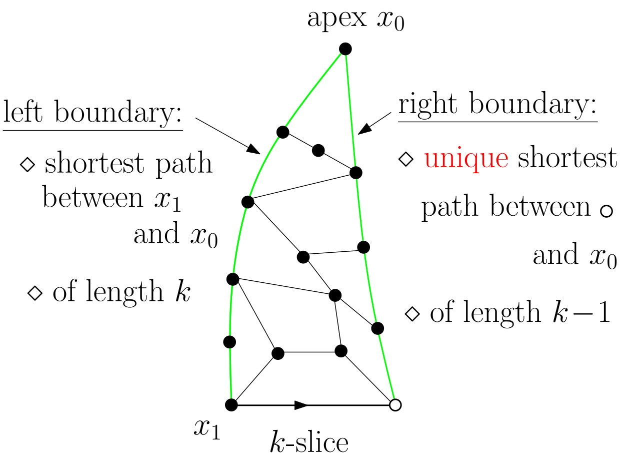
What is the statistics $\mathcal{L}(d)$ and $\mathcal{V}(d)$ in the ensemble $\mathcal{Q}_{k,N}$ for a given $d < k$?

The coding of maps by slices

Upon cutting the map along the leftmost shortest path from x_1 to x_0 (taking the marked edge $k \to k-1$ as first step), we transform it into a so-called k-slice

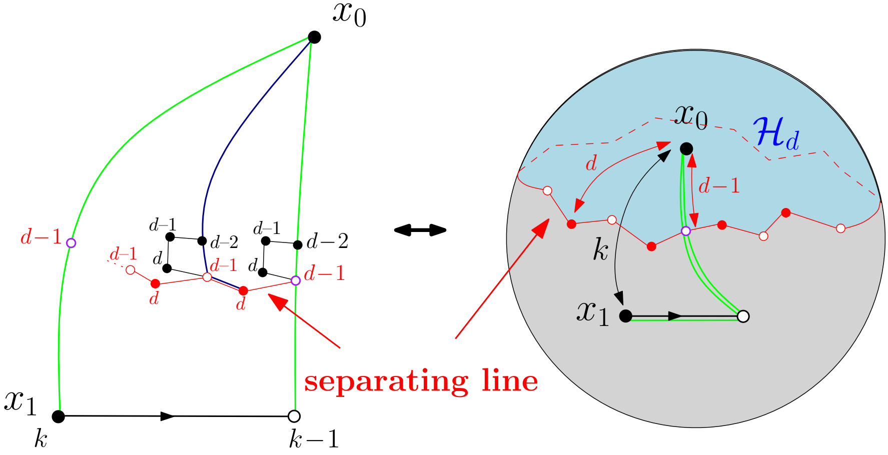


k-slice = "quadrangulation with a boundary" of total length $2k$



The coding is one-to-one and the distances from x_0 are preserved.

A construction of the hull on the k-slice



Slice generating functions (let N be unfixed)

$T_k \equiv T_k(g)$: generating function for ℓ-slices with $2 \le \ell \le k$ with a weight g per face

$\rightarrow$ cut the ℓ-slice along the separating line at distance $\ell - 1$

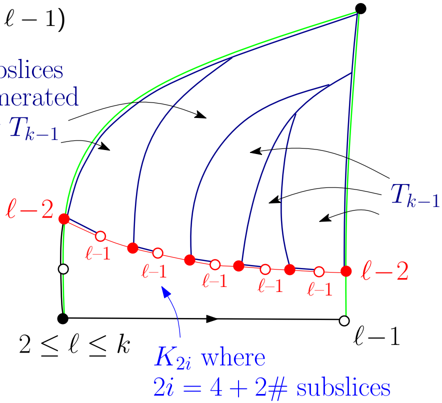
(i.e. visiting vertices at distance $\ell - 2$ and $\ell - 1$)

Upon defining

$$\mathcal{K}(T) \equiv \mathcal{K}(T; g) = \sum_{i \ge 2} K_{2i}(g) T^{i-2}$$

where $K_{2i}(g)$ enumerates particular quadrangulations with a boundary of length $2i$, then we have the recursion:

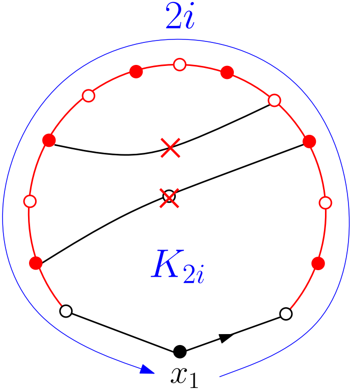
$$T_k = \mathcal{K}(T_{k-1})$$



Explicit form of $\mathcal{K}(T)$

$$\mathcal{K}(T) = \sum_{i\geq 2} K_{2i}(g) T^{i-2}$$

where K_{2i}enumerates quadrangulations with a boundary of length $2i$ with some constraints



idem Tutte (1962)

Explicit form of $\mathcal{K}(T)$

Introducing the parametrization

$$g = g(x) = \frac{x(1+x+x^2)}{(1+4x+x^2)^2} \qquad T_\infty = T_\infty(x) = \frac{x(1+4x+x^2)}{(1+x+x^2)^2}$$

with $0 \le x \le 1$, we find that $\mathcal{K}(T) = T_\infty \times \kappa\left(\frac{T}{T_\infty}\right)$ where

$$\kappa(\tau) = \frac{1}{2x(1+x(x+\tau))}\Big\{x(1-3x+x^2-x^3)+x(1+6x+x^2+x^3)\tau - x^2\tau^2\}$$
$$-(1-\tau)\left(1-\sqrt{(1+x+x^2+x^3+x^4)^2-2x^2(1+3x+5x^2+3x^3+x^4)\tau+x^4\tau^2}\right)\Big\}$$

so that we have for any (small enough) U:

$$\mathcal{K}\left(T_\infty \frac{(1-U\,x^{-1})(1-U\,x^4)}{(1-U\,x)(1-U\,x^2)}\right) = T_\infty \frac{(1-U)(1-U\,x^5)}{(1-U\,x^2)(1-U\,x^3)}$$

Solution of the recursion

$$\mathcal{K}\left(T_\infty \frac{(1-U\,x^{-1})(1-U\,x^4)}{(1-U\,x)(1-U\,x^2)}\right) = T_\infty \frac{(1-U)(1-U\,x^5)}{(1-U\,x^2)(1-U\,x^3)}$$

Setting

$$T_k = T_\infty \frac{(1-U_k)(1-U_k\,x^5)}{(1-U_k\,x^2)(1-U_k\,x^3)}$$

we may reformulate the recursion as

$$T_k = \mathcal{K}(T_{k-1}) \Rightarrow U_k = x\,U_{k-1}$$

with initial condition $U_1 = 1$ (since $T_1 = 0$) $\Rightarrow U_k = x^{k-1}$

$$T_k(g) = T_\infty(x) \frac{(1-x^{k-1})(1-x^{k+4})}{(1-x^{k+1})(1-x^{k+2})} \qquad \text{with } x \text{ s.t. } g = g(x)$$

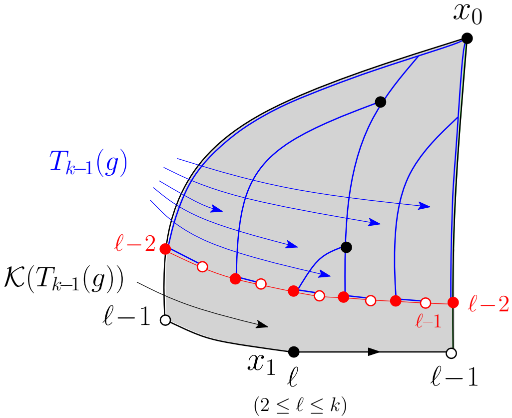
If we wish to enumerate quadrangulations with $d(x_0, x_1) = k$, we must fix $\ell = k$ and the desired g.f. is

$$W(g;k) = T_k(g) - T_{k-1}(g)$$

Slice generating functions controlling the hull geometry

$T_k(g)$ enumerates ℓ-slices with $2 \leq \ell \leq k$ and satisfies

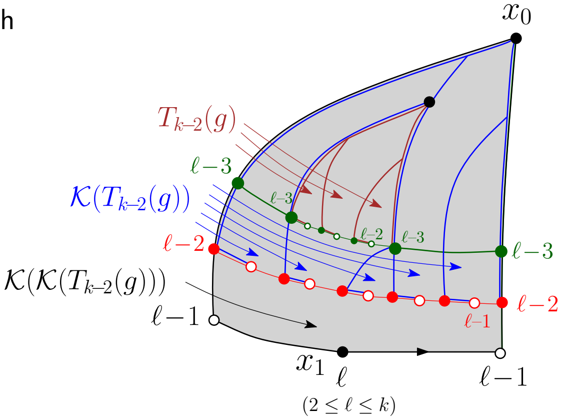
$$T_k(g) = \mathcal{K}\left(T_{k-1}(g)\right)$$



Slice generating functions controlling the hull geometry

$T_k(g)$ enumerates ℓ-slices with $2 \leq \ell \leq k$ and satisfies

$$T_k(g) = \mathcal{K}\left(\mathcal{K}\left(T_{k-2}(g)\right)\right)$$



Slice generating functions controlling the hull geometry

To enumerate ℓ-slices with $2 \leq \ell \leq k$ with an extra weight

$$\rho^{\mathcal{V}(\ell-2)}\,\alpha^{\mathcal{L}(\ell-2)}$$

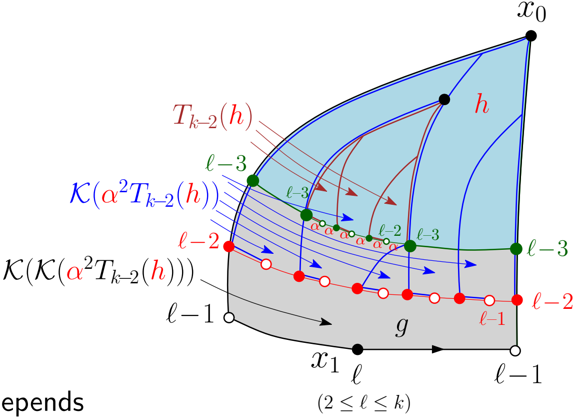
we simply have to consider

$$\mathcal{K}\left(\mathcal{K}\left(\alpha^2\; T_{k-2}(h)\right)\right)$$

where

$$h = \rho\, g$$

NB: here $\mathcal{K}(T) \equiv \mathcal{K}(T;g)$ depends on g only



More generally, if we wish to enumerate ℓ-slices with $2 \leq \ell \leq k$, and for $1 \leq m \leq k-1$ a weight

$$\rho^{\mathcal{V}(\ell-m)}\,\alpha^{\mathcal{L}(\ell-m)}$$

we simply have to consider

$$\underbrace{\mathcal{K}\Big(\cdots\Big(\mathcal{K}\Big(}_{m\ \text{times}}\alpha^2\,T_{k-m}(h)\Big)\Big)\Big)$$

NB: with the convention $\mathcal{V}(\ell-m)=\mathcal{L}(\ell-m)=0$ if $\ell \leq m$

If we wish to enumerate quadrangulations with $d(x_0,x_1)=k$ and a weight $\rho^{\mathcal{V}(d)}\,\alpha^{\mathcal{L}(d)}$, we must fix $\ell=k$ and $m=k-d$

The desired generating function is (setting $h=\rho\,g$)

$$Z(h,\alpha,g;d,k)\equiv\underbrace{\mathcal{K}\Big(\cdots\Big(\mathcal{K}\Big(}_{k-d\ \text{times}}\alpha^2\,T_d(h)\Big)\Big)\Big)-\underbrace{\mathcal{K}\Big(\cdots\Big(\mathcal{K}\Big(}_{k-d\ \text{times}}\alpha^2\,T_{d-1}(h)\Big)\Big)\Big)$$

Using

$$\underbrace{\mathcal{K}(\cdots(\mathcal{K}(}_{k-d\ \text{times}}T_\infty\frac{(1-U\,x^{-1})(1-U\,x^4)}{(1-U\,x)(1-U\,x^2)}))) = T_\infty\frac{(1-U\,x^{k-d-1})(1-U\,x^{k-d+4})}{(1-U\,x^{k-d+1})(1-U\,x^{k-d+2})}$$

we get $$\underbrace{\mathcal{K}(\cdots(\mathcal{K}(}_{k-d\ \text{times}}\alpha^2\,T_d(h)))) = T_\infty\frac{(1-U_d\,x^{k-1})(1-U_d\,x^{k+4})}{(1-U_d\,x^{k+1})(1-U_d\,x^{k+2})}$$

where $U_d = U_d(x,y,\alpha)$ is defined via

$$T_\infty(x)\frac{(1-U_d\,x^{d-1})(1-U_d\,x^{d+4})}{(1-U_d\,x^{d+1})(1-U_d\,x^{d+2})} = \alpha^2\,T_d(h) = \alpha^2\,T_\infty(y)\frac{(1-y^{d-1})(1-y^{d+4})}{(1-y^{d+1})(1-y^{d+2})}$$

where x and y satisfy $g(x)=g$ and $g(y)=h$. Finally

$$Z(h,\alpha,g;d,k) = T_\infty(x)\left(\frac{(1-U_d\,x^{k-1})(1-U_d\,x^{k+4})}{(1-U_d\,x^{k+1})(1-U_d\,x^{k+2})} - \frac{(1-U_{d-1}\,x^{k-2})(1-U_{d-1}\,x^{k+3})}{(1-U_{d-1}\,x^{k})(1-U_{d-1}\,x^{k+1})}\right)$$

Universal laws for large maps

Recall that our aim is the statistics of the hull at distance d in the ensemble $\mathcal{Q}_{k,N}$ of quadrangulations with N faces, with two marked vertices x_0 et x_1 at graph distance $d(x_0, x_1) = k$ (and a marked edge $k \to k-1$)

To obtain universal laws for the hull geometry, we will in practice study this statistics only in the limit of infinitely large maps. More precisely:

- we first let $N \to \infty$
- in a second step, we then let k and d become large with a fixed ratio

$$u = d/k \ , \qquad 0 \leq u \leq 1$$

NB: we will not consider here the statistics obtained in another interesting universal regime where N, k and d tend simultaneously to ∞ with d/k and $k/N^{1/4}$ fixed

The $N \to \infty$ limit

Consider the generating function $W(g;k)$ for quadrangulations with two marked vertices at distance k (and a marked edge $k \to k-1$). It encodes the number of maps in the ensemble $\mathcal{Q}_{k,N}$, given by

$$[g^N]W(g;k)$$

The large N limit of this number is easily obtained from the singular behavior of $W(g;k)$, which occurs when $x \to 1$, i.e for $g \to g(1) = \frac{1}{12}$. Setting

$$g = \frac{1}{12}\left(1 - \epsilon^2\right)$$

we find an expansion of the form

$$W(g;k) = \mathfrak{w}_0(k) + \mathfrak{w}_2(k)\epsilon^2 + \mathfrak{w}_3(k)\epsilon^3 + O(\epsilon^4)$$

so that

$$W(g;k)|_{\text{sing.}} = \mathfrak{w}_3(k)\ (1 - 12\,g)^{3/2}$$

and

$$[g^N]W(g;k) \sim \frac{3}{4}\ \frac{12^N}{N^{5/2}} \times \mathfrak{w}_3(k)$$

From our explicit form of $W(g;k)$, we easily obtain

$$\mathfrak{w}_3(k) = \frac{4\left(k^2+2k-1\right)\left(5k^4+20k^3+27k^2+14k+4\right)}{35k(k+1)(k+2)} \; .$$

In particular for large k,

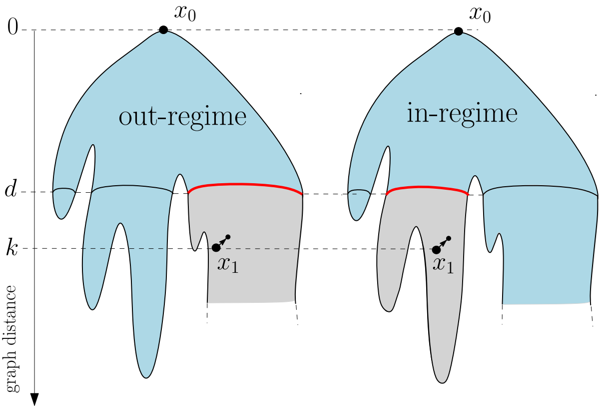
$$\mathfrak{w}_3(k) \underset{k\to\infty}{\sim} \frac{4}{7}\, k^3$$

The out- and the in-regime

For $N \to \infty$, the volume of the hull may itself be infinite !

- the "out-regime": $\mathcal{V}(d)$ is finite

- the "in-regime": $\mathcal{V}(d)$ is infinite (in which case the complementary of the hull has a finite volume with probability 1)



Let us denote for short $Z(h, g) \equiv Z(h, \alpha, g; k, d)$ and $n \equiv \mathcal{V}(d)$. Then the quantity of interest in the out-regime is the $N \to \infty$ behavior of

$$\sum_n [g^{N-n} h^n] Z(h, g)\, \rho^n$$

Setting as before $g = \frac{1}{12}\left(1 - \epsilon^2\right)$ we have an expansion

$$Z(h, g) = \mathfrak{z}_0(h) + \mathfrak{z}_2(h)\epsilon^2 + \mathfrak{z}_3(h)\epsilon^3 + O(\epsilon^4)$$

from which we deduce the large N behavior

$$[g^{N-n} h^n] Z(h, g) \underset{N \to \infty}{\sim} [h^n]\mathfrak{z}_3(h) \times \frac{3}{4} \frac{12^{N-n}}{\sqrt{\pi} N^{5/2}}$$

$$\sum_n [g^{N-n} h^n] Z(h, g)\, \rho^n \underset{N \to \infty}{\sim} \frac{3}{4} \frac{12^N}{\sqrt{\pi} N^{5/2}} \sum_n [h^n]\mathfrak{z}_3(h) \times \left(\frac{\rho}{12}\right)^n$$

Namely:
$$\sum_n [g^{N-n}h^n] Z(h,g)\,\rho^n \underset{N\to\infty}{\sim} \frac{3}{4}\frac{12^N}{\sqrt{\pi}N^{5/2}} \mathfrak{z}_3\left(\frac{\rho}{12}\right)$$
and upon normalization

$$E\left[\rho^{\mathcal{V}(d)}\,\alpha^{\mathcal{L}(d)}\,\mathbf{1}_{\mathcal{V}(d)\ \text{finite}}\right] = \frac{\mathfrak{z}_3\left(\frac{\rho}{12},\alpha;k,d\right)}{\mathfrak{w}_3(k)}$$

Similarly, for the in-regime, we consider the small η expansion
$$Z(h,g) = \mathfrak{j}_0(g) + \mathfrak{j}_2(g)\eta + \mathfrak{j}_3(g)\eta^3 + O(\eta^4)$$
where η is defined via $h = \frac{1}{12}\left(1-\eta^2\right)$ (i.e. $y = 1-\sqrt{6}\eta + \cdots$)

This yields
$$\sum_n [g^m h^{N-m}] Z(h,g) \underset{N\to\infty}{\sim} \frac{3}{4}\frac{12^N}{\sqrt{\pi}N^{5/2}} \mathfrak{j}_3\left(\frac{1}{12}\right)$$
and

$$E\left[\alpha^{\mathcal{L}(d)}\,\mathbf{1}_{\mathcal{V}(d)\ \text{infinite}}\right] = \frac{\mathfrak{j}_3\left(\frac{1}{12},\alpha;k,d\right)}{\mathfrak{w}_3(k)}$$

The probability to be in the out- or in the in-regime

We find that the probability to be in the out- or in-regime are respectively

$$E\left[\mathbf{1}_{\mathcal{V}(d)\ \text{finite}}\right] = \frac{1}{\mathfrak{w}_3(k)} \times$$

$$\Bigg(\frac{1}{105(2d+3)(k+1)^2(k+2)^2} \times \Big((2d+3)(k-1)(k+1)(k+2)(k+4)\left(15k^4+90k^3+237k^2+306k+140\right)$$

$$-(2k+3)(d-1)(d+1)(d+2)(d+4)\left(15d^4+90d^3+237d^2+306d+140\right)$$

$$-\frac{1}{105(2d+1)k^2(k+1)^2} \times \Big((2d+1)(k-2)k(k+1)(k+3)\left(15k^4+30k^3+57k^2+42k-4\right)$$

$$-(2k+1)(d-2)d(d+1)(d+3)\left(15d^4+30d^3+57d^2+42d-4\right)\Big)\Bigg)$$

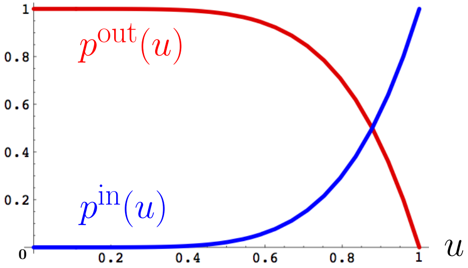
$$E\left[\mathbf{1}_{\mathcal{V}(d)\ \text{infinite}}\right] = \frac{1}{\mathfrak{w}_3(k)} \left(\frac{(2k+3)(d-1)(d+1)(d+2)(d+4)\left(15d^4+90d^3+237d^2+306d+140\right)}{105(2d+3)(k+1)^2(k+2)^2} \right.$$

$$\left. - \frac{(2k+1)(d-2)d(d+1)(d+3)\left(15d^4+30d^3+57d^2+42d-4\right)}{105(2d+1)k^2(k+1)^2} \right)$$

For k and d large, with $u = d/k$ fixed, we get

$$p^{\text{out}}(u) \equiv E\left[\mathbf{1}_{\mathcal{V}(d)\ \text{finite}}\right] = \frac{1}{4}\left(4 - 7u^6 + 3u^7\right)$$

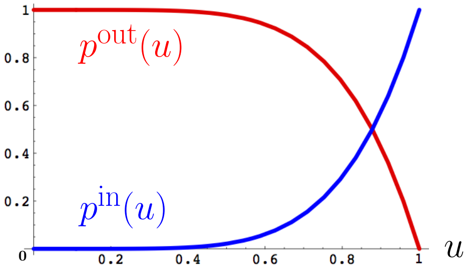
$$p^{\text{in}}(u) \equiv E\left[\mathbf{1}_{\mathcal{V}(d)\ \text{infinite}}\right] = \frac{1}{4}(7 - 3u)\,u^6$$



Universality: same expression for other families of maps (triangulations, Eulerian triangulations)

For $u \to 0$, i.e. $k \gg d$, x_1 lies in the infinite outgrowth at distance $> d$ with probability 1

For $u \to 1$, i.e. $k \sim d$, x_1 lies just below the boundary of some outgrowth at distance $> d$ but this is the infinite one with a probability which tends to 0 since the length of the boundary of this infinite outgrowth ($\sim d^2$) is negligible w.r.t. the total length of all the boundaries of all the outgrowths at distance $> d$ ($\sim d^3$)



Universality: same expression for other families of maps (triangulations, Eulerian triangulations)

Laws for the hull perimeter $\mathcal{L}(d)$

We can get similarly $E\left[\alpha^{\mathcal{L}(d)}\,\mathbf{1}_{\mathcal{V}(d)\ \text{finite}}\right]$ and $E\left[\alpha^{\mathcal{L}(d)}\,\mathbf{1}_{\mathcal{V}(d)\ \text{infinite}}\right]$

For large d and k, $\mathcal{L}(d)$ scale as d^2 so we set

$$L(d) \equiv \frac{\mathcal{L}(d)}{d^2}$$

Setting $\alpha = e^{-\tau/d^2}$, and performing an inverse Laplace transform w.r.t. τ, we deduce the large d and k conditional probability densities:

$$D^{\text{out}}(L,u) \equiv \frac{1}{dL}\frac{E[\mathbf{1}_{L\leq L(d)<L+dL}\ \mathbf{1}_{\mathcal{V}(d)\ \text{finite}}]}{E[\mathbf{1}_{\mathcal{V}(d)\ \text{finite}}]}$$

$$D^{\text{in}}(L,u) \equiv \frac{1}{dL}\frac{E[\mathbf{1}_{L\leq L(d)<L+dL}\ \mathbf{1}_{\mathcal{V}(d)\ \text{infinite}}]}{E[\mathbf{1}_{\mathcal{V}(d)\ \text{infinite}}]}$$

we find

$$D^{\mathrm{out}}(L,u) = \tfrac{2(1-u)^4}{c\sqrt{\pi}u}\,\tfrac{e^{-BX}}{4-7u^6+3u^7}\big(-2\sqrt{X}((X-10)X-2) + e^X\sqrt{\pi}X(X(2X-5)+6)\big(1-\mathrm{erf}\big(\sqrt{X}\big)\big)\big)$$

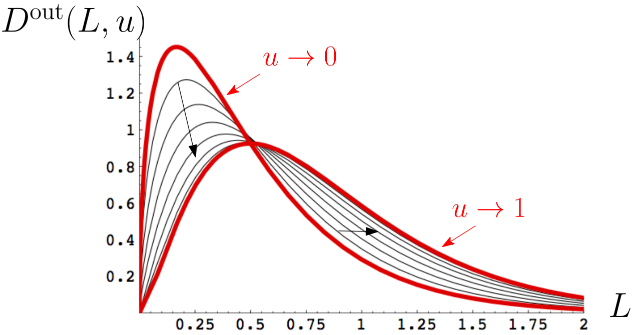
$$D^{\mathrm{in}}(L,u) = \tfrac{2}{c\sqrt{\pi}(1-u)^2}\,\tfrac{e^{-BX}}{u(7-3u)}(BX+2)\big(2\sqrt{X}(X+1)-e^X\sqrt{\pi}X(2X+3)\big(1-\mathrm{erf}\big(\sqrt{X}\big)\big)\big)$$

where $X = \dfrac{L}{c\;B}$, $\qquad B = \dfrac{(1-u)^2}{u^2}$

and with $c = 1/3$ for quadrangulations.

Universality: same expression for other families of maps up to the global normalization c.
We find $c = 1/2$ for triangulations and $c = 1/4$ for Eulerian triangulations.

The out-regime



$$\lim_{u\to 0} D^{\mathrm{out}}(L,u) = 2\sqrt{L}\,\frac{e^{-\frac{L}{c}}}{c^{3/2}\sqrt{\pi}}$$

$$\lim_{u\to 1} D^{\mathrm{out}}(L,u) = \frac{4}{3}(\sqrt{L})^3\,\frac{e^{-\frac{L}{c}}}{c^{5/2}\sqrt{\pi}}$$

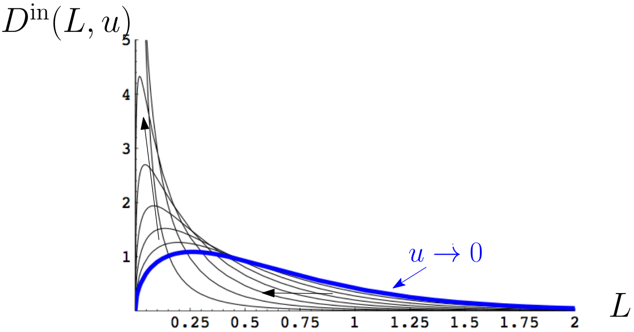
For $u \to 0$, i.e. $k \gg d$, we recover the probability density for the length at distance d separating a marked vertex (x_0) from infinity in infinitely large vertex-pointed quadrangulations, see Krikun (2005), Curien & Le Gall (2014).

For $u \to 1$, i.e. $k \simeq d$, the marked vertex x_1 must be "just below" this line separating x_0 from infinity and the number of acceptable positions being proportional to L, we have:

$$\lim_{u \to 1} D^{\mathrm{out}}(L,u) \ \propto \ L \times \lim_{u \to 0} D^{\mathrm{out}}(L,u)$$

which fixes the expression of $\lim_{u \to 1} D^{\mathrm{out}}(L,u)$ by normalization.

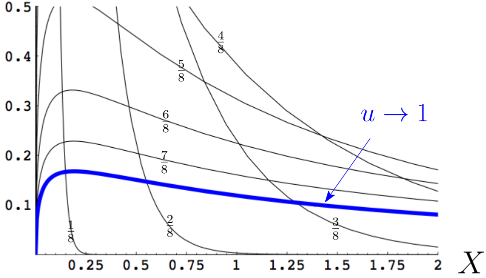
The in-regime



$$\lim_{u\to 0} D^{\text{in}}(L,u) = \frac{4}{7}\sqrt{L}\,(2c+L)\,\frac{e^{-\frac{L}{c}}}{c^{5/2}\sqrt{\pi}}$$

x_1 lies in a very long "finger". Explanation for this simple formula ?

$c\,B\,D^{\rm in}(c\,B\,X,u)$

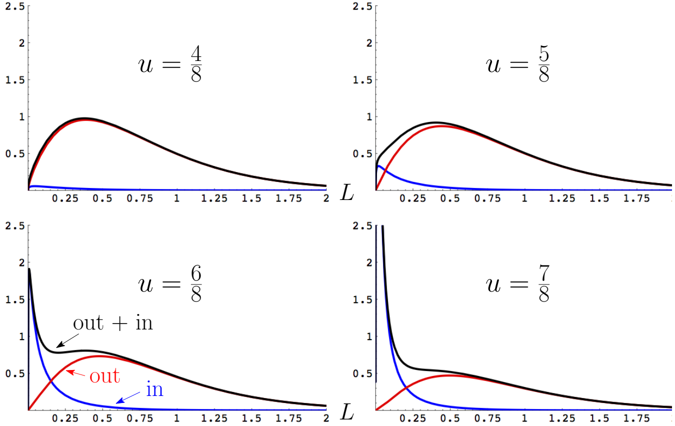


$$X(k,d) \equiv \frac{L(d)}{c\,B} \text{ with } B = \frac{(1-u)^2}{u^2}\ , \text{ namely } X(k,d) = \frac{\mathcal{L}(d)}{c\,(k^2-d^2)}$$

$$\lim_{u\to 1} c\,B\,D^{\rm in}(c\,B\,X,u) = \frac{2\sqrt{X}(X+1) - e^X\sqrt{\pi}X(2X+3)\left(1-\mathrm{erf}\left(\sqrt{X}\right)\right)}{\sqrt{\pi}}$$

NB: all the moments of this distribution are infinite

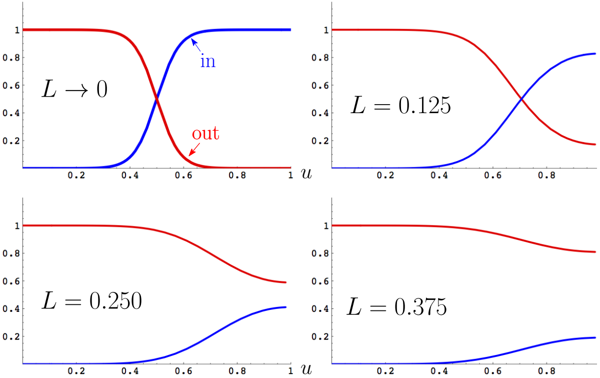
The relative contributions of the out- and in-regimes to the total probability density for L



$$D^{\text{tot}}(L,u) = p^{\text{out}}(u)\, D^{\text{out}}(L,u) + p^{\text{in}}(u)\, D^{\text{in}}(L,u)$$

Probability to be in the out- or in-regime, knowing $u = d/k$ and $L = \mathcal{L}(d)/d^2$

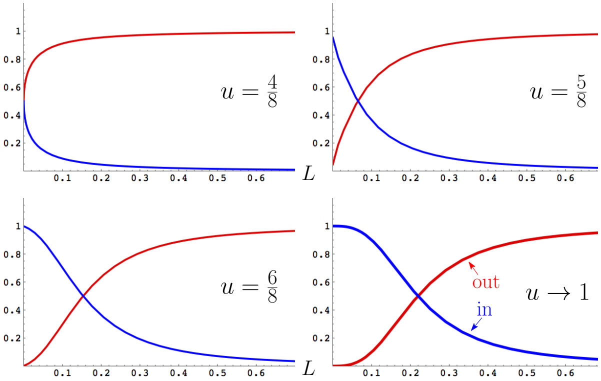
as a function of u



For $L \to 0$, the probability to be in the out-regime, knowing u, tends to

$$\frac{(1-u)^6}{(1-2u+2u^2)\,(1-4u+5u^2-2u^3+u^4)}$$

as a function of L



For $u \to 1$, the probability to be in the out-regime, knowing L, tends to

$$\frac{28L^3}{6c^3 + 3Lc^2 + 28L^3}$$

Joint law for $\mathcal{V}(d)$ and $\mathcal{L}(d)$ in the out-regime

We can now compute $E\left[\rho^{\mathcal{V}(d)}\,\alpha^{\mathcal{L}(d)}\,\mathbf{1}_{\mathcal{V}(d)\ \text{finite}}\right]$

For large d and k, $\mathcal{V}(d)$ scale as d^4 so we set $V(d) \equiv \dfrac{\mathcal{V}(d)}{d^4}$

Setting $\alpha = e^{-\tau/d^2}$ and $\rho = e^{-\sigma/d^4}$, we get, for large d and k:

$$E\left[e^{-\sigma\,V(d)-\tau\,L(d)}\ \mathbf{1}_{\mathcal{V}(d)\ \text{finite}}\right] = \frac{(1-u)^6}{u^3} \times \frac{(f\sigma)^{3/4}\cosh\left(\frac{1}{2}(f\sigma)^{1/4}\right)}{8\sinh^3\left(\frac{1}{2}(f\sigma)^{1/4}\right)} \times M\big(\mu(\sigma,\tau,u)\big)$$

where $M(\mu) = \dfrac{1}{\mu^4}\left(3\mu^2-5\mu+6+\dfrac{4\mu^5+16\mu^4-7\mu^2-40\mu-24}{4(1+\mu)^{5/2}}\right)$

and $\mu(\sigma,\tau,u) = \dfrac{(1-u)^2}{u^2}\left(c\,\tau\,+\dfrac{\sqrt{f\sigma}}{4}\left(\coth^2\left(\dfrac{1}{2}(f\sigma)^{1/4}\right)-\dfrac{2}{3}\right)\right)-1$

with $f=36$ (quandrangulations), 192 (triangulations), 16 (Eulerian triangulations)

Define $$E^{\text{out}}\left[e^{-\sigma\, V(d)-\tau\, L(d)}\right] \equiv \frac{E\left[e^{-\sigma\, V(d)-\tau\, L(d)}\ \mathbf{1}_{\mathcal{V}(d)\ \text{finite}}\right]}{E\left[\mathbf{1}_{\mathcal{V}(d)\ \text{finite}}\right]}$$

$$\lim_{u\to 0} E^{\text{out}}\left[e^{-\sigma\, V(d)-\tau\, L(d)}\right]$$
$$= \frac{(f\sigma)^{3/4}\cosh\left(\frac{1}{2}(f\sigma)^{1/4}\right)}{8\sinh^3\left(\frac{1}{2}(f\sigma)^{1/4}\right)\left(c\,\tau + \frac{\sqrt{f\sigma}}{4}\left(\coth^2\left(\frac{1}{2}(f\sigma)^{1/4}\right)-\frac{2}{3}\right)\right)^{3/2}}$$

We recover a result by Curien and Le Gall (2014)

$$\lim_{u\to 1} E^{\text{out}}\left[e^{-\sigma\, V(d)-\tau\, L(d)}\right]$$
$$= \frac{(f\sigma)^{3/4}\cosh\left(\frac{1}{2}(f\sigma)^{1/4}\right)}{8\sinh^3\left(\frac{1}{2}(f\sigma)^{1/4}\right)\left(c\,\tau + \frac{\sqrt{f\sigma}}{4}\left(\coth^2\left(\frac{1}{2}(f\sigma)^{1/4}\right)-\frac{2}{3}\right)\right)^{5/2}}\ .$$

$$= \lim_{u\to 0} E^{\text{out}}\left[L(d)\, e^{-\sigma\, V(d)-\tau\, L(d)}\right]/E^{\text{out}}[L(d)]$$

The law for the volume $V(d)$, knowing the value of $L(d)$

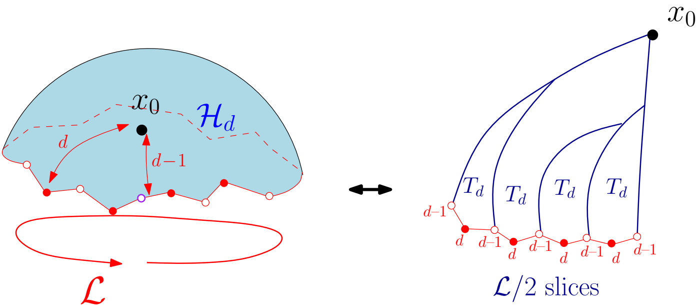
$$E^{\mathrm{out}}\left[e^{-\sigma\, V(d)}\Big| L(d) = L\right] = \frac{(f\sigma)^{3/4}\cosh\left(\frac{1}{2}(f\sigma)^{1/4}\right)}{8\sinh^3\left(\frac{1}{2}(f\sigma)^{1/4}\right)}\, e^{-\frac{L}{c}\left(\frac{\sqrt{f\sigma}}{4}\left(\coth^2\left(\frac{1}{2}(f\sigma)^{1/4}\right)-\frac{2}{3}\right)-1\right)}$$

Note that this quantity is independent of u and reproduces, for any u, the expression found by Ménard (2016)

We have in particular

$$E^{\mathrm{out}}\left[\mathcal{V}(d)\Big|\mathcal{L}(d) = \mathcal{L}\right] = \frac{f(c\,d^4 + \mathcal{L}\,d^2)}{240c}$$

Simple explanation:



enumerated by $(T_d(h))^{\frac{\mathcal{L}}{2}} - (T_{d-1}(h))^{\frac{\mathcal{L}}{2}}$.

For $N \to \infty$, the hull faces receive a weight $h = \frac{1}{12} e^{-\sigma/d^4}$, i.e. we have $y \simeq 1 - (f\sigma)^{1/4}/d$ (with $f = 36$). For large d:

$$T_d(h) \simeq \frac{2}{3}\left(1 - 6\frac{\sqrt{f\sigma}}{4}\left(\coth^2\left(\frac{1}{2}(f\sigma)^{1/4}\right) - \frac{2}{3}\right)\frac{1}{d^2}\right)$$

and therefore:

$$(T_d(h))^{\frac{\mathcal{L}}{2}} \simeq \left(\frac{2}{3}\right)^{\frac{\mathcal{L}}{2}} e^{-3L\frac{\sqrt{f\sigma}}{4}\left(\coth^2\left(\frac{1}{2}(f\sigma)^{1/4}\right)-\frac{2}{3}\right)}$$

We also have the large d behavior:

$$T_d(h) - T_{d-1}(h) \simeq \frac{8}{d^3}\frac{(f\sigma)^{3/4}\cosh\left(\frac{1}{2}(f\sigma)^{1/4}\right)}{8\sinh^3\left(\frac{1}{2}(f\sigma)^{1/4}\right)}$$

and eventually:

$$T_d^{\frac{\mathcal{L}}{2}} - T_{d-1}^{\frac{\mathcal{L}}{2}} \simeq \left(\frac{2}{3}\right)^{\frac{\mathcal{L}}{2}} \frac{6\,L}{d}\frac{(f\sigma)^{3/4}\cosh\left(\frac{1}{2}(f\sigma)^{1/4}\right)}{8\sinh^3\left(\frac{1}{2}(f\sigma)^{1/4}\right)} e^{-3L\frac{\sqrt{f\sigma}}{4}\left(\coth^2\left(\frac{1}{2}(f\sigma)^{1/4}\right)-\frac{2}{3}\right)}$$

Normalization (divide by the value at $\sigma = 0$):

$$\rightarrow \quad \frac{(f\sigma)^{3/4}\cosh\left(\frac{1}{2}(f\sigma)^{1/4}\right)}{8\sinh^3\left(\frac{1}{2}(f\sigma)^{1/4}\right)} e^{-3L\left(\frac{\sqrt{f\sigma}}{4}\left(\coth^2\left(\frac{1}{2}(f\sigma)^{1/4}\right)-\frac{2}{3}\right)-1\right)}$$

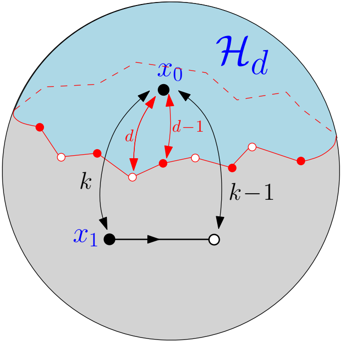
Conclusion

- For any T, we now how to evaluate

$$\underbrace{\mathcal{K}(\cdots(\mathcal{K}(}_{k-d \text{ times}}T)))$$

$\rightarrow$ allows to "decouple" the blue part (encoded in T) from the gray part (encoded in the $(k-d)$ operators $\mathcal{K}$).

By changing T, one could in principle "dress" the hull with additional degrees of freedom (or additional constraints).



- At large N, two regimes according to which of the blue or gray part in infinite $\rightarrow$ two sets of universal laws depending on the ratio d/k