Renormalization and disorder : a simple toy model

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Inhomogeneous Random Systems

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Paris January 2022

Collaborators

- Retaux 2014
- Chen, Hu, Lifshits, Shi $\geq$ 2017
- Dagard $\geq$ 2018

Derrida, Retaux, *J. Stat. Phys. 2014*

Chen, Dagard, Derrida, Hu, Lifshits, Shi, *Ann. of Prob. 2021*

Hu, Mallein, Pain, *Comm. Math. Phys. 2020*

Chen, Dagard, Derrida, Shi, *Journal of Physics A 2020*

OUTLINE

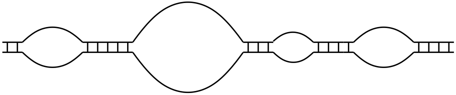
The Poland Sheraga model

The toy model

The Poland Scheraga 1966

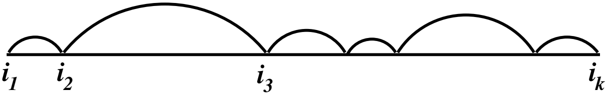
The Poland Scheraga model

Model of DNA denaturation



Model of depinning

The Poland Scheraga model

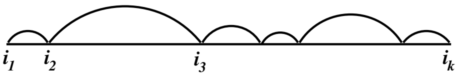


- the contact energy at position i is ϵ_i
- these contact energies are i.i.d. (quenched disorder)
- the weight of a loop of length n is

$$\omega(n) \sim \frac{s^n}{n^c} \qquad (s = 1)$$

$$Z_L = \sum_{k \geq 2} \sum_{1 < i_2 \cdots < i_{k-1} < L} \omega(i_2 - i_1) \cdots \omega(i_k - i_{k-1}) \exp\left[-\frac{\epsilon_{i_1} + \epsilon_{i_2} + \cdots \epsilon_{i_k}}{T} \right]$$

Phase transition in the Poland Scheraga model



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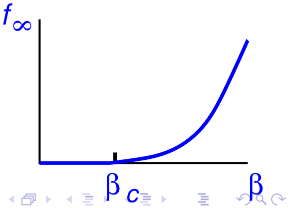
The free energy $F_L = \log Z_L$ In the thermodynamic limit

$$f_\infty = \lim_{L \to \infty} \frac{F_L}{L}$$

- $\beta < \beta_c$ $f_\infty = 0$ the unpinned phase
- $\beta > \beta_c$ $f_\infty > 0$ the pinned phase



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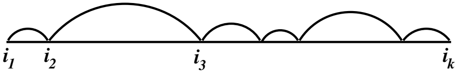

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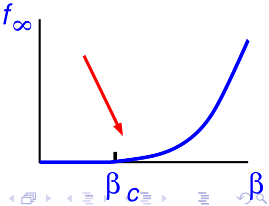
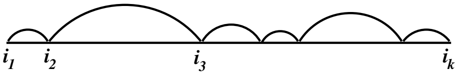


Phase transition in the pure case $\epsilon_i = \epsilon$



Fisher 1984

T_c is known

$$\exp\left[\frac{\epsilon}{T_c}\right] = \sum_{n\geq 1} \omega(n)$$

- For $c > 2$ the transition is first order

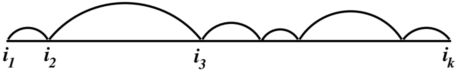
$$f_\infty \sim (T_c - T)$$

- For $1 < c < 2$ the transition is second order

$$f_\infty \sim (T_c - T)^{\frac{1}{c-1}}$$

$\Rightarrow$ Specific heat exponent $\boxed{\alpha_{\text{pure}} = \frac{2c-3}{c-1}}$

Solution of the pure case $\epsilon_i = \epsilon$



$$Z_L = \omega(L) e^{-2\beta\epsilon} + e^{-\beta\epsilon} \sum_{\ell=1}^{L-1} \omega(\ell)\ Z_{L-\ell}$$

Generating functions

$$\Omega(\lambda) = \sum_{\ell} \omega(\ell)\, \lambda^{\ell} \qquad ; \qquad \sum_{L} Z_L\, \lambda^L = \frac{\Omega(\lambda)\ e^{-2\beta\epsilon}}{1 - \Omega(\lambda)\ e^{-\beta\epsilon}}$$

$$\omega(n) \sim \frac{1}{n^c} \quad \Rightarrow \quad \Omega(\lambda) = \begin{cases} C_1 + C_2(1-\lambda)^{c-1} + \cdots & \text{for} \quad 1 < c < 2 \\ C_1 + C_2(1-\lambda) + C_3(1-\lambda)^{c-1} + \cdots & \text{for} \quad c > 2 \end{cases}$$

Some results in the disordered case

The irrelevant case $\alpha_{\text{pure}} < 0 \quad \Rightarrow \quad \alpha_{\text{disorder}} = \alpha_{\text{pure}}$

Harris 1974, Alexander, Giacomin, Lacoin, Toninelli , $\cdots$ since 2008

The transition is always smooth $\quad \Rightarrow \quad \alpha_{\text{disorder}} \leq 0$

Giacomin - Toninelli 2006

$\alpha_{\text{pure}} = 0$ is marginal relevant

Derrida - Hakim - Vannimenus 1992

Giacomin - Lacoin - Toninelli 2010 , Berger - Lacoin 2018

Strong disorder : infinite order transition

$$f_\infty \sim \exp[-K\,(T_c - T)^{-\nu}] \quad \text{with} \quad \nu = \begin{cases} 1/2 \\ 1 \end{cases}$$

Tang-Chaté 2001

Monthus 2017

The hierarchical lattice

Derrida - Hakim - Vannimenus 1992

- $L = 2^n$
- all loops have lengths 2^k with $k = 0, 1, 2, 3, \cdots$
- $Z_0^{(j)} = \exp\left[-\frac{\epsilon_j}{T}\right]$

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$$f_\infty = \lim_{n \to \infty} \frac{\log Z_L}{L}$$

$f_\infty > 0$ is the pinned phase ; $f_\infty = 0$ is the unpinned phase

$$\alpha_{\text{pure}} = \frac{\log 2 + 2\log(b-1) - 2\log b}{\log 2 + \log(b-1) - \log b}$$

(for $b > 2$)

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$f_\infty \sim$ ... 1/2 Tang-Chaté 2001

1 Monthus 2017

THE SAME

The hierarchical lattice ($X_n = \log Z_{2^n}$)

$$X_{n+1} = G\left(X_n^{(1)} + X_n^{(2)}\right) \quad \Leftarrow \quad Z_{2L} = \frac{Z_L^{(1)} Z_L^{(2)} + b - 1}{b}$$

with

$$G(X) = X + \log\left(\frac{1 + (b-1)e^{-X}}{b}\right)$$

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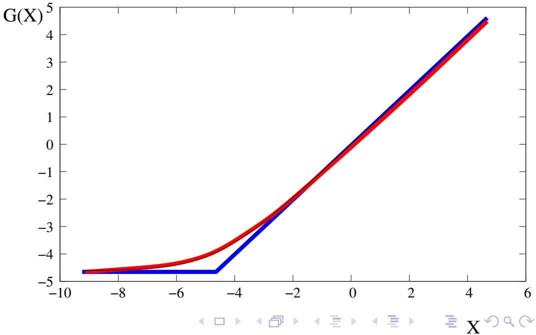
The toy model

$$X_{n+1} = G\left(X_n^{(1)} + X_n^{(2)}\right)$$

with

$$G(X) = \max(X, -a)$$



The toy model

The toy model

Two ingredients:

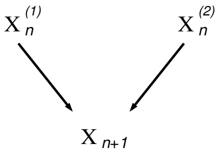
- Start with an infinite sequence i.i.d. random variables

$$X_0^{(1)} \; \ldots \; X_0^{(j)} \ldots$$

distributed according to a distribution $P_0(X)$.

- A non-linear function G to iterate these variables

$$X_{n+1}^{(j)} = G\left(X_n^{(2j-1)} + X_n^{(2j)}\right)$$



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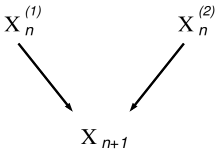


Interpretation:

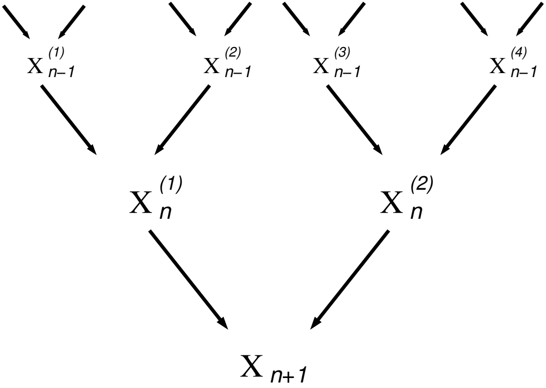
System size $L = 2^n$; Free energy $\log Z_L \equiv X_n$

Main question:

What is $Y = \lim \frac{X_n}{2^n} \equiv f_\infty$?

The toy model

Collet - Glaser - Eckmann - Martin 1984



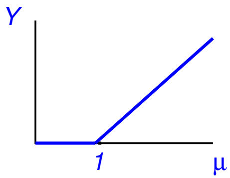
$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \;,\; 0\,]$$

Main question

$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \;,\; 0\,]$$

What is the limit Y of $\frac{X_n}{2^n}$

$P_0(x) = \delta_{x,\mu}$



Pure

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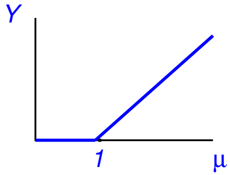
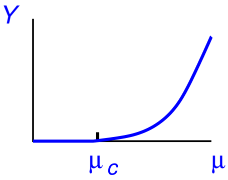


Pure

$$P_0(x) = (1-\lambda)\,\delta_{x,0} + \lambda\,\delta_{x,\mu}$$



Disordered

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What is the limit Y *of* $\frac{X_n}{2^n}$

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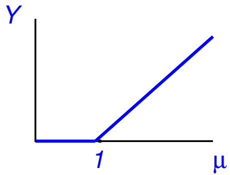

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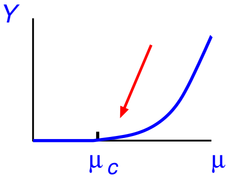


Disordered

Renormalization

$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \,,\; 0\,]$$

Exact renormalization (special case X_n are integers)

$P_0(x)$ is given ; $P_n \rightarrow P_{n+1}$

Define the generating function $H_n(z) = \sum_X P_n(X)\, z^X$

$$H_{n+1}(z) = \frac{H_n(z)^2 - H_n(0)^2}{z} + H_n(0)^2$$

A few facts

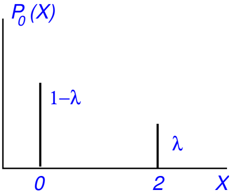
$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \; , \; 0\,]$$

$$\boxed{H_{n+1}(z) = \frac{H_n(z)^2 - H_n(0)^2}{z} + H_n(0)^2}$$

- A phase transition

Collet - Glaser - Eckmann - Martin 1984

For example



$$H_0(z) = 1 - \lambda + \lambda z^2$$

$$\Rightarrow \quad \boxed{\lambda_c = \frac{1}{5}}$$

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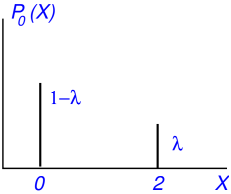

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- A one parameter family of fixed points

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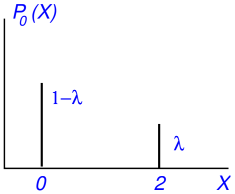

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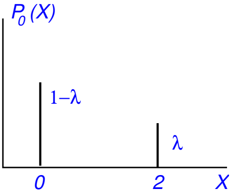

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- A phase transition of the Berezinski Kosterlitz Thouless type

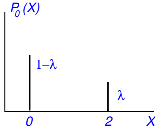
The critical behavior

$$X_{n+1} = \max[\,X_n^{(1)} + X_n^{(2)} - 1\ ,\ 0\,]$$

$$H_n(z) = \sum_X P_n(X) z^X$$

A phase transition given by $\boxed{2H'(2) - H(2) = 0}$

Collet - Glaser - Eckmann - Martin 1984



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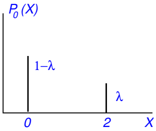

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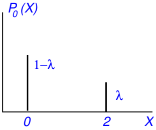

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$$2H'(2) - H(2) \sim \lambda - \lambda_c$$

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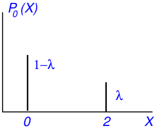
Collet - Glaser - Eckmann - Martin 1984



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$$2H'(2) - H(2) \leq 0$$

$$\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \to 0$$

The critical behavior

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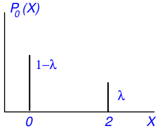

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$$2H'(2) - H(2) \sim \lambda - \lambda_c$$

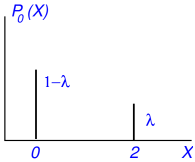
$$2H'(2) - H(2) \leq 0 \qquad\qquad 2H'(2) - H(2) > 0$$

$$\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \to 0 \qquad\qquad \boxed{\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \simeq \exp\left[-\frac{A}{\sqrt{\lambda - \lambda_c}}\right]}$$

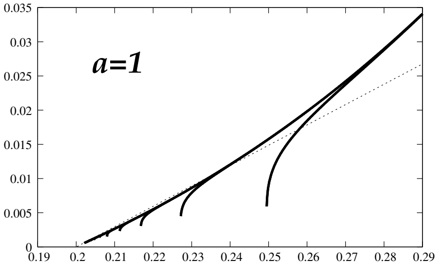
D. - Retaux 2014

An essential singularity ?

$$\lim_{n\to\infty} \frac{\langle X_n \rangle}{2^n} \simeq \exp\left[-\frac{A}{\sqrt{\lambda-\lambda_c}}\right] \quad \Leftrightarrow \quad \left(\log \frac{\langle X_n \rangle}{2^n}\right)^{-2} \propto \ (\lambda - \lambda_c)$$



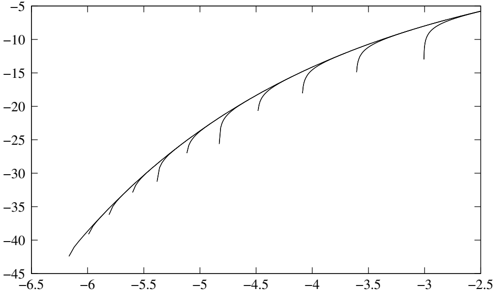
$$\lambda_c = \frac{1}{5}$$



$$\left(\log \frac{\langle X_n \rangle}{2^n}\right)^{-2} \quad \text{versus} \quad \lambda$$

A power law singularity ?

$$\frac{\langle X_n \rangle}{2^n} \propto (\lambda - \lambda_c)^{\chi} \quad \Leftrightarrow \quad \log \frac{\langle X_n \rangle}{2^n} \sim \chi \log(\lambda - \lambda_c)$$



$$\log \frac{\langle X_n \rangle}{2^n} \quad \text{versus} \quad \log(\lambda - \lambda_c)$$

Chen - Dagard - Derrida - Hu - Lifshits - Shi 2021

Initial distribution $P_0(X) = (1-\lambda)\delta_X + \lambda Q(X)$

$$\text{Particular case:} \quad Q(X) \sim \frac{C}{X^\gamma \, 2^X}$$

- If $Q(X)$ decays fast enough $(\gamma > 4)$ then $\lambda_c > 0$ and

$$\lim_n \frac{\langle X_n \rangle}{2^n} = \exp\left(-\frac{1}{(\lambda - \lambda_c)^{\frac{1}{2}+o(1)}}\right)$$

- If $2 < \gamma < 4$ then $\lambda_c > 0$

$$\lim_n \frac{\langle X_n \rangle}{2^n} = \exp\left(-\frac{1}{(\lambda - \lambda_c)^{\nu+o(1)}}\right) \quad \text{with } \nu = \frac{1}{\gamma - 2}$$

- If $\gamma \leq 2$ then $\lambda_c = 0$ (because $H'(2) = \infty$)

The critical Regime

Derrida - Retaux 2014

$$P_n(X) = 2^{-X} \; \epsilon^2 \; R(\epsilon X, \epsilon n) \qquad \textit{for} \qquad X > 0$$

and

$$P_n(0) = 1 - \sum_{X \geq 1} P_n(X)$$

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Derrida - Retaux 2014

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then one can show that for ϵ small

$$\boxed{\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2} \int_0^x R(x_1,\tau) \; R(x - x_1, \tau) \; dx_1}$$

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Still a difficult problem

Criticality

$$\int_0 R(X,\tau)\, X dX = 1$$

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\ R(x-x_1,\tau)\ dx_1$$

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For

$$R(x,\tau) = A(\tau)\exp[-B(\tau)x]$$

one gets

$$\frac{dA(\tau)}{d\tau} = -B(\tau)A(\tau) \quad ; \quad \frac{dB(\tau)}{d\tau} = -\frac{A(\tau)}{2}$$

Berezinsky Kosterlitz Thouless renormalization

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Berezinsky Kosterlitz Thouless renormalization

$$R(x,\tau) = 4\frac{k^2}{\sin(k(\tau+\tau_0))^2}\ \exp\left[-\frac{2k\ x}{\tan(k(\tau+\tau_0))}\ \right]$$

($k \to 0$ is the critical case)

Other solutions

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\; R(x-x_1,\tau)\; dx_1$$

Physical solutions

$$R = \sum_{i=1}^{n} A_i(\tau)\, e^{-B_i(\tau)x}$$

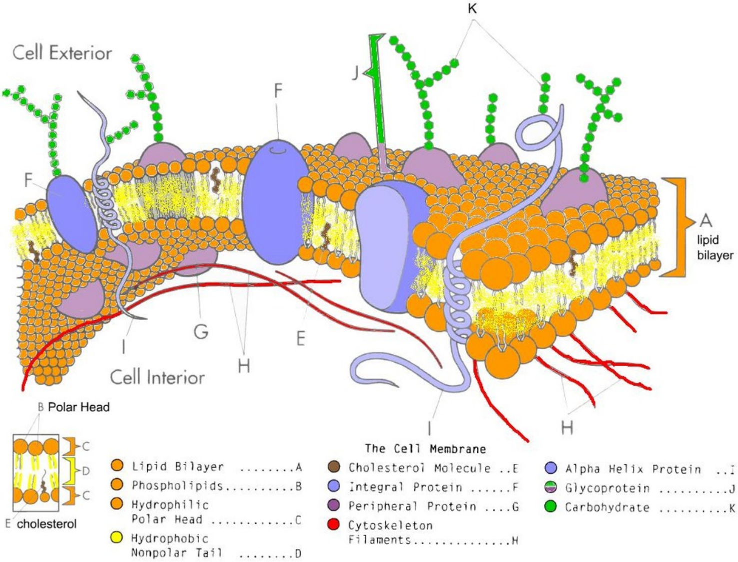
with

$$\frac{dB_i}{d\tau} = -\frac{A_i}{2} \qquad ; \qquad \frac{dA_i}{d\tau} = -B_i A_i - \sum_{j\neq i} \frac{A_i A_j}{B_i - B_j}$$

Unphysical solutions

$$R = \frac{4}{\tau^2} e^{-\frac{3x}{\tau}} \left[3\cos\left(\frac{\sqrt{3}x}{\tau}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}x}{\tau}\right)\right]$$

The critical manifold



Other solutions

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\ R(x-x_1,\tau)\ dx_1$$

Scaling solutions along the critical manifold $(\int R(X)\ X\ dX = 1)$

$$R = \frac{1}{t^2}\, G\left(\frac{x}{t}\right)$$

Then G should satisfy

$$(1+x)G'(x) + 2G(x) + \frac{1}{2}\int_0^x G(x_1)\ G(x-x_1)\ dx_1 = 0$$

- $G(z) = 4\, e^{-2z}$

Other solutions

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\; R(x-x_1,\tau)\; dx_1$$

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- $G(z) = 4\, e^{-2z}$
- Other scaling solution for $2 < \gamma < 4$: $\quad G(0) = \frac{\gamma(\gamma-2)}{2}$
  Taking the the Laplace transform $H(p) = \int_0^\infty G(z)\, e^{-pz}\, dz$

$$H(p) - pH'(p) + pH(p) + \frac{1}{2}H(p)^2 - G(0) = 0$$

This is a non-linear equation!

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\; R(x-x_1,\tau)\; dx_1$$

$$R = \frac{1}{t^2}\, G\left(\frac{x}{t}\right) \qquad\qquad H(p) = \int_0^\infty G(z)\, e^{-pz} dz$$

Introducing a function $y(p)$ such that

$$H(p) = -1 - p - i\,p\,\frac{y'(ip/2)}{y(ip/2)}$$

then y is solution of

$$p^2 y'' + p y' + \left(p^2 - \frac{(\gamma-1)^2}{4}\right) y = 0$$

So $y(p)$ is a Bessel function !

$$G(x) \sim x^{-\gamma} \qquad \text{for} \quad x \to \infty$$

Open questions

- X_n real, half integer

$$X_{n+1} = \max[\, X_n^{(1)} + X_n^{(2)} - 1 \;,\; 0\,]$$

- Analysis of

$$\frac{\partial R(x,\tau)}{\partial \tau} = \frac{\partial R(x,\tau)}{\partial x} + \frac{1}{2}\int_0^x R(x_1,\tau)\; R(x-x_1,\tau)\; dx_1$$

- Convolution 2 $\rightarrow$ Convolution 3 :
  $2 < \gamma < 4$ becomes $1.5 < \gamma < 2.6$

- Going back to the hierarchical model

$$X_n^{(j)} = G\left(X_n^{(2j-1)} + X_n^{(2j)} \right)$$

- Going back to the Poland Scheraga model

- Other tree models with B. K. T. transition