Asymptotics of multi-species ASEP

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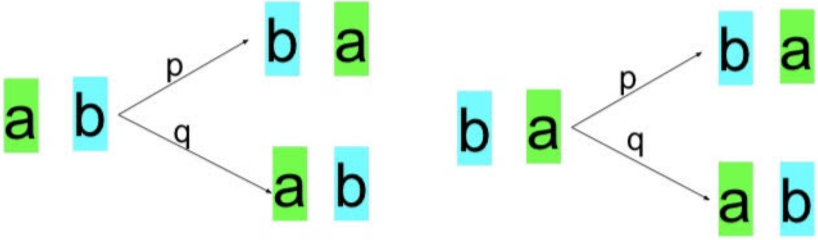
Biased Card Shuffling



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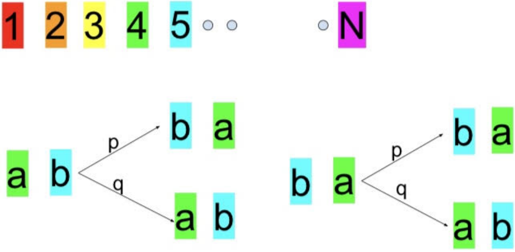


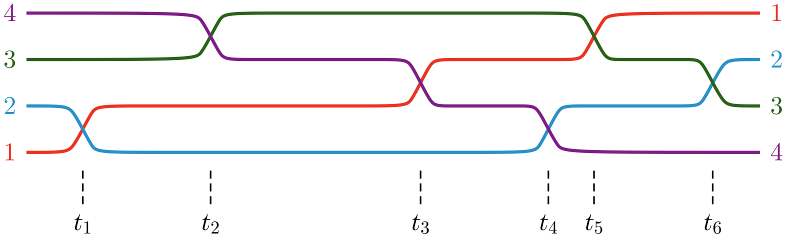
$a < b$, $p = 1$, $q = 0$.

Continuous time: Updates happen according to independent Poisson processes on $\mathbb{R}_{\geq 0}$ attached to each pair of neighboring positions.

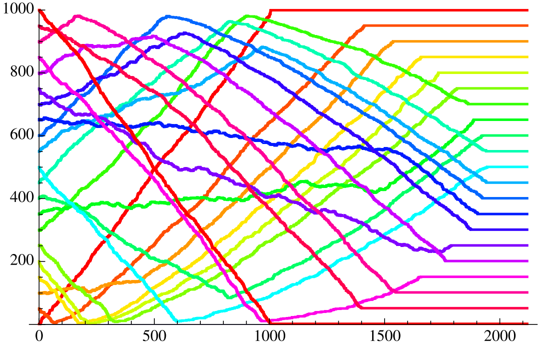
Question: When the sorting stops?

Multispecies TASEP on an interval

- Interval $\{1, 2, \ldots, N\}$. Symmetric group S_N.
- Each transposition $(i, i+1)$ has independent exponential clock.
- When the clock rings, we swap particles at i and $i+1$, but only **if** it will **increase** the number of color-position inversions.



Angel-Holroyd-Romik-08: What's happening as N becomes large?

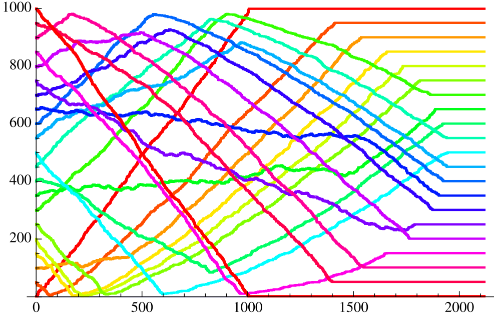


Picture from Angel-Holroyd-Romik-08.
Only 21 out of 1000 trajectories shown.

Theorem. [Angel-Holroyd-Romik] Set $\gamma_y = 1 + 2\sqrt{y(1-y)}$.
If $U_N(k)$ is the **last time the swap** $(k, k+1)$ **happens**, then

$$\frac{U_N(k) - N\gamma_{k/N}}{N^{1/3}(\gamma_{k/N})^{2/3}\left(\frac{k}{N}(1-\frac{k}{N})\right)^{-1/6}} \xrightarrow[N\to\infty]{d} F_2, \qquad \text{(Tracy-Widom distribution)}$$

Proof is based on coupling with TASEP with step initial condition and the result of Johansson'00.



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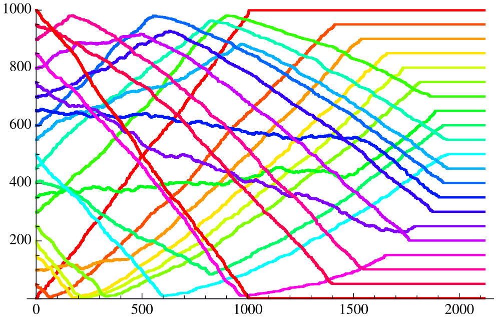
Question. Set T_N^{OSP} — the time when the systems **stops**

[AHR-08]: We have $T_N^{\mathrm{OSP}} \approx 2N$. What are **the fluctuations**?



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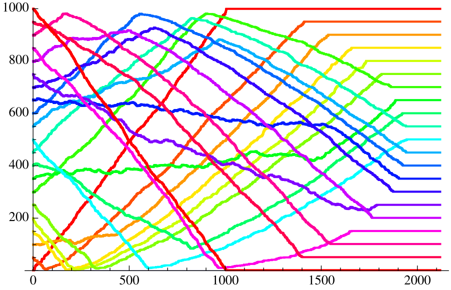
Theorem. Bufetov-Gorin-Romik'20

$$\frac{T_N^{\text{OSP}} - 2N}{2^{1/3} N^{1/3}} \xrightarrow[N \to \infty]{d} F_1,$$

where F_1 is another Tracy-Widom distribution.



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Theorem. (Bufetov-Gorin-Romik-20)

$$\frac{T_N^{\mathrm{OSP}} - 2N}{2^{1/3} N^{1/3}} \xrightarrow[N\to\infty]{d} F_1,$$

Proof is based on symmetries of interacting particle systems Borodin-Gorin-Wheeler'19, Galashin'20; also we prove some of conjectures from Bisi-Cunden-Gibbons-Romik'20.

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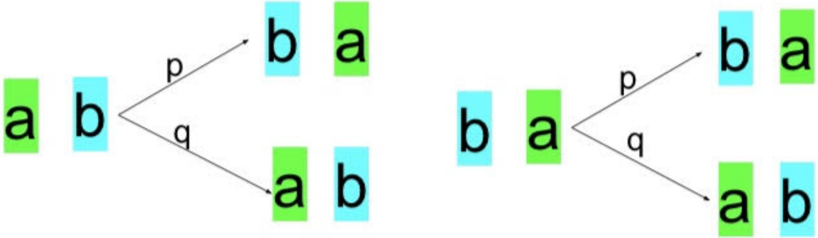

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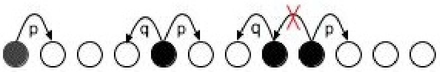
ASEP on a finite interval

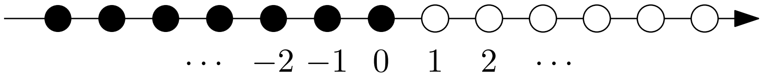


ASEP = Asymmetric Simple Exclusion Process.

There are k_N particles on $\tilde{\mathbb{Z}}_N = \{1, 2, \ldots, N\}$ which evolve in time. There are two Poisson processes of rates p and $q < p$ associated with each particle, $p + q = 1$.

Each particle jumps one step to the right with rate p, and jumps one step to the left with rate q, if the neighboring positions are vacant. If the position is occupied by another particle, the jump does not happen.

All Poisson processes are independent.



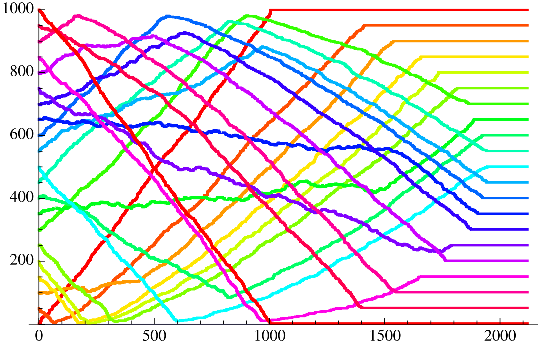
$q = 0$ (totally asymmetric simple exclusion process = TASEP).

Consider a standard (two-color = particles and holes) TASEP started with the step initial condition. Let $h^{tasep}(x, t)$ be the number of particles that are to the right of x at time t.

Johansson'00

$$\frac{h^{tasep}(0, t) - t/4}{-t^{1/3}2^{-4/3}} \xrightarrow[t\to\infty]{d} F_2,$$

where in the right-hand side stands the F_2 Tracy-Widom distribution.



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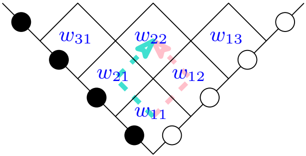
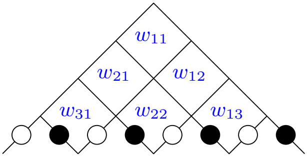




Connection with last passage percolation

Flat initial condition: Baik-Rains'99, Sasamoto'05, Borodin-Ferrari-Prahofer-Sasamoto'07

$$\frac{x_0(t) - t/4}{-2^{2/3} t^{1/3}} \xrightarrow[t\to\infty]{d} F_1,$$

It turns out that there exist exact distribution identities which relate this single-species problem with a multi-species problem. They exist due to inherent algebraic structure behind the model.

Hecke algebra

$W = S_n$, $s_i = (i, i+1)$.

$L(w) :=$ number of inversions in $w \in W$.

Hecke algebra: $\{T_w\}_{w \in W}$ — linear basis

$$\begin{cases} T_s T_w = T_{sw}, & \text{if } L(sw) = L(w) + 1 \\ T_s T_w = (1-q) T_w + q T_{sw}, & \text{if } L(sw) = L(w) - 1. \end{cases}$$

The linear map $I : \mathcal{H} \to \mathcal{H}$

$$I : \sum_w a_w T_w \to \sum_w a_w T_{w^{-1}}$$

satisfies

$$I(h_r h_{r-1} \dots h_2 h_1) = I(h_1) I(h_2) \dots I(h_r), \qquad h_i \in \mathcal{H}.$$

Random walk on Hecke algebra

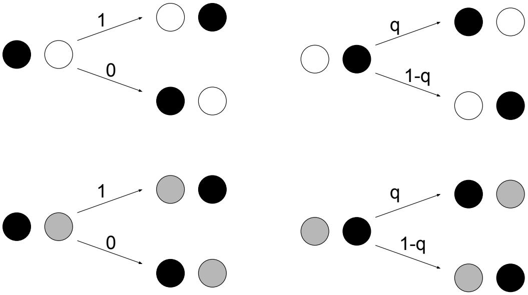
Generators $\{G_1, \ldots, G_k\}$, each of these generators has an independent exponential clock. When the clock s rings, we multiply G_s to the current position of the random walk $P \in \mathcal{H}$ — our new position is $G_s P$. This is a *random walk on Hecke algebra*.

An element of Hecke algebra

$$h := \sum_w \kappa_w T_w, \qquad \kappa_w \geq 0, \ \sum_w \kappa_w = 1,$$

can be interpreted as a random element of W. Random walk on Hecke algebra generates the random walk on W.

Multi-species ASEP



We consider particles of various types (=classes, colors, species).

Set of types is linearly ordered, and a particle of a smaller type interacts with a particle of a larger type as a particle with a hole.

Particular case: configurations are given by permutations $\pi : \mathbb{Z}_n \to \mathbb{Z}_n$, where $\pi(j)$ is encoding the type of a particle standing at j.

Multi-species ASEP / Hecke algebra

$W = S_n$, generators: $\{T_{s_i}\}_{i=1}^{n-1}$. Equivalent language for the description of ASEP: Vocabulary

- Random multi-species configuration — element of Hecke algebra
- Update —- multiplication by T_s
- ASEP evolution — element of S_n generated by random walk on Hecke algebra
- Projection to fewer colors — projection to cosets of parabolic subgroups
- Class-position symmetry — involution I swaps w and w^{-1}.

Other Coxeter groups generate ASEP with a source (hyperoctahedral group), ASEP on a ring (affine Weyl group $\tilde{A}_n$).

Multi-species ASEP / Hecke algebra

$W = S_n$, generators: $\{T_{s_i}\}_{i=1}^{n-1}$. Equivalent language for the description of ASEP.

- Multi-species ASEP is generated by Hecke algebra: Alcaraz-Rittenberg'93, Alcaraz-Droz-Henkel-Rittenberg'93, ..., Lam'11 , Cantini-de Gier-Wheeler'15, ...
- Color-position symmetry and applications for asymptotic analysis: Angel-Holroyd-Romik'08 (TASEP, $q = 0$), Amir-Angel-Valko'08 (ASEP), Borodin-Bufetov'19 (inhomogeneous stochastic six vertex model).
  Explanation through Hecke algebra: Bufetov'20, Galashin'20; a closely related proof Kuan'20.
- Hidden symmetries: Borodin-Gorin-Wheeler'19, Galashin'20, Bisi-Cunden-Gibbons-Romik'20, Dauvergne'20, Bufetov-Korotkikh'20, Zhang'21.

- Amir-Angel-Valko'08: Joint distribution of second class particles started with step initial condition in multispecies TASEP.
- Borodin-Bufetov'19: second class particle in multispecies ASEP with deformed initial condition.
  Bufetov-P. L. Ferrari'20: second class particle in the TASEP shock under a variety of scalings.
- Other generators of a random walk on Hecke algebra
  Bufetov'20. Asymptotic behavior of second class particle in multispecies q-TAZRP with deformed initial conditions.
  Second-class particle in ASEP with a source and deformed initial condition (comes from BC-Hecke algebra).

The results about limit behavior of second class particles continue the line of research from P. A. Ferrari-Kipnis'95, P. A. Ferrari-Goncalves- Martin'08 (results about limit behavior of second class particle started from a particular initial condition, ASEP), Cator-Pimentel'13 (second class particle started from arbitrary initial condition, TASEP).

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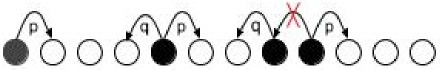
  Second-class particle in ASEP with a source and deformed initial condition (comes from BC-Hecke algebra).
- Bufetov-Nejjar'20: Cutoff profile for a single-species ASEP on segment.

ASEP on a finite interval



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All Poisson processes are independent.

Cutoff

Ergodic Markov chain with finitely many states. S— state space, ξ — initial configuration, Q_t^ξ — the distribution of the Markov chain started from ξ at time t.

There is a unique stationary distribution π. We measure the *total variance distance*:

$$||Q_t^\xi - \pi||_{TV} := \frac{1}{2}\sum_{w \in S} |Q_t^\xi(w) - \pi(w)| = \max_{A \subset S} |Q_t^\xi(A) - \pi(A)|.$$

$$d(t) := \max_{\xi \in S} ||Q_t^\xi - \pi||_{TV}$$

Mixing time:

$$T^{mix}(\varepsilon) := \inf\{t : d(t) \le \varepsilon\}.$$

Cutoff

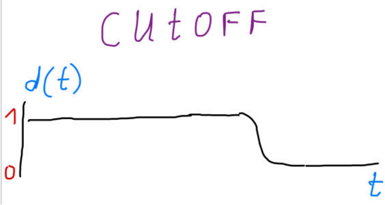
A *sequence* of Markov chains depending on N.

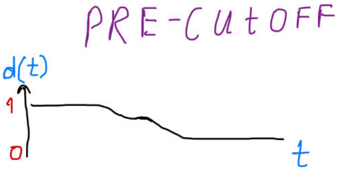
Cutoff: for any $\varepsilon > 0$:

$$\lim_{N\to\infty} \frac{T_N^{mix}(\varepsilon) - T_N^{mix}(1-\varepsilon)}{T_N^{mix}(1/4)} = 0.$$

Pre-cutoff:

$$\sup_{\varepsilon} \limsup_{N\to\infty} \frac{T_N^{mix}(\varepsilon) - T_N^{mix}(1-\varepsilon)}{T_N^{mix}(1/4)} < \infty.$$





Cutoff profile

A sequence of Markov chains exhibits a cutoff at time $f(N)$ with window of order $g(N)$ if

$$\lim_{c \to +\infty} \limsup_{N \to \infty} d_N(f(N) + cg(N)) = 0,$$

$$\lim_{c \to -\infty} \liminf_{N \to \infty} d_N(f(N) + cg(N)) = 1,$$

(for $g(N) << f(N)$).

This cutoff has profile $\mathcal{F}(c)$ if

$$\lim_{N \to \infty} d_N(f(N) + cg(N)) = \mathcal{F}(c).$$

ASEP on a finite interval

There is a unique stationary measure for ASEP on a finite interval.

Previous results

- Diaconis-Ram'00: a discrete time ASEP (systematic scan Methropolis algorithm; colored vertex model) exhibits a pre-cutoff. Method: representations of Hecke algebra.
- Benjamini-Berger-Hoffman-Mossel'02: continuous time ASEP (as defined above) exhibits a pre-cutoff. Method: link with ASEP on an infinite lattice. Hydrodynamics.
- Labbe-Lacoin'16: continuous time ASEP exhibits cutoff. Method: link with ASEP on an infinite lattice. Hydrodynamics.

Cutoff profile for ASEP

Theorem (Bufetov-Nejjar'20)

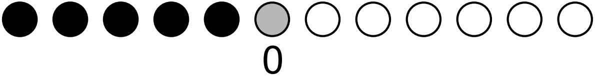
For ASEP on an interval of length N with k_N particles, assume that $k_N/N \to \alpha \in (0;1)$, as $N \to \infty$. We have

$$\lim_{N\to\infty} d_N\left(\frac{N\left(1+2\sqrt{\alpha(1-\alpha)}\right)+cN^{1/3}}{p-q}\right) = 1 - F_{GUE}\left(cf(\alpha)\right),$$

where

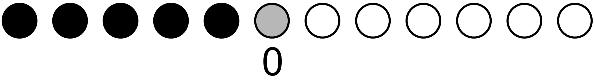
$$f(\alpha) := \frac{(\alpha(1-\alpha))^{1/6}}{\left(\sqrt{\alpha}+\sqrt{1-\alpha}\right)^{4/3}}.$$

and F_{GUE} is a distribution function of the (GUE) Tracy-Widom distribution.



Let us start with this initial condition. Let $S_1(t)$ be the position of the second class particle at time t.

Asymptotics of $S_1(t)$?

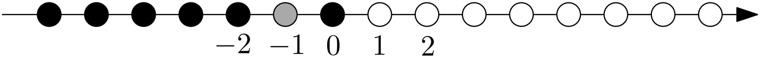


Let us start with this initial condition. Let $S_1(t)$ be the position of the second class particle at time t.

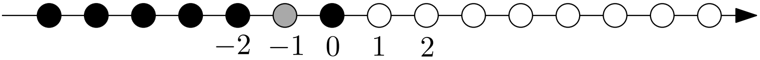
$$\lim_{t\to\infty} \mathrm{Prob}\left(\frac{S_1(t)}{t} < x\right) = d(-x) = \frac{1}{2}\left(1+\frac{x}{1-q}\right).$$

Uniform distribution on $[-(1-q);(1-q)]$.

P.A. Ferrari-Kipnis'95, P.A. Ferrari-Goncalves-Martin'08.



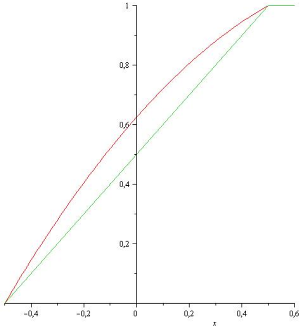
The asymptotic distribution of the second class particle ?

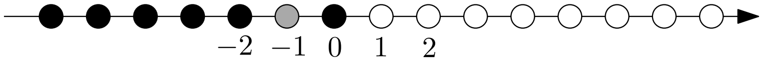


(Borodin-Bufetov'19) The asymptotic distribution of the second class particle

$$\lim_{t\to\infty} \mathrm{Prob}\left(\frac{S_1(t)}{t} < x\right) = d(-x) + (1-q)d(-x)(1-d(-x)).$$

Note the nontrivial dependence on q.





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- $q = 0$: TASEP, Cator-Pimentel'13: for general initial conditions.
- for a class of initial configurations and general q: Borodin-Bufetov'19
- Bufetov'20: Similar results for a second class particle for half-line ASEP with a source, and a second class particle in q-TAZRP.