1/3 – 2/3 scaling in the Ising model.

Yvan Velenik

Université de Genève

— Introduction —

Ising model

- **Box:** $\mathrm{B}_N = \{-N+1, \dots, N\}^2$
- $\oplus$ **boundary condition:**

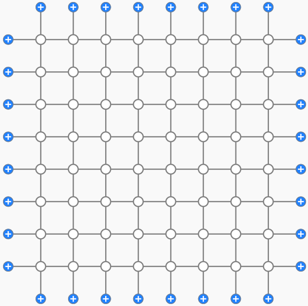
$$\Omega_N^{\oplus} = \{\sigma = (\sigma_i)_{i \in \mathbb{Z}^2} \in \{\pm 1\}^{\mathbb{Z}^2} \ : \ \forall i \notin \mathrm{B}_N, \sigma_i = 1\}$$

- **Hamiltonian:** For $\sigma \in \Omega_N^{\oplus}$,

$$\mathscr{H}_N(\sigma) = -\beta \sum_{\substack{\{i,j\} \cap \mathrm{B}_N \neq \varnothing \\ i \sim j}} \sigma_i \sigma_j$$

- **Gibbs measure:** Probability measure on $\Omega_N^{\oplus}$ s.t.

$$\mu_{N;\beta}^{\oplus}(\sigma) = \frac{1}{\mathscr{Z}_{N;\beta}^{\oplus}} \mathrm{e}^{-\mathscr{H}_N(\sigma)}$$



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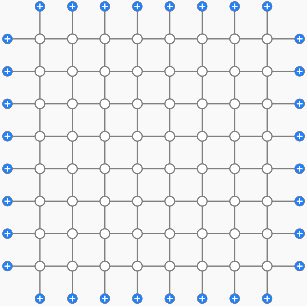

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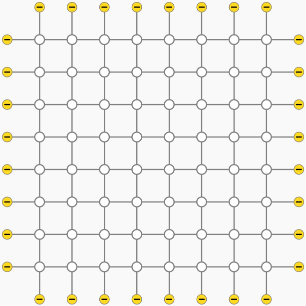
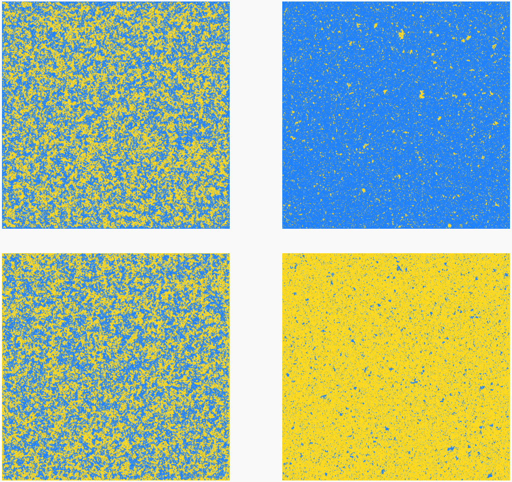
- Extends straightforwardly to other boundary conditions.
  For instance, the ⊖ **boundary condition**: $\mu_{N;\beta}^{-}, \dots$



Phase transition

Let $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$. **Typical configurations** at $\beta \in [0,\infty)$ for $N > N_0(\beta)$:

$\beta < \beta_c$	$\beta > \beta_c$



under $\mu^{+}_{N;\beta}$ (top row)

under $\mu^{-}_{N;\beta}$ (bottom row)

— I. Equilibrium Crystal Shapes —

Equilibrium Crystal Shapes

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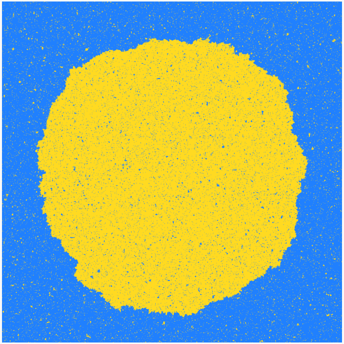

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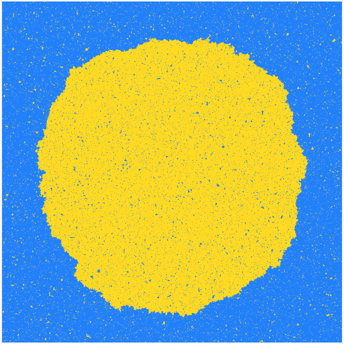
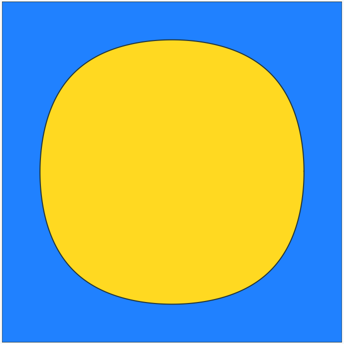
- Typical configurations contain a **unique macroscopic droplet** of ⊖ phase, whose shape becomes deterministic in the continuum limit. Limiting shape is the **Wulff shape**.





Well understood for the planar Ising model since the 1990s:
[Dobrushin, Kotecký, Shlosman '92], [Pfister '91], [Ioffe '94, '95], [Pfister, V. '97], [Ioffe, Schonmann '98], …

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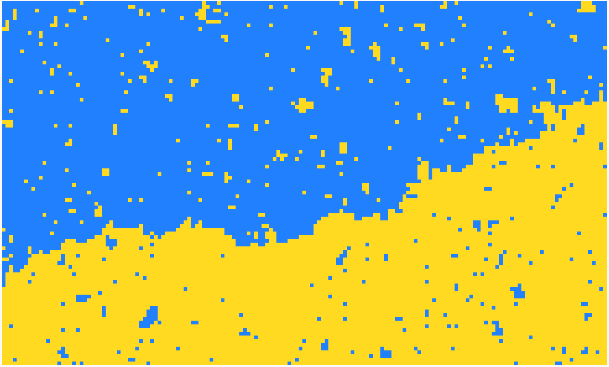
- [Alexander '01], [Uzun, Alexander '03], [Hammond '11, '12] analyzed the **local roughness**, that is, the fluctuations away from its (random) convex hull. These fluctuations live at the $N^{1/3}$ **scale.**

Let us discuss the latter in more detail...

Equilibrium Crystal Shapes: local roughness

Several relevant quantities:



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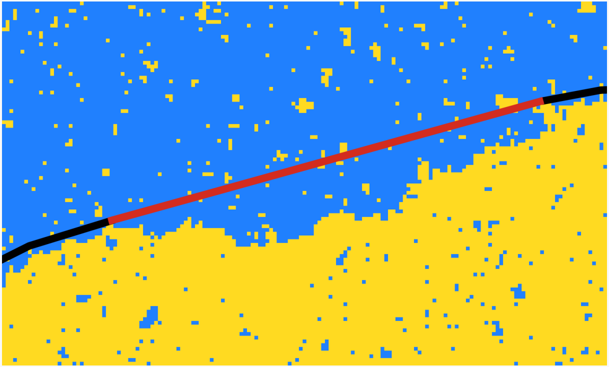

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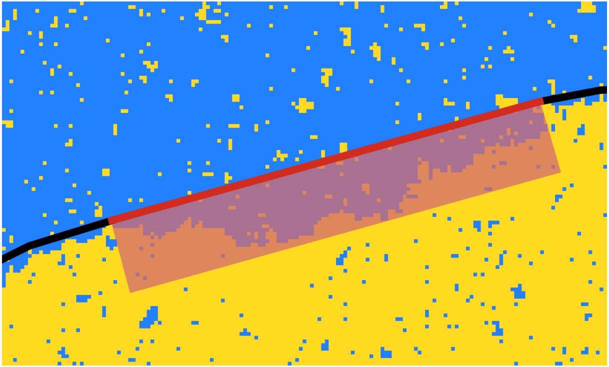

- ▷ the maximal and the average length of an affine piece (facet) of the convex hull
- ▷ the maximal and the average inward deviation of the droplet

- It is **proved** in [Hammond '11, '12] that
  - the maximal length of a facet of the convex hull is $\Theta\left(N^{2/3}(\log N)^{1/3}\right)$
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- It is **conjectured** (see, e.g., [Uzun, Alexander '03]) that
  - the average length of a facet of the convex hull is $\Theta\left(N^{2/3}\right)$
  - the average local roughness is $\Theta\left(N^{1/3}\right)$

Equilibrium Crystal Shapes: heuristics behind this scaling

- The main ingredients are:
  - the **diameter** of the Wulff shape is $\sim N$
  - its **curvature** is uniformly bounded away from 0 and ∞
  - typical **distance between successive points** of the convex hull $= R$
  - **diffusive behavior** of the droplet boundary on small scales

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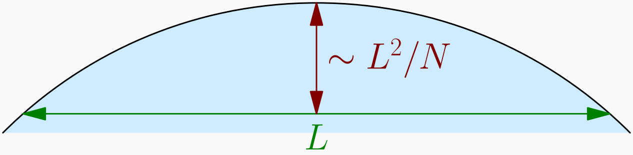

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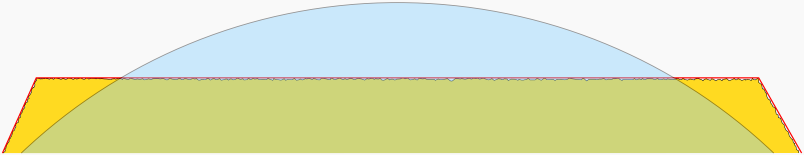

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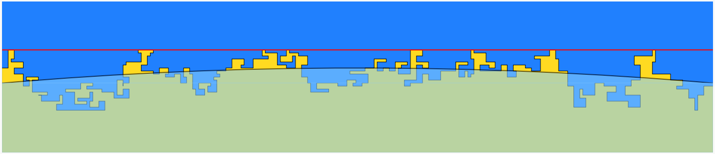

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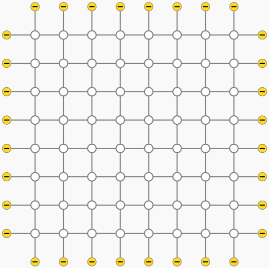
  - **diffusive behavior** of the droplet boundary on small scales
- Compare the inward deviation $\sim \sqrt{R}$ with the curvature-induced deviation R^2/N.
- We conclude that $\sqrt{R} \sim R^2/N$, that is: $R \sim N^{2/3}$

— II. Metastability —

- Let us consider the model with ⊖ boundary condition



but add to the Hamiltonian a **magnetic field term**

$$-h \sum_{i \in B_N} \sigma_i$$

with $\boldsymbol{h} > \boldsymbol{0}$.

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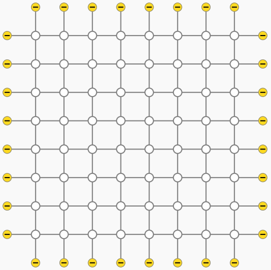

but add to the Hamiltonian a **magnetic field term**

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with $\boldsymbol{h > 0}$.

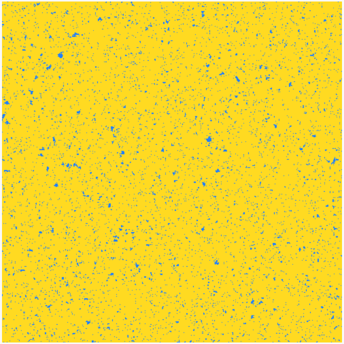
- This induces a **competition between the boundary condition and the magnetic field**:

effect of the boundary condition $\sim N$ effect of the field $\sim hN^2$

competition if $h \sim 1/N$

Metastability: typical configurations

- Let $h = \lambda/N$. [Schonmann and Shlosman 1996] proved: $\exists \lambda_c \in (0, \infty)$ such that



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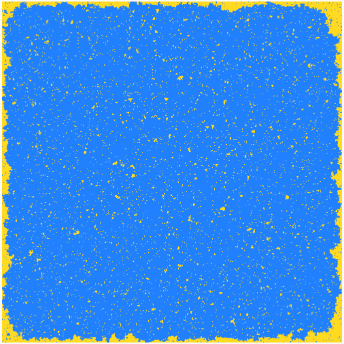

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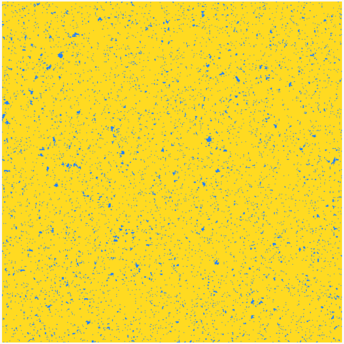

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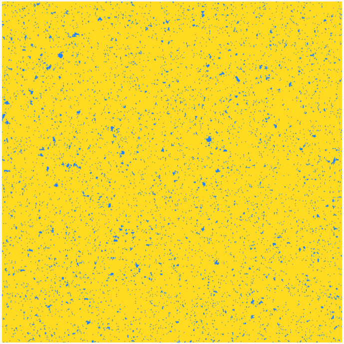

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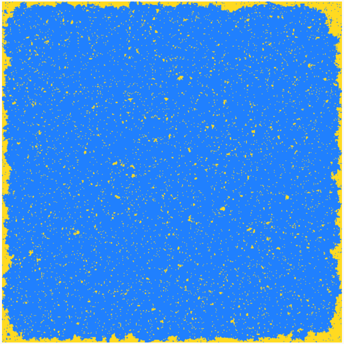

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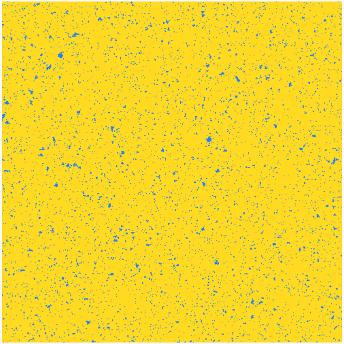

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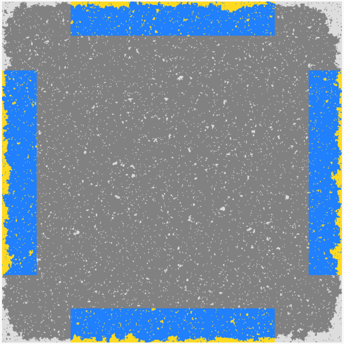

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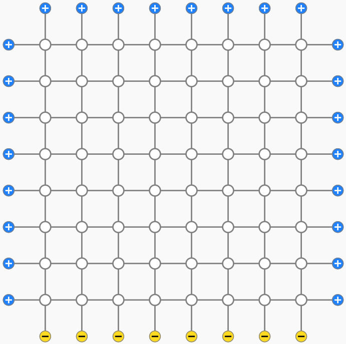
- **Conjecture:** Fluctuations along the walls are of order $N^{1/3}$ and the limiting process (under $(N^{1/3}, N^{2/3})$ scaling) is a Ferrari–Spohn diffusion.

— III. Critical prewetting —

Critical prewetting: the settings

We consider the boundary condition



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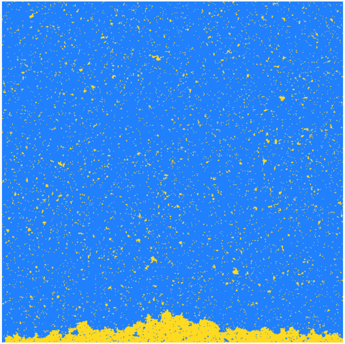

$h = 0$

mesoscopic layer

average width $= \Theta(N^{1/2})$

(scaling limit = Brownian excursion)

[Ioffe, Ott, V., Wachtel '20]

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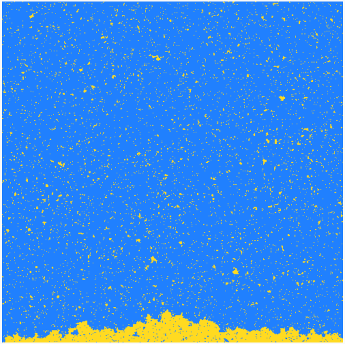

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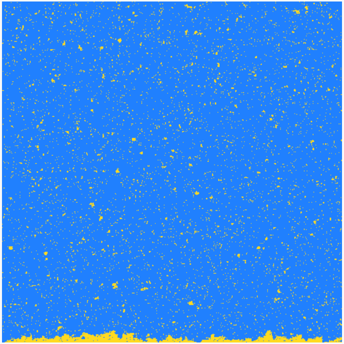
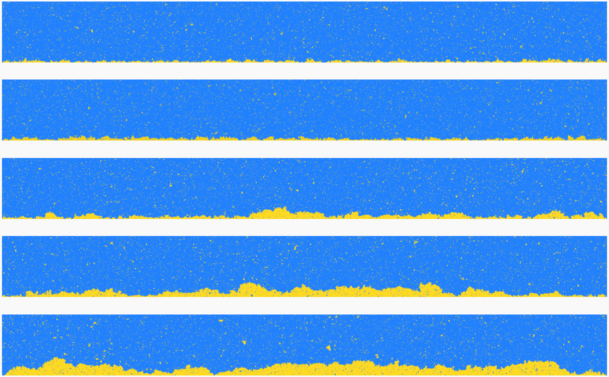


$h > 0$

microscopic layer

average width $= \Theta(1)$

- The width of the layer diverges as h decreases towards 0:



Critical prewetting: layer growth

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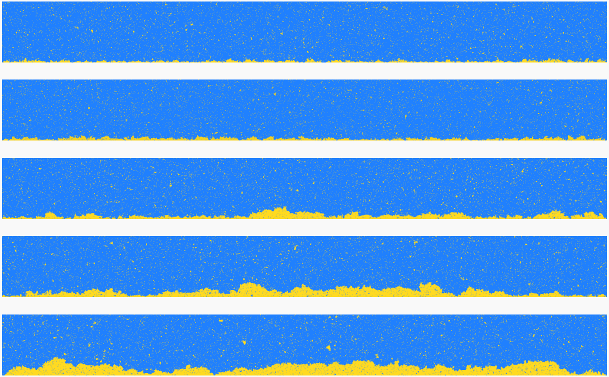


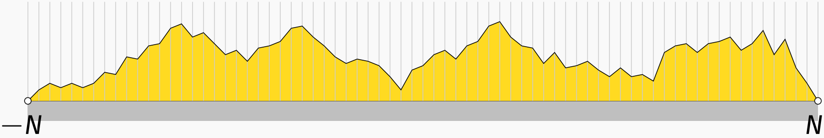
- To get a meaningful scaling limit and mimic the Schonmann–Shlosman setting, we choose $h = h(N)$ to be of the form

$$h = \frac{\lambda}{N}$$

for some $\lambda > 0$.

Critical prewetting: earlier rigorous results

- This type of problem was first studied for **effective models** in
  - [Abraham, Smith 1986] specific integrable model: width $\sim N^{1/3}$, corr. length $\sim N^{2/3}$
  - [Hryniv, V. 2004] general class: width $\sim N^{1/3}$, correlation length $\sim N^{2/3}$
  - [Ioffe, Shlosman, V. 2015] general class: weak convergence to Ferrari–Spohn diffusion



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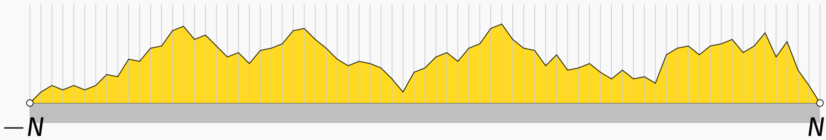

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- Results for the **2d Ising model** were obtained in
  - [V. 2004] width $\sim N^{1/3+o(1)}$
  - [Ganguly, Gheissari 2021] width $\sim N^{1/3}$ (and various other global estimates)

- The relevant **Ferrari–Spohn diffusion** in the present context is the diffusion on $(0, \infty)$ with generator

$$L_\beta = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r^2} + \frac{\varphi_0'}{\varphi_0}\frac{\mathrm{d}}{\mathrm{d}r}$$

and Dirichlet boundary condition at 0.

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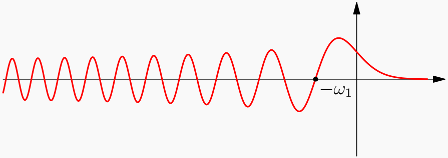
- Above, $\varphi_0(r) = \mathrm{Ai}\big((4\lambda m_\beta^* \sqrt{\chi_\beta})^{1/3}\, r - \omega_1\big)$, where Ai is the **Airy function** and $-\omega_1$ its first zero.



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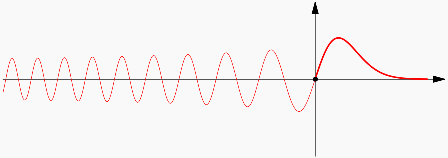

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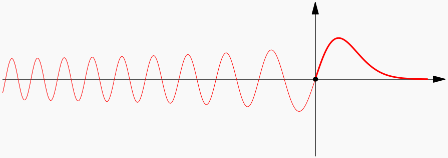

- The quantities appearing above are:
  - m_β^* is the **spontaneous magnetization**
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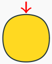

Critical prewetting: the Ferrari–Spohn diffusion

- The relevant **Ferrari–Spohn diffusion** in the present context is the diffusion on $(0, \infty)$ with generator
$$L_\beta = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r^2} + \frac{\varphi_0'}{\varphi_0}\frac{\mathrm{d}}{\mathrm{d}r}$$
and Dirichlet boundary condition at 0.

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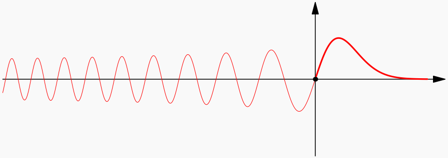

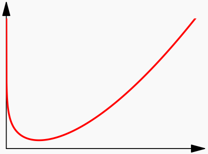

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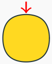
  - χ_β is the **curvature of the Wulff shape** at its apex. 

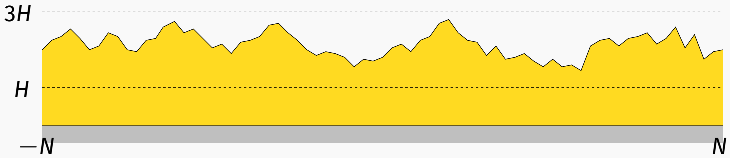
Critical prewetting: main result

- The following is **proved** in [Ioffe, Ott, Shlosman, V. 2021]:
  - Consider the Ising model on $\mathbb{Z}^2$.
  - Fix $\beta > \beta_c$ and $\lambda > 0$.
  - Rescale the interface
    - horizontally by $N^{-2/3}$
    - vertically by $\chi_\beta^{-1/2} \cdot N^{-1/3}$
  - Then, as $N \to \infty$, its distribution converges weakly to that of the trajectories the stationary Ferrari–Spohn diffusion introduced in the previous slide.

Critical prewetting: Heuristic argument

Let us conclude this part with a heuristic explanation of the origin of the $N^{1/3}$ scaling in the effective model.

- Consider a path staying in the tube $[-N, N] \times [H, 3H]$ for some fixed $H > 0$.



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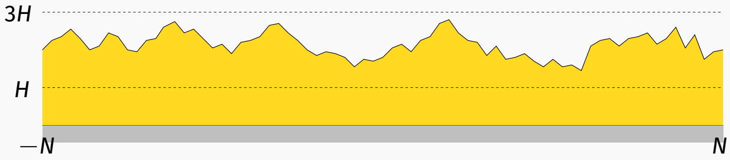

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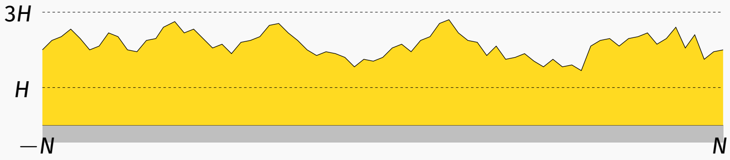

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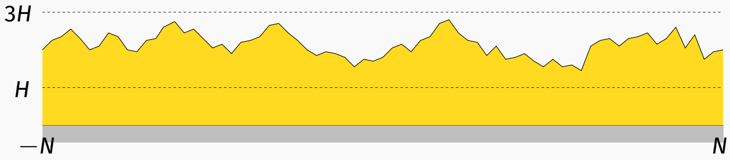

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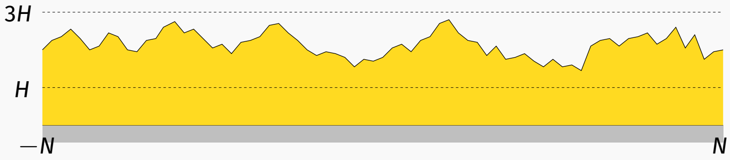

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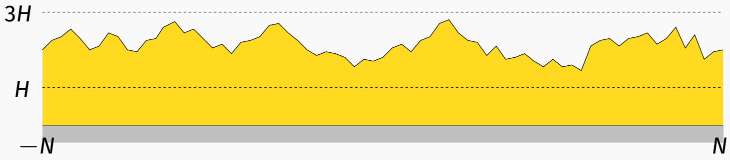

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- This argument can be turned into a rigorous proof (for effective models).

— IV. Inverse correlation length —

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- We are interested in the **truncated 2-point function**

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- It is known that $\nu_{\beta,h} > 0$ if and only if $(\beta, h) \neq (\beta_\mathrm{c}, 0)$.

Inverse correlation length: non-analytic behavior as $h \downarrow 0$

- **Work in progress** [Ott, V.]**:** For all $\beta > \beta_c$, as $h \downarrow 0$,

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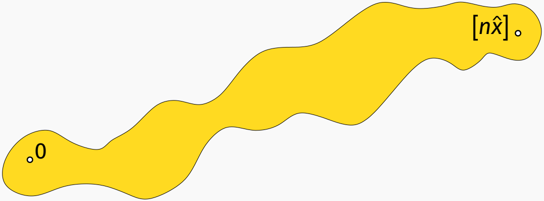
- It is the only case in this talk for which **planarity is expected to be essential**: for a non-planar two-dimensional system (or when $d \geq 3$), it is expected that, for all $\beta > \beta_c$ as $h \downarrow 0$,

$$\nu_{\beta,h}(\hat{x}) = \nu_{\beta,0}(\hat{x}) + \Theta(h).$$

Inverse correlation length: heuristics

- The main contribution to the covariance between the spins at 0 and $[n\hat{x}]$ is due to the presence of a large contour surrounding both vertices.



- For the same reason as in the previous part (prewetting), the typical width of the contour is of order $h^{-1/3}$.
- This leads to a magnetic-field cost of order $h \cdot h^{-1/3} n \sim h^{2/3} n$.
- This picture can be made precise, for instance, in the random-current representation of the model.

Thank you for your attention!