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Fluctuations of shocks in the asymmetric simple exclusion process

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1. The exclusion process: some basics

TASEP: a model in the KPZ class

- **TASEP:** Totally Asymmetric Simple Exclusion Process

- Configurations

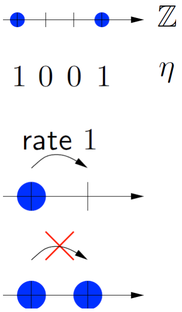
$$\eta = \{\eta_j\}_{j\in\mathbb{Z}},\ \eta_j = \begin{cases} 1, & \text{if } j \text{ is occupied,} \\ 0, & \text{if } j \text{ is empty.} \end{cases}$$

- Dynamics

  Independently, particles jump on the right site with rate 1, provided the right is empty:

$$\mathcal{L}f(\eta) = \sum_{j\in\mathbb{Z}} \eta_j(1-\eta_{j+1})[f(\eta^{j,j+1}) - f(\eta)].$$

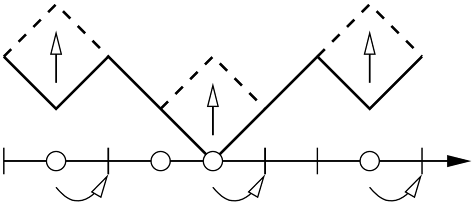
- Ordering is preserved: if particles are initially at $\cdots x_2(0) < x_1(0) < \cdots$, then at any later times $\cdots x_2(t) < x_1(t) < \cdots$.



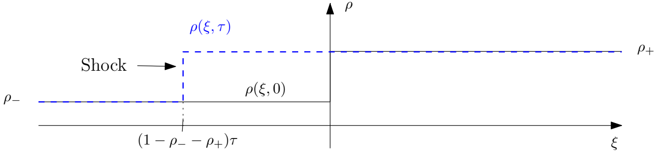
- Height function associated with TASEP particles

(a) $h(0,0) = 0,$

(b) $h(j,t) - h(j-1,t) = 1 - 2\eta_j(t)$



- Hydrodynamic scaling: $h_{\mathrm{ma}}(\xi,\tau) := \lim_{\varepsilon\to 0} \varepsilon h(\varepsilon^{-1}\xi, \varepsilon^{-1}\tau)$.
- Macroscopic density: $\frac{\partial}{\partial\xi} h_{\mathrm{ma}}(\xi,\tau) =: 1 - 2\rho(\xi,\tau)$.
- Burgers equation: $\frac{\partial}{\partial\tau}\rho + \frac{\partial}{\partial\xi}\rho(1-\rho) = 0$.



- Example: $\rho(\xi,0) = \begin{cases} \rho_- & \text{if } \xi < 0, \\ \rho_+ & \text{if } \xi \geq 0, \end{cases}$ with $\rho_- < \rho_+$.

  Then

$$\rho(\xi,\tau) = \begin{cases} \rho_- & \text{if } \xi < (1-\rho_- - \rho_+)\tau, \\ \rho_+ & \text{if } \xi \geq (1-\rho_- - \rho_+)\tau. \end{cases}$$

- The discontinuity in the density is called a shock.

- Macroscopic slope: $u(\xi,\tau)=\frac{\partial}{\partial\xi}h_{\rm ma}(\xi,\tau)$.
- Macroscopic speed of growth: $v(u)$.
- PDE for u:

$$\frac{\partial}{\partial\tau}u+a(u)\frac{\partial}{\partial\xi}u=0 \quad \text{with} \quad a(u)=-\frac{\partial}{\partial u}v(u).$$

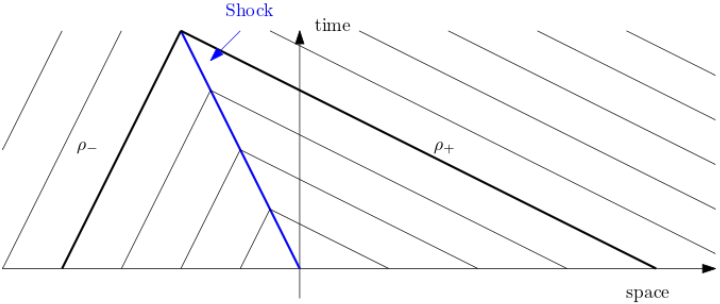
- The characteristic lines are solutions of the PDE which satisfy

$$\frac{\partial\xi}{\partial\tau}=a(u) \text{ and } \frac{\partial u}{\partial\tau}=0.$$

- Example: TASEP has $v(u)=\frac{1}{2}(1-u^2)$ form which $a(u)=u$, so the characteristic lines for initial condition with density ρ has speed $a(u)=u=1-2\rho$:

$$\xi(\tau)=\xi(0)+u\tau=\xi(0)+(1-2\rho)\tau.$$

- Let $\rho_- < \rho_+$. The shock position is the intersection of the characteristic lines.



Space-time correlations

- TASEP with stationary initial condition: Bernoulli-ρ.
- Two-point function

$$S(j,t) := \mathbb{E}(\eta_j(t)\eta_0(0)) - \rho^2$$

- Second class particle: let $Z(t)$ be the position of a second class particle starting at 0. Then

$$\mathbb{P}(Z(t)=j) = \chi^{-1} S(j,t) \quad \text{with } \chi = \rho(1-\rho).$$

- Second class particle is a microscopic version of the shock.
- Scaling function: Prähofer,Spohn'02

$$S(j,t) \simeq \frac{\chi}{4}\frac{1}{2\chi^{1/3}t^{2/3}} f_{\rm KPZ}\left(\frac{(j-(1-2\rho)t)}{2\chi^{1/3}t^{2/3}}\right).$$

- Space-time correlations are non-trivial in a $t^{2/3}$-neighborhood of characteristic lines.

- Along characteristic lines, decorrelation is over time of $\mathcal{O}(t)$.
- Example: TASEP with density ρ (stationary or deterministic).
- In terms of height function we have:

Theorem (Corwin,Ferrari,Péché'10)

Let us fix $\nu < 1$. *Then, for any* $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathbb{P}\left(\left|\frac{h((1-2\rho)t,t)-h((1-2\rho)(t-t^\nu),(t-t^\nu))-C(\rho)t^\nu}{t^{1/3}}\right| \geq \varepsilon\right) = 0.$$

- In terms of particle positions, a similar argument gives:

Theorem (Ferrari,Nejjar'19)

Let us fix $\nu < 1$. *Then, for any* $\varepsilon > 0$,

$$\lim_{t\to\infty} \mathbb{P}\left(\left|\frac{x_{\rho^2 t}(t)-x_{\rho^2(t-t^\nu)}(t-t^\nu)-(1-2\rho)t^\nu}{t^{1/3}}\right| \geq \varepsilon\right) = 0.$$

2. Particle fluctuations around the shocks

- Step initial conditions: $x_n(0) = -n+1$, $n \geq 0$. For any $\alpha \in (0,1)$: Johansson'00

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_{\alpha t}(t) - (1-2\sqrt{\alpha})t}{-\sigma(\alpha)t^{1/3}} \leq s\right) = F_{\mathrm{GUE}}(s).$$

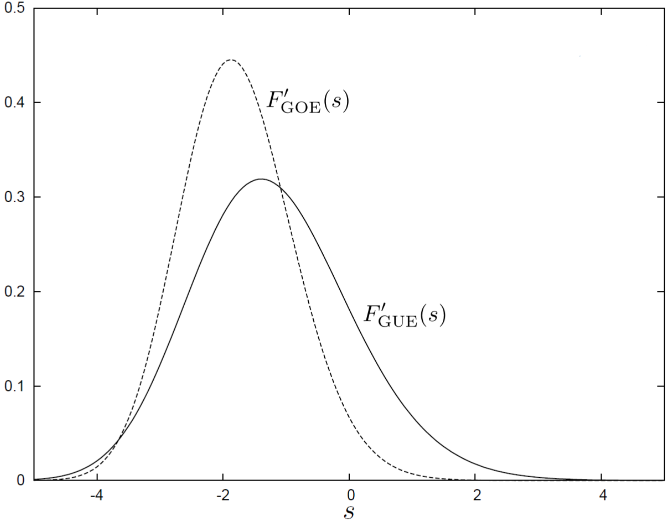
- Flat initial conditions: $x_n(0) = -2n$, $n \in \mathbb{Z}$.

Sasamoto'05;Borodin,Ferrari,Prähofer,Sasamoto'07

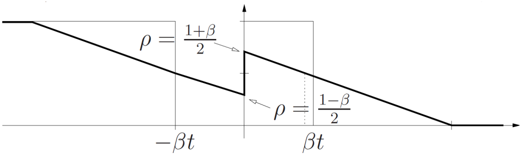
$$\lim_{t\to\infty} \mathbb{P}\left(\frac{x_0(t) - t/4}{-2^{-2/3}t^{1/3}} \leq s\right) = F_{\mathrm{GOE}}(s).$$

- F_{GUE} and F_{GOE} are the Tracy-Widom distribution function arising as limiting distribution of the largest eigenvalue in the Gaussian Unitary / Orthogonal Ensemble of random matrices.

Densities of F_{GUE} and F_{GOE}



- The influence of the shock on the particle fluctuation is of $\mathcal{O}(t^{1/3})$.



Theorem (Ferrari-Nejjar'13)

Fix $\beta \in (0,1)$, $x_n(0) = -n - \lfloor \beta t \rfloor$ *for* $n \geq 1$ *and* $x_n(0) = -n$ *for* $-\lfloor \beta t \rfloor \leq n \leq 0$. *Then, for* $\nu = (1-\beta)^2/4$,

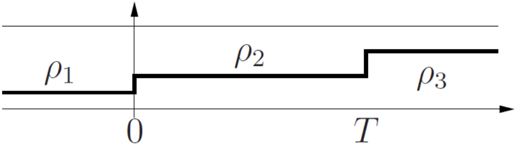
$$\lim_{t\to\infty} \mathbb{P}(x_{\nu t + \xi t^{1/3}}(t) \geq -s t^{1/3}) = F_{\mathrm{GUE}}\left(\frac{s-\xi/\rho_1}{\sigma_1}\right) F_{\mathrm{GUE}}\left(\frac{s-\xi/\rho_2}{\sigma_2}\right)$$

for explicit constants $\rho_1, \rho_2, \sigma_1, \sigma_2$.

- Note the product form of the distribution function.

- Let $\rho_1 < \rho_2 < \rho_3$ be three densities and T a large parameter. Consider the initial condition

$$x_n(0) = \begin{cases} -\lfloor n/\rho_1 \rfloor, & \text{for } n \geq 0, \\ -\lfloor n/\rho_2 \rfloor, & \text{for } -\lfloor \rho_2 T \rfloor \leq n \leq -1, \\ T - \lfloor (n + \lfloor \rho_2 T \rfloor)/\rho_3 \rfloor, & \text{for } n < \lfloor \rho_2 T \rfloor. \end{cases}$$

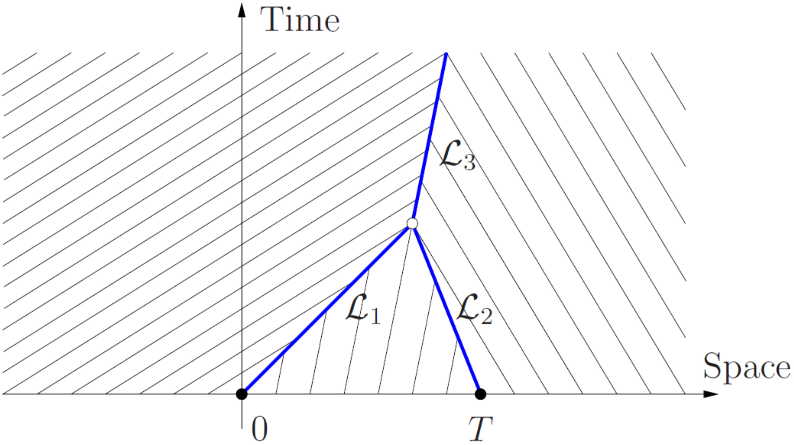


- The two shocks have trajectories

$$s_1(t) = (1 - \rho_1 - \rho_2)t, \quad s_2(t) = T + (1 - \rho_2 - \rho_3)t$$

which merge at time

$$t = T/(\rho_3 - \rho_1).$$


Time
$\mathcal{L}_3$
$\mathcal{L}_1$
$\mathcal{L}_2$
Space
0
T

- Scaling around the shock:
  - time $t = \frac{T}{\rho_3-\rho_1} + \tau T^{1/3}$
  - position $X = \frac{(1-\rho_1-\rho_2)T}{\rho_3-\rho_1}$
  - particle label $N = \frac{\rho_1\rho_2 T}{\rho_3-\rho_1} + uT^{1/3}$.

Theorem (Ferrari-Nejjar'19)

With the above notations,

$$\lim_{T\to\infty} \mathbb{P}\left(\frac{x_N(t)-X}{-T^{1/3}} \leq s\right) = \prod_{k\in\{1,2,3\}} F_{\mathrm{GOE}}\left(\frac{s-u/\rho_k+(1-\rho_k)\tau}{\sigma_k}\right)$$

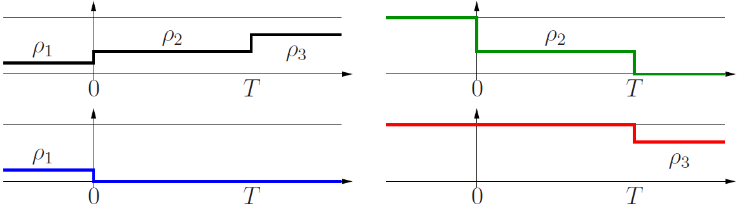
for some explicit constants $\sigma_1, \sigma_2, \sigma_3$.

- For a time span of order $T^{1/3}$ the fluctuations is a product of three F_{GOE} distributions.
- Before and after that, the fluctuations are a product of two F_{GOE} distributions.

- Decomposition on sub-problems: one shows Seppäläinen'98

$$x_N(t) = \min\{x_N^{(1)}(t), x_N^{(2)}(t), x_N^{(3)}(t)\}$$

where $x_N^{(1)}(0)$, $x_N^{(2)}(0)$, $x_N^{(3)}(0)$, have initial density:



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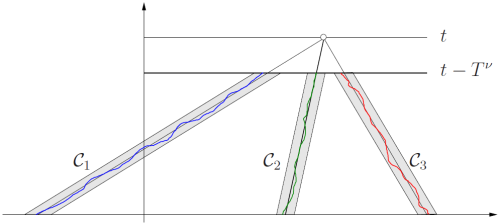
where $x_N^{(1)}(0)$, $x_N^{(2)}(0)$, $x_N^{(3)}(0)$

- Slow-decorrelation for $\nu \in (2/3, 1)$: for some N_ℓ, c_ℓ,

$$x_N^{(\ell)}(t) = x_{N_\ell}^{(\ell)}(t - T^\nu) + c_\ell T^\nu + o(T^{1/3}).$$

- Localization: $x_{N_\ell}^{(\ell)}(t - T^\nu)$ for $\ell = 1, 2, 3$ are asymptotically independent: correlations only in $\mathcal{C}_\ell$ of width $\mathcal{O}(T^{2/3})$.



3. Shock (second class particle) fluctuations

- Initial condition: Bernoulli(ρ_-)-Bernoulli(ρ_+) product measure with

$$\mathbb{P}(\eta_j(0)=1) = \begin{cases} \rho_- & \text{if } j<0, \\ \rho_+ & \text{if } j\geq 0. \end{cases}$$

Theorem (Ferrari-Fontes'94)

Fluctuations of the second class particle $Z(t)$ starting at 0: for $\rho_- < \rho_+$,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{Z(t)-(1-\rho_- - \rho_+)t}{\sigma(\rho_-,\rho_+)t^{1/2}} \leq s\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s} e^{-x^2/2}dx$$

for some explicit constant σ.

- Speed of the shock is $1-\rho_- - \rho_+$.
- The $t^{1/2}$ (and Gaussian nature) is actually coming from the Brownian scaling of the initial conditions, not of the KPZ-type dynamics of TASEP.

- For non-random initial conditions, the fluctuations are reduced to a $t^{1/3}$ scale.
- Initial condition: particles with density ρ_- on $\mathbb{Z}_-$ and ρ_+ on $\mathbb{Z}_+$:

$$x_n(0) = -\lfloor n/\rho_- \rfloor \text{ for } n > 0$$
$$x_n(0) = -\lfloor n/\rho_+ \rfloor \text{ for } n < 0,$$

and put a second class particle at the origin: $Z(0) = 0$.

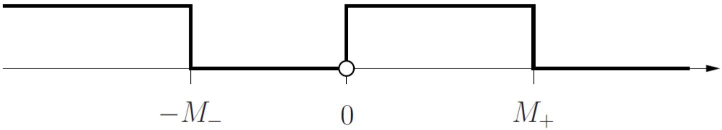
Theorem (Ferrari, Ghosal, Nejjar'17)

For some explicit constants c_1, c_2 of ρ_-, ρ_+,

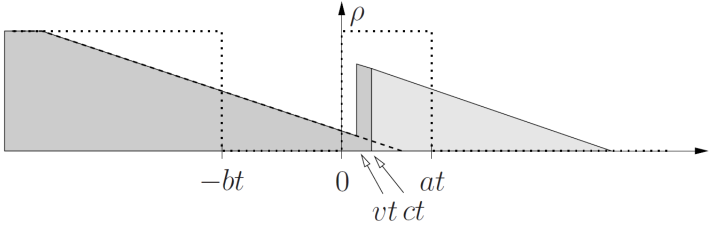
$$\lim_{t\to\infty} \frac{Z(t) - (1 - \rho_- - \rho_+)t}{t^{1/3}} \stackrel{\mathcal{D}}{=} c_1 \xi^{(1)}_{\rm GOE} - c_2 \xi^{(2)}_{\rm GOE}$$

where $\xi^{(1)}_{\rm GOE}$ and $\xi^{(2)}_{\rm GOE}$ are independent GOE Tracy-Widom distributed random variables.

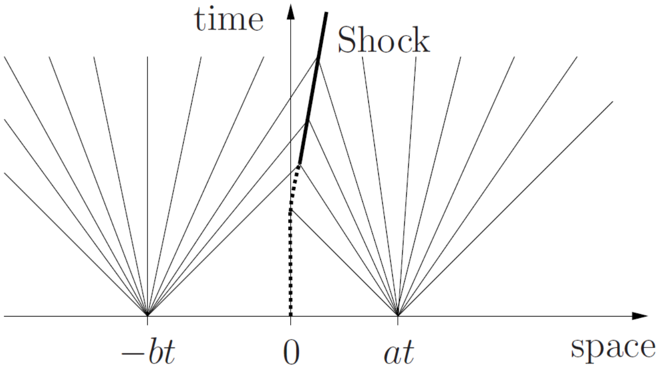
- Initial condition: TASEP particles at $(-\infty, -M_-) \cup [1, M_+]$
Second class particle $Z(t)$ initially at 0.



- Macroscopic picture: let $0 < a < b < 1$ and set $M_+ = at$, $M_- = bt$



Characteristic lines for the GUE-GUE shock



- We have a variety of scaling, depending on how the sizes M_- and M_+ depend on the observation time t.
- Case 1: let $M_+ = at$, $M_- = bt$ for some $a, b \in (0,1)$.
- Speed of shock: $v = \frac{(a-b)(a+b-2)}{2(a+b)}$.

Theorem (Bufetov,Ferrari'20)

Assume that $2 - a - 2\sqrt{a} < b < 2\sqrt{a} - a$. Then, for some explicit constants c_1, c_2,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{Z(t) - vt}{t^{1/3}} \le s\right) \stackrel{\mathcal{D}}{=} c_1 \xi^{(1)}_{\rm GUE} - c_2 \xi^{(2)}_{\rm GUE}$$

where $\xi^{(1)}_{\rm GUE}$ and $\xi^{(2)}_{\rm GUE}$ are two independent GUE Tracy-Widom distributed random variables.

- Cases 2-3: let $M_+ = at^\delta$, $M_- = bt^\delta$ for some $a, b > 0$.

Theorem (Bufetov,Ferrari'20)

Case 2: If $\delta \in (2/3, 1)$, *with* $X_t = \frac{(a-b)}{(a+b)}t + \frac{a+b}{2}t^\delta$. *Then for some explicit constant* c_3,

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{Z(t)-X_t}{t^{4/3-\delta}} \leq s\right) \stackrel{\mathcal{D}}{=} c_3(\xi^{(1)}_{\mathrm{GUE}} - \xi^{(2)}_{\mathrm{GUE}})$$

where $\xi^{(1)}_{\mathrm{GUE}}$ *and* $\xi^{(2)}_{\mathrm{GUE}}$ *are two independent GUE Tracy-Widom.*
Case 3: If $\delta \in (0, 2/3)$, *let* $v = \frac{a-b}{a+b}$. *Then*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{Z(t)-vt}{t^{1-\delta/2}} \leq s\right) \stackrel{\mathcal{D}}{=} \mathcal{N}\left(0, \frac{4ab}{(a+b)^3}\right)$$

where $\mathcal{N}(0, \sigma^2)$ *is a centered normal distributed random variable with variance* σ^2.

- Case 4: let $M_+ = M_- = a2^{1/3}t^{2/3}$ for some $a > 0$.

Theorem (Borodin,Bufetov'19)

Then

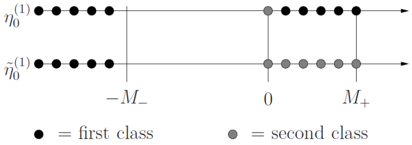
$$\lim_{t\to\infty} \mathbb{P}\left(\frac{Z(t)}{2^{1/3}t^{2/3}} \leq s\right) = \mathbb{P}\left(\mathcal{A}_2(a-s) - \mathcal{A}_2(-a-s) \geq 4as\right)$$

where $\mathcal{A}_2$ is the Airy$_2$ process.

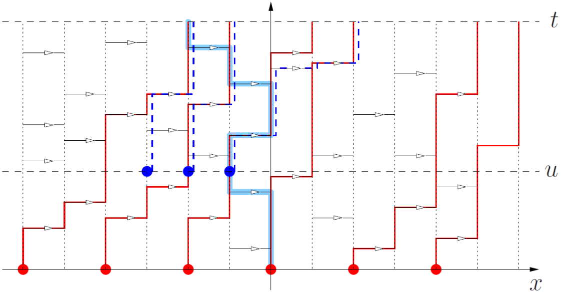
- As $a \to 0$: $\mathcal{A}_2(a-s) - \mathcal{A}_2(-a-s) \sim \sqrt{2}\mathcal{B}(2a)$ ($\mathcal{B}$ is standard Brownian motion).
- As $a \to \infty$: $\mathcal{A}_2(a-s)$ and $\mathcal{A}_2(-a-s)$ becomes independent F_{GUE} random variables.

4. A few more words on the methods

- For all cases we use slow-decorrelation and localization
- For the first mentioned results, we used the link to last passage percolation.
- For the more recent results like the shock collision, we made working directly in the space-time picture.
- For the last results with Alexey, we used multi-colored (multi-type) TASEP and a symmetry theorem by Borodin-Bufetov, allowing to link the position of the second class particle in terms of various height TASEP functions without second class particles.

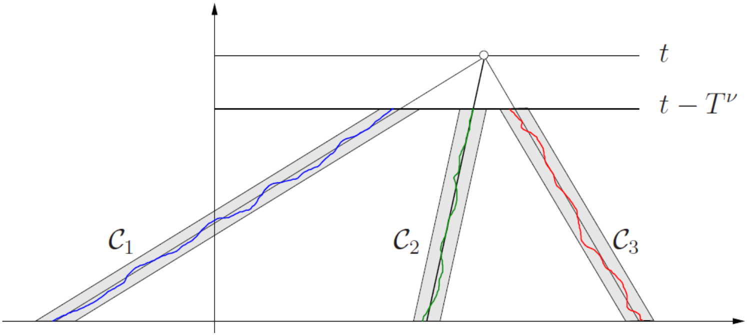


- For particle N at time t, we construct the backwards path $x_{N(u)}(u)$, $u : t \to 0$ (analogue of geodesics for LPP). The label $N(u)$ satisfies:
  - $N(t) = N$,
  - $N(u) \to N(u) - 1$ is at time u a jump of particle $N(u)$ is suppressed.
- At any time $u \in [0, t]$, resetting the system to a step-initial condition at position $x_{N(u)}(u)$ leads to the same position of particle N at time t.



Localization: use of backwards paths

- As input we need estimates on the tails of the distribution functions of $x_n(t)$
- We show that with high probability backwards paths are localized in a $\mathcal{O}(t^{2/3})$ region around the characteristic lines.



Gaussian fluctuations

- For the results of Gaussian fluctuations, we use comparison to stationarity, the analogue in LPP by Cator,Pimentel'15.
- Backwards paths of x, y with initial configuration 1: $\pi_{1,x}, \pi_{1,y}$
- Backwards paths of x, y with initial configuration 2: $\pi_{2,x}, \pi_{2,y}$
- If $\pi_{1,x} \cap \pi_{2,y} \neq \emptyset$, then $h_2(y,t) - h_2(x,t) \geq h_1(y,t) - h_1(x,t)$
- Strategy: sandwich the backwards paths for step-initial condition (or other) between two stationary ones, which under scaling converges to the same Brownian law.