Invariant measures for the KPZ equation

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Inhomogeneous Random Systems

KPZ equation on arbitrary domains

This talk is about invariant measures for the [Kardar-Parisi-Zhang] equation

$$\partial_t h = \frac{1}{2}\partial_{xx} h + \frac{1}{2}\big(\partial_x h\big)^2 + \xi, \quad t \geqslant 0, \quad x \in \mathbb{X}.$$

Question: Are there invariant measures ? Can one classify them all ? How to describe them precisely ? The answer depends on the space $\mathbb{X}$.

We may consider

- The whole line $\mathbb{X} = \mathbb{R}$
- A torus $\mathbb{X} = \mathbb{R}/\mathbb{Z}$
- An interval $\mathbb{X} = [0, L]$ with boundary conditions
- A half-line $\mathbb{X} = \mathbb{R}_+$

The simple case: $\mathbb{R}$ ou $\mathbb{R}/\mathbb{Z}$

- Assume that $\mathbb{X} = \mathbb{R}$. For a large class of initial conditions, $h(t,x) \sim \frac{-t}{24}$, so we do not expect that the law of $h(t,x)$ can be time invariant.

  No invariant measures in $C(\mathbb{R})$ but there exist invariant measures in $C(\mathbb{R})/\{x \mapsto c\}_{c\in\mathbb{R}}$ modulo a constant.

- If $h(0,x) = B_x^{(\mu)}$ a Brownian motion with drift μ, then for all time $t > 0$, as processes in x,

$$h(t,x) - h(t,0) \overset{(d)}{=} B_x^{(\mu)}.$$

  [Bertini-Giacomin 1997, Funaki-Quastel 2014] The law of $h(t,0)$ is non-trivial

$$h(t,0) = \frac{-t}{24} + \text{fluctuations } O(t^{1/3}).$$

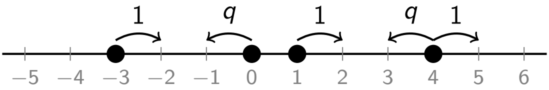
- On the torus $\mathbb{R}/\mathbb{Z}$, the Brownian motion is the unique invariant measure [Hairer-Mattingly 2016].

Plan of the talk

1. How to find invariant measures on $\mathbb{R}$? Using ASEP!
2. KPZ equation on a segment and its invariant measures
3. Some proof ideas (Matrix-product ansatz, harnesses, Liouville Hamiltonian)
4. Various conjectures: Large scale limits, invariant measures of the KPZ fixed point, invariant measures on $\mathbb{R}_+$.

ASEP

ASEP (asymmetric simple exclusion process) is a continuous Markov process on $\{0,1\}^{\mathbb{Z}}$, whose transition rates depend on an asymmetry parameter q.



- For any $\varrho \in [0,1]$, the measure $\mathrm{Ber}(\varrho)^{\otimes \mathbb{Z}}$ is invariant.
- Define a height function $H(t,x)$ so that

$$H(t,x) - H(t,x-1) = \begin{cases} 1 & \text{if site } x \text{ is occupied.} \\ -1 & \text{if site } x \text{ is empty.} \end{cases}$$

and $H(t,0)$ is the number of particles which have crossed the origin.

KPZ equation on $\mathbb{R}$

Solutions of the KPZ equation

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}\big(\partial_x h\big)^2 + \xi, \quad t \geqslant 0, x \in \mathbb{R}$$

are defined through the multiplicative noise stochastic heat equation

$$\partial_t Z(t,x) = \tfrac{1}{2}\Delta \; Z(t,x) + Z(t,x)\; \xi(t,x), \quad t > 0, x \in \mathbb{R},$$

where ξ is a space-time Gaussian white noise. In dimension 1, a solution $Z(t,x)$ solves

$$Z(t,x) = Z_0 * p_t(x) + \int_0^t ds \int_{\mathbb{R}} d_y p_{t-s}(y,x) Z(s,y)\xi(s,y),$$

where $p_t(y,x)$ is the standard heat kernel.

Definition

$h := \log(Z)$ is **the** solution of the KPZ equation with initial data $\log Z_0$.

Convergence ASEP $\to$ KPZ

Let $\mathcal{Z}_t(x) = q^{\frac{1}{2}H(t,x)-\nu t}$, where $\nu = (1-\sqrt{q})^2$. For $q = e^{-\varepsilon}$, when $\varepsilon \to 0$

$$\mathcal{Z}_{\epsilon^{-4}t}(\epsilon^{-2}x) \Longrightarrow Z(t,x),$$

the solution of

$$\partial_t Z(t,x) = \tfrac{1}{2}\Delta Z(t,x) + Z(t,x)\,\xi(t,x).$$

ASEP height function converges to a solution of KPZ equation.
[Bertini-Giacomin 1997]

Rmk: Under $\mathrm{Ber}(\varrho)^{\otimes\mathbb{Z}}$, the height function converges to a Brownian motion (with drift), up to a global shift.

KPZ equation on a segment

Consider the KPZ equation on the segment $[0, L]$,

$$\partial_t h = \tfrac{1}{2}\partial_{xx} h + \tfrac{1}{2}\big(\partial_x h\big)^2 + \xi.$$

For the solution to be unique, one needs to impose boundary conditions. Since $h(x,t) \sim -ct$ cannot be fixed, it is natural to impose a Newman type condition

$$\partial_x h(t,0) = A, \quad \partial_x h(t,L) = B.$$

Physically, $\partial_x h$ corresponds to the density in ASEP. These parameters also have a natural interpretation when $Z(t,x) = e^{h(t,x)}$ is viewed as the partition function of a directed polymer.

! $h(t,x)$ is not differentiable...

Boundary conditions

$h = \log Z$ yields

$$\partial_t Z(t,x) = \tfrac{1}{2}\Delta Z(t,x) + Z(t,x)\,\xi(t,x).$$

On $Z(t,x)$, boundary conditions become

$$\partial_x Z(t,0) = AZ(t,0), \quad \partial_x Z(t,L) = BZ(t,L).$$

Definition ([Corwin-Shen 2016])

$h(t,x)$ solves the KPZ equation on $[0,L]$ with boundary parameters $u \in \mathbb{R}$ and $v \in \mathbb{R}$ if:
For all $t > 0$, $x \in [0,L]$, $h(t,x) = \log Z(t,x)$ and

$$Z(t,x) = \int_0^L dy Z_0(y) p_t^{u,v}(x,y) + \int_0^t ds \int_0^L d_y p_{t-s}^{u,v}(x,y) Z(s,y)\xi(s,y),$$

where $p_t^{u,v}(x,y)$ is the heat kernel on $[0,L]$ with boundary conditions

$$\begin{cases} \partial_x p_t^{u,v}(x=0,y) = (u-\frac{1}{2})p_t^{u,v}(0,y), \\ \partial_x p_t^{u,v}(x=L,y) = (-v+\frac{1}{2})p_t^{u,v}(L,y). \end{cases}$$

Invariant measures on a segment

Theorem ([Corwin-Knizel 2021])

Fix $u, v \in \mathbb{R}$ such that $u + v > 0$ and $L = 1$.

1. *There exist a stationary process $h_{u,v}$. Its finite dimensional marginals $(h_{u,v}(x_1), \ldots, h_{u,v}(x_k))$ are characterized by a Laplace transform formula*

$$\mathbb{E}\left[\prod_{i=1}^{k} e^{-s_i(h_{u,v}(x_i)-h_{u,v}(x_{i-1}))}\right] = \textit{Some formula.}$$

2. *When $u + v = 0$, $h_{u,v}$ is a Brownian motion with drift $u = -v$.*

Motivations for looking at a simpler characterization:

- It's not clear how to extend to $u + v < 0$.
- The symmetry in u, v is not really apparent.
- The formula for $u + v = 0$ does not clearly degenerates to the Gaussian Laplace transform.

For any continuous process X_t, let

$$A_t(X) = \int_0^t e^{-2X_s} ds.$$

Theorem ([Bryc-Kuznetsov-Wang-Wesołowski], [B.- Le Doussal], [Bryc-Kuznetsov] 2021)

Fix $u, v \in \mathbb{R}$ such that $u + v > 0$. Fix $L = 1$.

$$h_{u,v}(x) = W_x + X_x,$$

where

- *W is a Brownian motion with diffusion coefficient $1/2$.* (From now on, all BM have diffusion coefficient $1/2$ and start from 0)
- *X is a continuous process, independent from W, whose law $\mathbb{P}_X$ is absolutely continuous w.r.t. the law $\mathbb{P}_B$ of the Brownian motion B.*

$$\begin{aligned}\frac{d\mathbb{P}_X}{d\mathbb{P}_B}(X) &= \frac{1}{\mathcal{Z}_{u,v}} \left(A_L(X)\right)^{-u} \left(A_L(X - X_L)\right)^{-v} \\ &= \frac{1}{\mathcal{Z}_{u,v}} \left(\int_0^L e^{-2X_s} ds\right)^{-u} \left(\int_0^L e^{-2(X_s - X_L)} ds\right)^{-v}\end{aligned}$$

This means that for functionals F,

$$\mathbb{E}_X[F(X)] = \frac{1}{\mathcal{Z}_{u,v}} \mathbb{E}_B \left[F(B) \left(A_L(B)\right)^{-u} \left(A_L(B - B_L)\right)^{-v} \right],$$

where B is a Brownian motion, and $A_L(B) = \int_0^L e^{-2B_s} ds$.

- Exchanging u et v has the same effect as reversing space, i.e. changing X_x into $x \mapsto X_{L-x} - X_L$.
- We have

  $$\left(A_L(X)\right)^{-u} \left(A_L(X - X_L)\right)^{-v} = e^{-2vX_L} A_L(X)^{-u-v}$$

  Hence by Cameron-Martin theorem, X is absolutely continuous w.r.t. a Brownian motion $B^{(-v)}$ with drift $-v$, and the Radon-Nikodym derivative is

  $$\frac{d\mathbb{P}_X}{d\mathbb{P}_{B^{(-v)}}}(X) = \frac{1}{\widetilde{\mathcal{Z}}_{u,v}} A_L(X)^{-u-v}.$$

- When $u + v = 0$, X is a Brownian motion.

Several conjectures

The Radon-Nikodym derivative

$$(A_L(X))^{-u}\,(A_L(X-X_L))^{-v}$$

makes perfect sense and is integrable even if $u+v<0$.

Conjecture ([B.- Le Doussal, 2021])

For any $u, v \in \mathbb{R}$, there exists a unique invariant process $h_{u,v}$, whose distribution is analytic in $u, v \implies$ Theorem holds for any $u, v \in \mathbb{R}$.

Conjecture ([B.- Le Doussal, 2021])

The Theorem holds for any interval length L.

Conjecture ([B.- Le Doussal, 2021])

The limits as $L \to +\infty$ are invariant measures for the KPZ equation on $\mathbb{R}_+$ (proof in progress).

Main steps

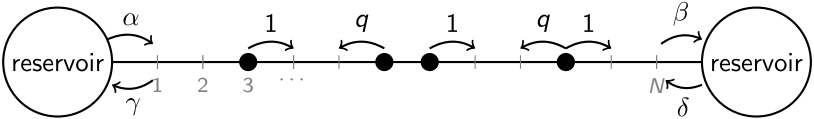
1. Find invariant measures for ASEP on a finite domain with boundary conditions (reservoirs) [Derrida-Evans-Hakim-Pasquier 1993]
2. Reformulate the result to obtain formulas [Bryc-Wesołowski 2018], in particular a Laplace transform
3. Study the KPZ equation limit [Corwin-Shen 2016] [Corwin-Knizel 2021] and prove that the limiting processes are invariant for the KPZ equation.
4. Invert the Laplace transform
   - [Bryc-Kuznetsov-Wang-Wesołowski 2021] first found another description.
   - [B.- Le Doussal 2021] found the statement just discussed (written in a physics paper) and made several conjectures.
   - Finally, [Bryc-Kuznetsov 2021] proved that for $u + v > 0$,

     [B.- Le Doussal 2021] $\iff$ [Bryc-Kuznetsov-Wang-Wesołowski 2021]

     and proved scaling limits conjectured in [B.- Le Doussal 2021].

Matrix product ansatz

Consider ASEP on $\{0,1\}^N$ with boundary parameters $\alpha, \beta, \gamma, \delta$.



We describe the state of the system by $\tau \in \{0,1\}^N$. The invariant measure $\mathbb{Q}$, determined by [Derrida-Evans-Hakim-Pasquier 1993], is such that for all $t_1, \dots, t_n \in \mathbb{C}$,

$$\mathbb{Q}\left[\prod_{j=1}^{N} t_j^{\tau_j}\right] = \frac{1}{K_N} w^T (E + t_1 \mathrm{D})(\mathrm{E} + t_2 \mathrm{D}) \dots (\mathrm{E} + t_N \mathrm{D})\, v$$

where

$$K_N = w^T (E + \mathrm{D})^N\, v$$

and E, D are infinite matrices, and w, v are vectors such that

$$DE - qED = D + E$$
$$w^T(\alpha E - \gamma D) = w^T$$
$$(\beta D - \delta E)v = v$$

Askey-Wilson orthogonal polynomials

- Finding representations, i.e. matrices E, D and explicit vectors u, v satisfying the relations, is non trivial. [Uchiyama-Sasamoto-Wadati 2003] found a representation using Askey-Wilson orthogonal polynomials.
- This allows to rewrite [Bryc-Wesołowski 2018]

$$\mathbb{Q}\left[\prod_{j=1}^{N} t_j^{\tau_j}\right] = \frac{\mathbb{E}\left[\prod_{j=1}^{N}(1+t_j+2\sqrt{t_j}AW_{t_j})\right]}{\mathbb{E}\left[\left(2+\sqrt{1-q}AW_1\right)^N\right]},$$

where AW_t, the Askey-Wilson process, is an auxiliary Markov process satisfying many interesting properties, for instance:

- The transition probabilities of AW_t are given by an explicit formula involving the Askey-Wilson density (product of ratios of many q-Pochhammer symbols).
- $n \mapsto P_n(AW_t; t)$ is a sequence of **orthogonal martingale polynomials** (where $P_n(x; t)$ denotes Askey-Wilson polynomials specialized appropriately). This means that $t \mapsto P_n(AW_t; t)$ is a martingale and $\mathbb{E}[P_n(AW_t; t)P_m(AW_t; t)] = 0$.

Martingales and harnesses

Harnesses

How did [Bryc-Wesołowski 2018] came up with this?

Definition

- A Harness H_t [Hammersley 67] is a stochastic process such that for $s < t < u$,

$$\mathbb{E}\left[H_t | \sigma(\mathcal{F}_{\leqslant s}, \mathcal{F}_{\geqslant u})\right] = \frac{t-s}{u_s} H_s + \frac{u-t}{u-s} H_u$$

  Conditionnally on what happens before s and after u, the expectation of H_t is the linear interpolation between H_s and H_u.

- A quadratic harness is a harness such that

$$\mathbb{E}\left[H_t^2 | \sigma(\mathcal{F}_{\leqslant s}, \mathcal{F}_{\geqslant u})\right] = Q_{s,u}(H_s, H_u)$$

  where $Q_{s,u}(x, y)$ is a quadratic form.

[Bryc-Wesołowski 2010] found that these conditions typically determine H_t, and solutions can be constructed using orthogonal martingale polynomials. The Askey-Wilson process provides a large class of quadratic harnesses...

Corwin-Knizel's result

[Corwin-Knizel 2021] studied the limit of formulas from [Bryc-Wesołowski 2018] under the KPZ equation scaling.

The auxiliary Markov process AW_t has a limit with explicit transition probabilities. In the one-point case,

$$\mathbb{E}[e^{-sh_{u,v}(y)}] = e^{\frac{s^2 y}{4}} \times \iint_0^\infty dt_1\, dt_2 p_0(t_2) p_{0,s}(t_2, t_1) e^{\frac{-1}{4}(t_1 y + t_2(L-y))},$$

where, for $u, v > 0$,

$$p_0(t) = \frac{(u+v)(u+v+1)}{8\pi} \frac{\Gamma\left(v \pm i\frac{\sqrt{t}}{2}\right)\Gamma\left(u \pm i\frac{\sqrt{t}}{2}\right)}{\sqrt{t}\Gamma\left(\pm i\sqrt{t}\right)}$$

$$p_{0,s}(t_2, t_1) = \frac{1}{8\pi} \frac{\Gamma\left(u - \frac{s}{2} \pm i\frac{\sqrt{t_1}}{2}\right)}{\Gamma\left(u \pm i\frac{\sqrt{t_2}}{2}\right)\sqrt{t_1}\Gamma\left(\pm i\sqrt{t_1}\right)} \frac{\Gamma\left(\frac{s}{2} \pm i\frac{\sqrt{t_2}}{2} \pm i\frac{\sqrt{t_1}}{2}\right)}{\Gamma(s)}$$

Remark: For $u < 0$ or $v < 0$ with $u + v \geqslant 0$, Laplace transform formulas are analytic continuations.

Liouville quantum mechanics I

- One may recognize expressions that typically arise in Liouville quantum mechanics, or when computing exponential functionals of the Brownian motion [Comtet, Monthus, Texier, and others in the 80's and 90's] [B.-Le Doussal 2020].
- The Liouville Hamiltonian, in dimension 1, is the operator

$$H = \frac{-1}{4}\frac{d^2}{dx^2} + e^{-2x}.$$

It is diagonalized by the family of eigenfuctions

$$\psi_k(x) = \sqrt{\frac{2}{\pi}}\frac{1}{|\Gamma(ik)|}K_{ik}(2e^{-x}),$$

where $K_{ik}(z)$ is a Bessel function, normalized so that

$$H\psi_k = \frac{k^2}{4}\psi_k, \quad \langle \psi_k | \psi_{k'} \rangle = \delta(k-k').$$

Liouville quantum mechanics II

We have a Cauchy-type identity, for $s > 0$

$$\int_{\mathbb{R}} dx e^{-sx} \psi_k(x) \psi_{k'}(x) = \frac{C}{8\pi} \frac{\Gamma\left(\frac{s}{2} \pm \frac{ik}{2} \pm \frac{ik'}{2}\right)}{\Gamma(s)},$$

which we recognize in the fomula for $p_{0,s}(t_2, t_1)$. We get

$$\mathbb{E}\left[e^{-sh_{u,v}(x)}\right] = e^{\frac{1}{4}s^2 y} \int dx_1 \int dx_2 \int dk_1 \int dk_2 e^{-2vx_2-(2u-s)x_1)}$$
$$\times \psi_{k_2}(x_2)\psi_{k_1}(x_1) \int dx_3 e^{-sx_3} \psi_{k_1}(x_3)\psi_{k_2}(x_3) e^{-x\frac{k_1^2}{2}-(L-x)\frac{k_2^2}{2}}$$

There exist analogous formulas for the Laplace transform of $(h_{u,v}(x_1), \ldots, h_{u,v}(x_d))$, which allow to write $h_{u,v}(x)$ as the sum of a Brownian motion and the Doob transform of a Markov process Y_x having explicit transitions.

This is the description proved for $u + v > 0$ in [Bryc-Kuznetsov-Wang-Wesołowski 2021]

Feynman-Kac

Using Feynman-Kac formula, the fundamental solution of $\partial_t u = -Hu$ can be written as

$$\int dk \psi_k(x') e^{-t\frac{k^2}{4}} \psi_k(x) = p_t(x, x') \mathbb{E}_B \left[e^{-\int_0^t e^{-2B(s)ds}} \right],$$

where the expectation is w.r.t. a Brownian bridge such that $B(0) = x$ et $B(t) = x'$. After interpreting a bit the formulas, we obtain

$$h_{u,v}(x) = W_x + Y_x - Y_0$$

where

- W is a Brownian motion.
- Y is independent from W, and its law $\mathbb{P}_Y$ is absolutely continuous w.r.t. to that of a Brownian B with free starting point. The Radon-Nikodym derivative is

$$\frac{1}{\mathcal{Z}_{u,v}} e^{-2uY_0 - 2vY_L} e^{-A_L(Y)},$$

where $A_L(Y) = \int_0^L e^{-2Y_s} ds$.

Final step

Write $X_s = Y_s - Y_0$ and integrate over the starting point Y_0 using the identity

$$\int_0^\infty \mathrm{d}z \; z^{u+v-1} e^{-zA} = A^{-u-v} \Gamma(u+v)$$

(think of A being the exponential Brownian functional)

The law of $X_s = Y_s - Y_0$ is absolutely continuous w.r.t. that of a Brownian B with $B(0) = 0$. The Radon-Nikodym derivative is

$$\frac{d\mathbb{P}_X}{d\mathbb{P}_B}(X) = \frac{1}{\tilde{\mathcal{Z}}_{u,v}} \left(A_L(X)\right)^{-u} \left(A_L(X - X_L)\right)^{-v}$$

où $A_L(X) = \int_0^L e^{-2X_s} ds$.

The process X was first studied by [Hariya-Yor, 2004] in the paper *Limiting distributions associated with moments of exponential Brownian functionals*, motivated by the Matsumoto-Yor identity.

Monsieur Jourdain...

"Our reference to Monsieur Jourdain (a character of Molière (1622–1673) [17]) in the title alludes to this point; Monsieur Jourdain discovers that he is practicing prose without being aware of it; analogously the following theorem shows that a number of authors have been dealing with harnesses:"

Harnesses, Lévy bridges and Monsieur Jourdain, R. Mansuy, M. Yor, 2005.

Universality

It is expected that all models in the KPZ class converge at large scale to a universal Markov process called **KPZ fixed point**.

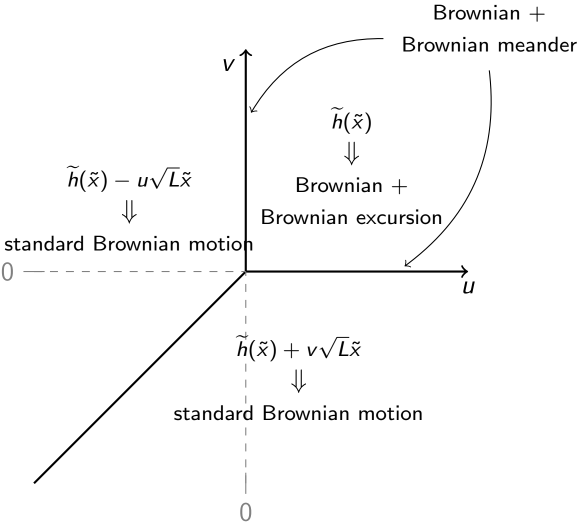
Its definition depends on which space we consider: $\mathbb{R}$, $\mathbb{R}/\mathbb{Z}$, $\mathbb{R}_+$, $[0, L]$, but it has been defined only on $\mathbb{R}$ [Matetski-Quastel-Remenik 2016]

Assuming the existence of this universal process and the convergence of KPZ class models towards it, invariant measures of any KPZ class model must converge to invariant measure of the corresponding KPZ fixed point. Hence we expect that

$$\tilde{h}(\tilde{x}) := \frac{1}{\sqrt{L}} h_{u,v}(L\tilde{x})$$

converge to universal processes.

The following convergences were non-rigorously derived in [B.-Le Doussal, 2021] and proved in [Bryc-Kuznetsov 2021].

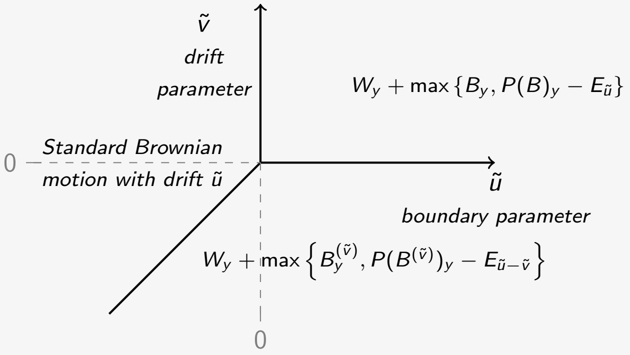


We recover the same result as for TASEP [Derrida-Enaud-Lebovitz 2004] and ASEP [Bryc-Wang 2018].

We may also take the $L \to +\infty$ limit to obtain invariant processes on $\mathbb{R}_+$, and then take the large scale limit to the KPZ fixed point on $\mathbb{R}_+$.

Conjecture

Invariant measures for ASEP on $\mathbb{N}$ – the existence of a two-parameter family is proved in [Liggett, 1975]– converge to



where $P(B)$ is the Pitman transform of a Brownian and $E_{\tilde{u}}$ and $E_{\tilde{u}-\tilde{v}}$ are independent exponential random variables.

Conclusion

Invariant measures for KPZ equation on a segment are given by re-weightings of the Brownian measure by exponential functionals, studied by many authors, in particular Hariya and Yor.

Perspectives

- **An important open problem:** Uniqueness/classification, ergodic limit theorems.
- **A mystery:** Direct understanding of the relation between Liouville field theory and KPZ? What about higher dimensions?

Thank you