Free Probability and Free Cumulants

Roland Speicher
Universität des Saarlandes
Saarbrücken

Free Probability and Free Cumulants
Something Old and Something New

Roland Speicher
Universität des Saarlandes
Saarbrücken

Section 1

Freeness

Some history

1985 Voiculescu introduces "freeness" in the context of isomorphism problem of free group factors

Some history

1985 Voiculescu introduces "freeness" in the context of isomorphism problem of free group factors

1991 Voiculescu discovers relation with random matrices (which leads, among others, to deep results on free group factors)

Some history

1985 Voiculescu introduces "freeness" in the context of isomorphism problem of free group factors

1991 Voiculescu discovers relation with random matrices (which leads, among others, to deep results on free group factors)

1994 Speicher and Nica develop a combinatorial theory of freeness, based on the notion of "free cumulants"

Some history

1985 Voiculescu introduces "freeness" in the context of isomorphism problem of free group factors

1991 Voiculescu discovers relation with random matrices (which leads, among others, to deep results on free group factors)

1994 Speicher and Nica develop a combinatorial theory of freeness, based on the notion of “free cumulants”

later ... many new results on operator algebras, eigenvalue distribution of random matrices, and much more ....

Definition of freeness

Definition

- Let $(\mathcal{A}, \varphi)$ be **non-commutative probability space**, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional (i.e., $\varphi(1) = 1$)

Definition of freeness

Definition

- Let $(\mathcal{A}, \varphi)$ be **non-commutative probability space**, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional (i.e., $\varphi(1) = 1$)
- Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are **free** or **freely independent**, if $\varphi(a_1 \cdots a_n) = 0$ whenever
  - $a_i \in \mathcal{A}_{j(i)}, \qquad j(i) \in I \quad \forall i,$
  - $j(1) \neq j(2) \neq \cdots \neq j(n)$
  - $\varphi(a_i) = 0 \quad \forall i$

Definition of freeness

Definition

- Let $(\mathcal{A}, \varphi)$ be **non-commutative probability space**, i.e., $\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is unital linear functional (i.e., $\varphi(1) = 1$)
- Unital subalgebras $\mathcal{A}_i$ $(i \in I)$ are **free** or **freely independent**, if $\varphi(a_1 \cdots a_n) = 0$ whenever
  - $a_i \in \mathcal{A}_{j(i)}, \qquad j(i) \in I \quad \forall i,$
  - $j(1) \neq j(2) \neq \cdots \neq j(n)$
  - $\varphi(a_i) = 0 \quad \forall i$
- Random variables $x_1, \dots, x_n \in \mathcal{A}$ are free, if their generated unital subalgebras $\mathcal{A}_i := \text{algebra}(1, x_i)$ are so.

What is freeness?

Remark

Freeness between x and y is an infinite set of equations relating various moments in x and y:

$$\varphi\Big(p_1(x)q_1(y)p_2(x)q_2(y)\cdots\Big) = 0$$

What is freeness?

Remark

Freeness between x and y is an infinite set of equations relating various moments in x and y:

$$\varphi\Big(p_1(x)q_1(y)p_2(x)q_2(y)\cdots\Big) = 0$$

Basic observation: freeness between x and y is actually a **rule for calculating mixed moments** in x and y from the moments of x and the moments of y:

$$\varphi\Big(x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots\Big) = \text{polynomial}\big(\varphi(x^i), \varphi(y^j)\big)$$

Example

If x and y are free, then we have

$$\varphi(x^m y^n) = \varphi(x^m) \cdot \varphi(y^n)$$

$$\varphi(x^{m_1} y^n x^{m_2}) = \varphi(x^{m_1+m_2}) \cdot \varphi(y^n)$$

Example

If x and y are free, then we have

$$\varphi(x^m y^n) = \varphi(x^m) \cdot \varphi(y^n)$$

$$\varphi(x^{m_1} y^n x^{m_2}) = \varphi(x^{m_1+m_2}) \cdot \varphi(y^n)$$

but also

$$\varphi(xyxy) = \varphi(x^2) \cdot \varphi(y)^2 + \varphi(x)^2 \cdot \varphi(y^2) - \varphi(x)^2 \cdot \varphi(y)^2$$

Example

If x and y are free, then we have

$$\varphi(x^m y^n) = \varphi(x^m) \cdot \varphi(y^n)$$

$$\varphi(x^{m_1} y^n x^{m_2}) = \varphi(x^{m_1+m_2}) \cdot \varphi(y^n)$$

but also

$$\varphi(xyxy) = \varphi(x^2) \cdot \varphi(y)^2 + \varphi(x)^2 \cdot \varphi(y^2) - \varphi(x)^2 \cdot \varphi(y)^2$$

Remark

Free independence is a rule for calculating mixed moments, analogous to the concept of independence for random variables.
Note: free independence is a different rule from classical independence; free independence occurs typically for **non-commuting random variables**, like operators on Hilbert spaces or (random) matrices.

Where does freeness show up?

Important occurrences

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$

Where does freeness show up?

Important occurrences

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$
- creation and annihilation operators on full Fock spaces

Where does freeness show up?

Important occurrences

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$
- creation and annihilation operators on full Fock spaces
- for many classes of random matrices

Where does free probability show up?

Important occurrences

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$
- creation and annihilation operators on full Fock spaces
- for many classes of random matrices
- black holes, tensor networks, fluctuations of Q-SSEP, eigenstate thermalization hypothesis, etc ...

Fundamental questions: why and what???

Important occurrences

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$
- creation and annihilation operators on full Fock spaces
- for many classes of random matrices
- black holes, tensor networks, fluctuations of Q-SSEP, eigenstate thermalization hypothesis, etc ...

Fundamental questions

- Why should this strange definition of freeness be something special – it looks so arbitrary?

Fundamental questions: why and what???

Important occurrences

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$
- creation and annihilation operators on full Fock spaces
- for many classes of random matrices
- black holes, tensor networks, fluctuations of Q-SSEP, eigenstate thermalization hypothesis, etc ...

Fundamental questions

- Why should this strange definition of freeness be something special – it looks so arbitrary?
- How can we understand those rules for mixed moments in a systematic way?

Fundamental questions: why and what???

Important occurrences

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$
- creation and annihilation operators on full Fock spaces
- for many classes of random matrices
- black holes, tensor networks, fluctuations of Q-SSEP, eigenstate thermalization hypothesis, etc ...

Fundamental questions

- Why should this strange definition of freeness be something special – it looks so arbitrary? —> **universal construction**
- How can we understand those rules for mixed moments in a systematic way?

Fundamental questions: why and what???

Important occurrences

- generators of the free group in the corresponding free group von Neumann algebras $L(\mathbb{F}_n)$
- creation and annihilation operators on full Fock spaces
- for many classes of random matrices
- black holes, tensor networks, fluctuations of Q-SSEP, eigenstate thermalization hypothesis, etc ...

Fundamental questions

- Why should this strange definition of freeness be something special – it looks so arbitrary? —> **universal construction**
- How can we understand those rules for mixed moments in a systematic way? —> **free cumulants**

Section 2

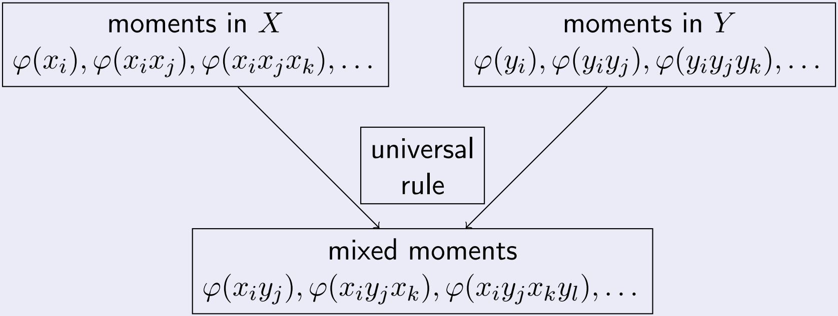
What is freeness?

A universal concept!

Universal products

How to get mixed moments in a universal way?

Input: moments of $\{x_i\}$ and moments of $\{y_i\}$
Output: mixed moments in $\{x_i, y_j\}$



Speicher 1997, Ben Ghorbal and Schürmann 2002

There are exactly two universal rules on states, which are associative, commutative and unital:

- the tensor product
- the reduced free product of Voiculescu

Speicher 1997, Ben Ghorbal and Schürmann 2002

There are exactly two universal rules on states, which are associative, commutative and unital:

- the tensor product
- the reduced free product of Voiculescu

So by general principles, there are only two possibilities for $\varphi(xyxy)$:

- $\varphi(xx)\varphi(yy)$
- $\varphi(xx)\varphi(y)\varphi(y) + \varphi(x)\varphi(x)\varphi(yy) - \varphi(x)\varphi(y)\varphi(x)\varphi(y)$

and then everything else is determined!

Speicher 1997, Ben Ghorbal and Schürmann 2002

There are exactly two universal rules on states, which are associative, commutative and unital:

- the tensor product —> **classical independence**
- the reduced free product of Voiculescu —> **free independence**

So by general principles, there are only two possibilities for $\varphi(xyxy)$:

- $\varphi(xx)\varphi(yy)$
- $\varphi(xx)\varphi(y)\varphi(y) + \varphi(x)\varphi(x)\varphi(yy) - \varphi(x)\varphi(y)\varphi(x)\varphi(y)$

and then everything else is determined!

Speicher 1997, Ben Ghorbal and Schürmann 2002

There are exactly two universal rules on states, which are associative, commutative and unital:

- the tensor product —> **classical independence**
- the reduced free product of Voiculescu —> **free independence**

So by general principles, there are only two possibilities for $\varphi(xyxy)$:

- $\varphi(xx)\varphi(yy)$
- $\varphi(xx)\varphi(y)\varphi(y) + \varphi(x)\varphi(x)\varphi(yy) - \varphi(x)\varphi(y)\varphi(x)\varphi(y)$

and then everything else is determined!

more possibilities

without unital: also boolean product
without commutative: also monotone product

Section 3

How do we understand freeness conceptually: free cumulants

Understanding the freeness rule: the idea of cumulants

- write moments in terms of other quantities, which we call **free cumulants**

Understanding the freeness rule: the idea of cumulants

- write moments in terms of other quantities, which we call **free cumulants**
- freeness is much easier to describe on the level of free cumulants: **vanishing of mixed cumulants**

Understanding the freeness rule: the idea of cumulants

- write moments in terms of other quantities, which we call **free cumulants**
- freeness is much easier to describe on the level of free cumulants: **vanishing of mixed cumulants**
- relation between moments and cumulants is given by summing over **non-crossing or planar partitions**

Non-crossing partitions

Definition

A **partition** of $\{1,\dots,n\}$ is a decomposition $\pi = \{V_1,\dots,V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq j), \qquad \bigcup_i V_i = \{1,\dots,n\}$$

Non-crossing partitions

Definition

A **partition** of $\{1, \ldots, n\}$ is a decomposition $\pi = \{V_1, \ldots, V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq j), \qquad \bigcup_i V_i = \{1, \ldots, n\}$$

The V_i are the **blocks** of $\pi \in \mathcal{P}(n)$.

Non-crossing partitions

Definition

A **partition** of $\{1, \dots, n\}$ is a decomposition $\pi = \{V_1, \dots, V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq j), \qquad \bigcup_i V_i = \{1, \dots, n\}$$

The V_i are the **blocks** of $\pi \in \mathcal{P}(n)$.
π is **non-crossing** if we do not have $p_1 < q_1 < p_2 < q_2$ such that p_1, p_2 are in same block, q_1, q_2 are in same block, but those two blocks are different.

Non-crossing partitions

Definition

A **partition** of $\{1, \ldots, n\}$ is a decomposition $\pi = \{V_1, \ldots, V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq j), \qquad \bigcup_i V_i = \{1, \ldots, n\}$$

The V_i are the **blocks** of $\pi \in \mathcal{P}(n)$.
π is **non-crossing** if we do not have $p_1 < q_1 < p_2 < q_2$ such that p_1, p_2 are in same block, q_1, q_2 are in same block, but those two blocks are different.

$$\mathbf{NC(n)} := \{\text{non-crossing partitions of } \{1,\ldots,n\}\}$$

Non-crossing partitions

Definition

A **partition** of $\{1, \ldots, n\}$ is a decomposition $\pi = \{V_1, \ldots, V_r\}$ with

$$V_i \neq \emptyset, \qquad V_i \cap V_j = \emptyset \quad (i \neq j), \qquad \bigcup_i V_i = \{1, \ldots, n\}$$

The V_i are the **blocks** of $\pi \in \mathcal{P}(n)$.
π is **non-crossing** if we do not have $p_1 < q_1 < p_2 < q_2$ such that p_1, p_2 are in same block, q_1, q_2 are in same block, but those two blocks are different.

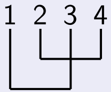
$$\mathbf{NC(n)} := \{\text{non-crossing partitions of } \{1,\ldots,n\}\}$$

Remark

$NC(n)$ is actually a lattice with refinement order.

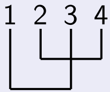
the only crossing partition for $n = 4$

$$\pi = \{(1,3),(2,4)\}$$

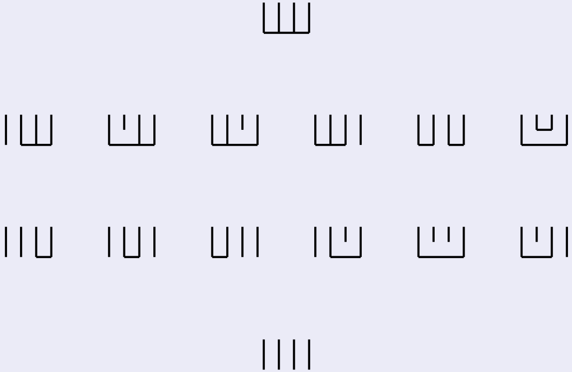


the only crossing partition for $n = 4$

$$\pi = \{(1,3),(2,4)\}$$



$NC(4)$: the 14 non-crossing partitions for $n = 4$



Moments and cumulants

Definition

For unital linear functional

$$\varphi : \mathcal{A} \to \mathbb{C}$$

we define **cumulant functionals** κ_n (for all $n \geq 1$)

$$\kappa_n : \mathcal{A}^n \to \mathbb{C}$$

as multi-linear functionals by moment-cumulant relation

$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi[a_1, \dots, a_n]$$

Moments and cumulants

Definition

For unital linear functional

$$\varphi : \mathcal{A} \to \mathbb{C}$$

we define **cumulant functionals** κ_n (for all $n \geq 1$)

$$\kappa_n : \mathcal{A}^n \to \mathbb{C}$$

as multi-linear functionals by moment-cumulant relation

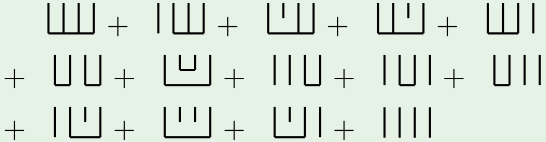
$$\varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi[a_1, \dots, a_n]$$

Remark

Note: classical cumulants are defined by a similar formula, where only $NC(n)$ is replaced by $\mathcal{P}(n)$

Example ($n = 4$)

$\varphi(a_1 a_2 a_3 a_4) =$



$$
\begin{aligned}
= \quad & \kappa_4(a_1, a_2, a_3, a_4) + \kappa_1(a_1)\kappa_3(a_2, a_3, a_4) \\
& + \kappa_1(a_2)\kappa_3(a_1, a_3, a_4) + \kappa_1(a_3)\kappa_3(a_1, a_2, a_4) \\
& + \kappa_3(a_1, a_2, a_3)\kappa_1(a_4) + \kappa_2(a_1, a_2)\kappa_2(a_3, a_4) \\
& + \kappa_2(a_1, a_4)\kappa_2(a_2, a_3) + \kappa_1(a_1)\kappa_1(a_2)\kappa_2(a_3, a_4) \\
& + \kappa_1(a_1)\kappa_2(a_2, a_3)\kappa_1(a_4) + \kappa_2(a_1, a_2)\kappa_1(a_3)\kappa_1(a_4) \\
& + \kappa_1(a_1)\kappa_2(a_2, a_4)\kappa_1(a_3) + \kappa_2(a_1, a_4)\kappa_1(a_2)\kappa_1(a_3) \\
& + \kappa_2(a_1, a_3)\kappa_1(a_2)\kappa_1(a_4) + \kappa_1(a_1)\kappa_1(a_2)\kappa_1(a_3)\kappa_1(a_4)
\end{aligned}
$$

Freeness $\hat{=}$ vanishing of mixed cumulants

Theorem (Speicher 1994)

The fact that $x_1, \dots, x_m$ *are free is equivalent to the fact that*

$$\kappa_n(x_{i(1)}, \dots, x_{i(n)}) = 0$$

whenever

- $1 \leq i(1), \dots, i(n) \leq m$
- *there are* p, q *such that* $i(p) \neq i(q)$ *(in particular,* $n \geq 2$*)*

Freeness $\hat{=}$ vanishing of mixed cumulants

Theorem (Speicher 1994)

The fact that $x_1, \dots, x_m$ *are free is equivalent to the fact that*

$$\kappa_n(x_{i(1)}, \dots, x_{i(n)}) = 0$$

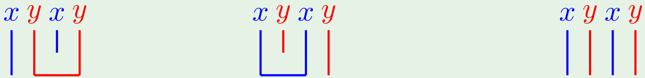
whenever

- $1 \leq i(1), \dots, i(n) \leq m$
- *there are* p, q *such that* $i(p) \neq i(q)$ *(in particular,* $n \geq 2$*)*

Example

If x and y are free then: $\varphi(xyxy) =$

$$\kappa_1(x)\kappa_1(x)\kappa_2(y,y) + \kappa_2(x,x)\kappa_1(y)\kappa_1(y) + \kappa_1(x)\kappa_1(y)\kappa_1(x)\kappa_1(y)$$



Sum of free variables: description via R-transform

Definition

Consider a random variable $x \in \mathcal{A}$. We define its **Cauchy transform** $G = G_x$ and its **$\mathcal{R}$-transform $\mathcal{R} = \mathcal{R}_x$** by

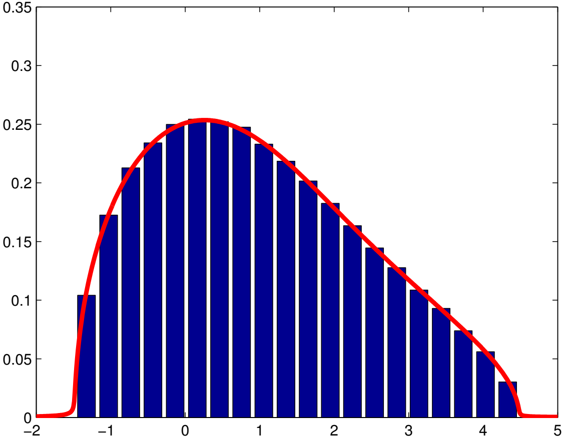
$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\varphi(x^n)}{z^{n+1}}, \qquad \mathcal{R}(z) = \sum_{n=1}^{\infty} \kappa_n(x, \dots, x) z^{n-1}$$

Theorem (Voiculescu 1986, Speicher 1994)

Then we have

- $\frac{1}{G(z)} + \mathcal{R}(G(z)) = z$
- $\mathcal{R}_{x+y}(z) = \mathcal{R}_x(z) + \mathcal{R}_y(z)$ *if x and y are free*

Eigenvalues of the sum of independent Gaussian and Wishart 3000×3000 random matrices



Product of free variables: description via S-transform

Theorem (Voiculescu 1987; Haagerup 1997; Nica, Speicher 1997)

Put

$$M_x(z) := \sum_{m=1}^{\infty} \varphi(x^m) z^m$$

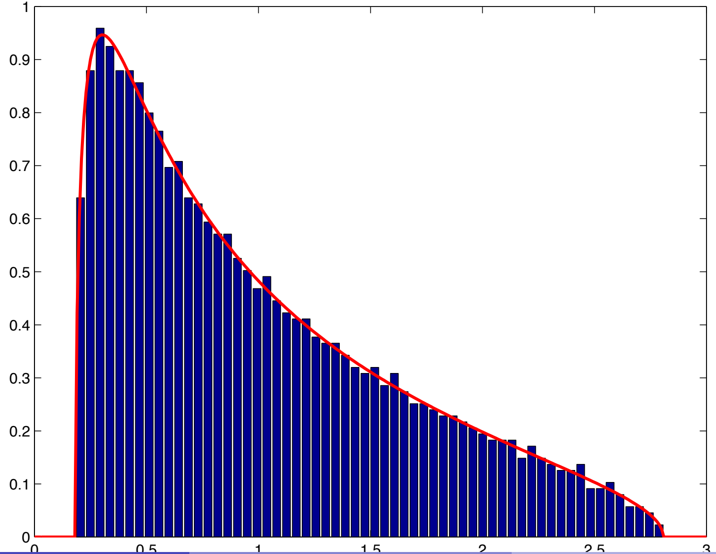
and define

$$S_x(z) := \frac{1+z}{z} M_x^{<-1>}(z) \qquad \textbf{S-transform of } x$$

Then: If x and y are free, we have

$$S_{xy}(z) = S_x(z) \cdot S_y(z).$$

Eigenvalues of the product of two independent Wishart 2000×2000 random matrices



Section 4

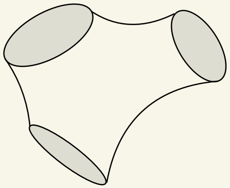
And now something new: Generalization to higher order and arbitrary genus

Topological numbers, indexed by genus g and numbers $l_1, \dots, l_n$ of points on n boundaries

$$\alpha_{g;l_1,\dots,l_n}$$

Topological numbers, indexed by genus g and numbers $l_1, \ldots, l_n$ of points on n boundaries

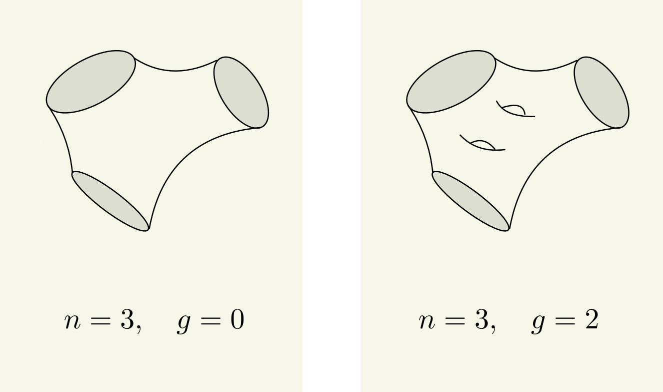
$$\alpha_{g;l_1,\ldots,l_n}$$



$n = 3, \quad g = 0$

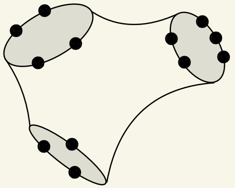
Topological numbers, indexed by genus g and numbers $l_1, \dots, l_n$ of points on n boundaries

$$\alpha_{g;l_1,\dots,l_n}$$



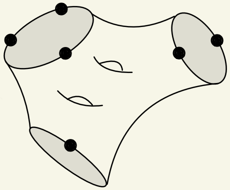
Topological numbers, indexed by genus g and numbers $l_1, \dots, l_n$ of points on n boundaries

$$\alpha_{g;l_1,\dots,l_n}$$



$n = 3, \quad g = 0$

$\alpha_{0;4,5,3}$

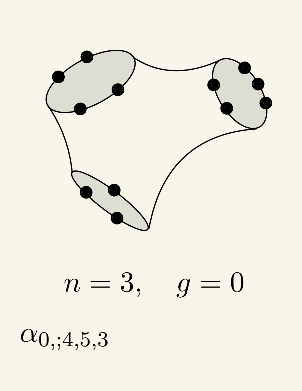


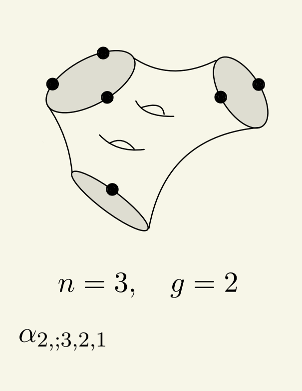
$n = 3, \quad g = 2$

$\alpha_{2;3,2,1}$

Topological linear functionals, indexed by genus g and numbers $l_1, \dots, l_n$ of points on n boundaries

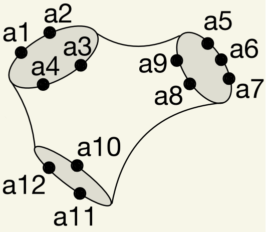
$\alpha_{g;l_1,\dots,l_n}(a_1, \dots, a_{l_1}, a_{l_1+1}, \dots, a_{l_1+\dots+l_n})$ $\quad a_i \in \mathcal{A}$ some algebra $\mathcal{A}$





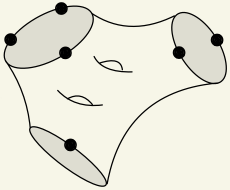
Topological linear functionals, indexed by genus g and numbers $l_1, \ldots, l_n$ of points on n boundaries

$\alpha_{g;l_1,\ldots,l_n}(a_1, \ldots, a_{l_1}, a_{l_1+1}, \ldots, a_{l_1+\cdots+l_n})$ $a_i \in \mathcal{A}$ some algebra $\mathcal{A}$



$n = 3, \quad g = 0$

$\alpha_{0,;4,5,3}[a_1, \cdots, a_{12}]$

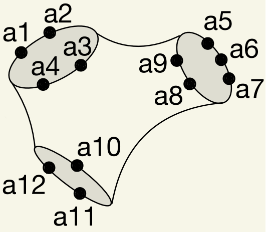


$n = 3, \quad g = 2$

$\alpha_{2,;3,2,1}$

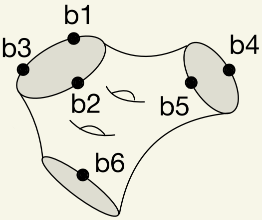
Topological linear functionals, indexed by genus g and numbers $l_1, \dots, l_n$ of points on n boundaries

$\alpha_{g;l_1,\dots,l_n}(a_1, \dots, a_{l_1}, a_{l_1+1}, \dots, a_{l_1+\dots+l_n})$ $a_i \in \mathcal{A}$ some algebra $\mathcal{A}$



$n = 3, \quad g = 0$

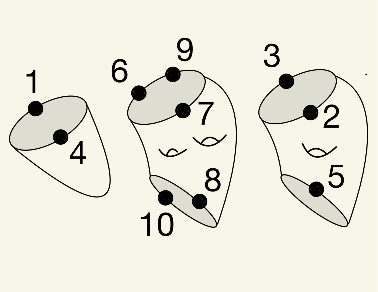
$\alpha_{0,;4,5,3}[a_1, \cdots, a_{12}]$



$n = 3, \quad g = 2$

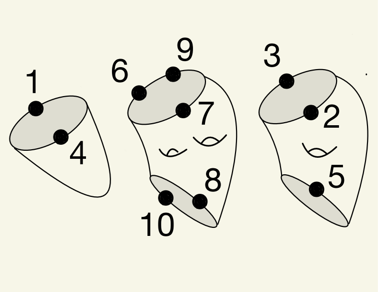
$\alpha_{2,;3,2,1}[b_1, b_2, \cdots, b_6]$

Encoding in surfaced permutations $\mathbb{PS}$ (or partitioned permutations with genus)



surfaced permutation: $(\mathcal{V}, \pi, g)$

Encoding in surfaced permutations $\mathbb{PS}$ (or partitioned permutations with genus)

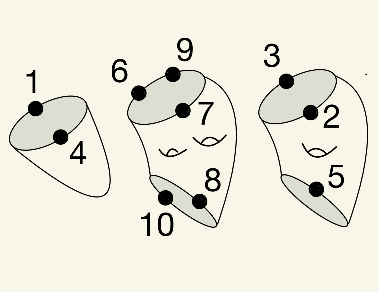


surfaced permutation: $(\mathcal{V}, \pi, g)$

- π is the permutation with cycles according to all the boundaries

$$(1,4)(2,3)(5)(6,9,7)(8,10)$$

Encoding in surfaced permutations $\mathbb{PS}$ (or partitioned permutations with genus)

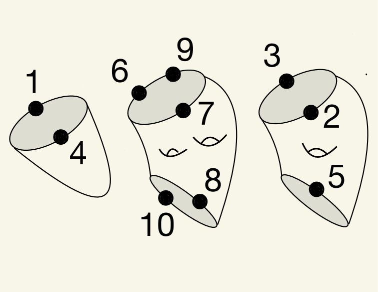


surfaced permutation: $(\mathcal{V}, \pi, g)$

- π is the permutation with cycles according to all the boundaries

$$(1,4) \quad (2,3)(5) \quad (6,9,7)(8,10)$$

Encoding in surfaced permutations $\mathbb{PS}$ (or partitioned permutations with genus)

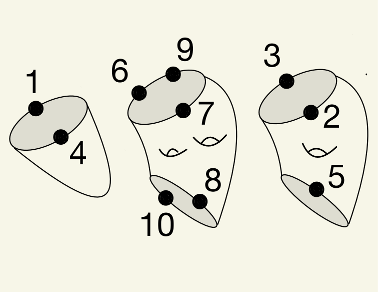


surfaced permutation: $(\mathcal{V}, \pi, g)$

- π is the permutation with cycles according to all the boundaries
- $\mathcal{V}$ is the partition of all cycles of π according to the connected components

$$\{(1,4)\} \quad \{(2,3)(5)\} \quad \{(6,9,7)(8,10)\}$$

Encoding in surfaced permutations $\mathbb{PS}$ (or partitioned permutations with genus)



surfaced permutation: $(\mathcal{V}, \pi, g)$

- π is the permutation with cycles according to all the boundaries
- $\mathcal{V}$ is the partition of all cycles of π according to the connected components
- g is the vector of genera of the components

$$\{(1,4)\} \quad \{(2,3)(5)\} \quad \{(6,9,7)(8,10)\} \qquad g = (0,2,1)$$

Multiplicative functions on surfaced permutations

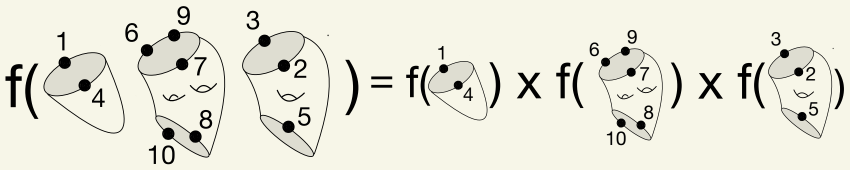
Definition

A function $f\colon \mathbb{PS} \to \mathbb{C}$ is called multiplicative if

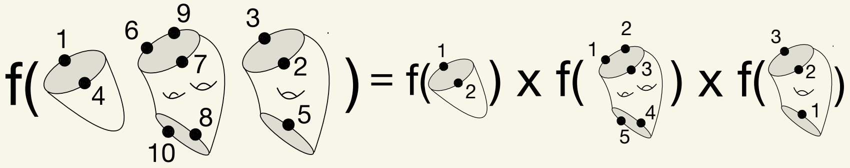
$$f(\mathcal{V}, \pi, g) = \prod_{B \in \mathcal{V}} f(B, \pi|_B, g|_B)$$

and if it is invariant under conjugation of π.

Example for multiplicativity

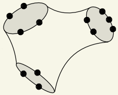


Example for multiplicativity



Our examples for such multiplicative functions: correlations coming from random matrices

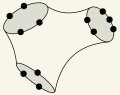
Let $A = (a_{ij})_{i,j=1}^N$ be random matrix. Let k_r be classical cumulants:



- $l_1 = 4$,
  $l_2 = 5$,
  $l_3 = 3$
- $\gamma_1 = (1, 2, 3, 4)$
- $\gamma_2 = (5, 6, 7, 8, 9)$
- $\gamma_3 = (10, 11, 12)$

Our examples for such multiplicative functions: correlations coming from random matrices

Let $A = (a_{ij})_{i,j=1}^N$ be random matrix. Let k_r be classical cumulants:



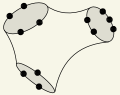
- $l_1 = 4$,
  $l_2 = 5$,
  $l_3 = 3$
- $\gamma_1 = (1, 2, 3, 4)$
- $\gamma_2 = (5, 6, 7, 8, 9)$
- $\gamma_3 = (10, 11, 12)$

- traces along cycles

$$k_n(\mathrm{Tr}(A^{l_1}), \dots, \mathrm{Tr}(A^{l_n})) = \sum_g N^{2-n-2g} \varphi_{g;l_1,\dots,l_n}$$

Our examples for such multiplicative functions: correlations coming from random matrices

Let $A = (a_{ij})_{i,j=1}^N$ be random matrix. Let k_r be classical cumulants:



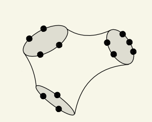
- $l_1 = 4$,
  $l_2 = 5$,
  $l_3 = 3$
- $\gamma_1 = (1, 2, 3, 4)$
- $\gamma_2 = (5, 6, 7, 8, 9)$
- $\gamma_3 = (10, 11, 12)$

- traces along cycles

$$k_3(\mathrm{Tr}(A^4), \mathrm{Tr}(A^5), \mathrm{Tr}(A^3)) = \sum_g N^{-1-2g} \varphi_{g;4,5,3}$$

Our examples for such multiplicative functions: correlations coming from random matrices

Let $A = (a_{ij})_{i,j=1}^N$ be random matrix. Let k_r be classical cumulants:



- $l_1 = 4,$
  $l_2 = 5,$
  $l_3 = 3$
- $\gamma_1 = (1, 2, 3, 4)$
- $\gamma_2 = (5, 6, 7, 8, 9)$
- $\gamma_3 = (10, 11, 12)$

- traces along cycles

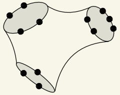
$$k_3(\mathrm{Tr}(A^4), \mathrm{Tr}(A^5), \mathrm{Tr}(A^3)) = \sum_g N^{-1-2g} \varphi_{g;4,5,3}$$

- product along cycles of matrix entries

$$k_n(\prod a_{j\gamma_1(j)}, \dots, \prod a_{j\gamma_n(j)}) = \sum_g N^{-d+2-n-2g} \kappa_{g,l_1,\dots,l_n}$$

Our examples for such multiplicative functions: correlations coming from random matrices

Let $A = (a_{ij})_{i,j=1}^N$ be random matrix. Let k_r be classical cumulants:



- $l_1 = 4$,
  $l_2 = 5$,
  $l_3 = 3$
- $\gamma_1 = (1, 2, 3, 4)$
- $\gamma_2 = (5, 6, 7, 8, 9)$
- $\gamma_3 = (10, 11, 12)$

- traces along cycles

$$k_3(\text{Tr}(A^4), \text{Tr}(A^5), \text{Tr}(A^3)) = \sum_g N^{-1-2g} \varphi_{g;4,5,3}$$

- product along cycles of matrix entries

$$k_3(a_{12}a_{23}a_{34}a_{41}, a_{56}a_{67}a_{78}a_{89}a_{95}, \dots) = \sum_g N^{-13-2g} \kappa_{g,4,5,3}$$

Moment-cumulant formulas

- "moments" are given by collection of numbers $(\varphi_{g;l_1,\dots,l_n})_{n;l_1,\dots,l_n;g}$
- "cumulants" are given by collection of numbers $(\kappa_{g;l_1,\dots,l_n})_{n;l_1,\dots,l_n;g}$

Moment-cumulant formulas

- "moments" are given by collection of numbers $(\varphi_{g;l_1,\dots,l_n})_{n;l_1,\dots,l_n;g}$
- "cumulants" are given by collection of numbers $(\kappa_{g;l_1,\dots,l_n})_{n;l_1,\dots,l_n;g}$
- moments and cumulants determine each other by combinatorial formulas
  - "moment-cumulant formula"

Moment-cumulant formulas

- “moments” are given by collection of numbers $(\varphi_{g;l_1,\dots,l_n})_{n;l_1,\dots,l_n;g}$
- “cumulants” are given by collection of numbers $(\kappa_{g;l_1,\dots,l_n})_{n;l_1,\dots,l_n;g}$
- moments and cumulants determine each other by combinatorial formulas
  - “moment-cumulant formula”
  - given by “convolution of multiplicative functions on surfaced permutations”

Moment-cumulant formulas

- "moments" are given by collection of numbers $(\varphi_{g;l_1,\dots,l_n})_{n;l_1,\dots,l_n;g}$
- "cumulants" are given by collection of numbers $(\kappa_{g;l_1,\dots,l_n})_{n;l_1,\dots,l_n;g}$
- moments and cumulants determine each other by combinatorial formulas
  - "moment-cumulant formula"
  - given by "convolution of multiplicative functions on surfaced permutations"
  - corresponding to "product of surfaced permutations"

skip details

Product of surfaced permutations

Definition

Let $(\mathcal{V}, \pi, g)$ and $(\mathcal{W}, \sigma, h)$ be surfaced permutations. We define their product to be

$$(\mathcal{V}, \pi, g) \odot (\mathcal{W}, \sigma, h) = (\qquad , \quad , \ )$$

Product of surfaced permutations

Definition

Let $(\mathcal{V}, \pi, g)$ and $(\mathcal{W}, \sigma, h)$ be surfaced permutations. We define their product to be

$$(\mathcal{V}, \pi, g) \odot (\mathcal{W}, \sigma, h) = (\mathcal{V} \vee \mathcal{W}, \quad , \quad)$$

where

- $\mathcal{V} \vee \mathcal{W}$ is the join of the two set partitions $\mathcal{V}$ and $\mathcal{W}$

Product of surfaced permutations

Definition

Let $(\mathcal{V}, \pi, g)$ and $(\mathcal{W}, \sigma, h)$ be surfaced permutations. We define their product to be

$$(\mathcal{V}, \pi, g) \odot (\mathcal{W}, \sigma, h) = (\mathcal{V} \vee \mathcal{W}, \pi\sigma, \quad)$$

where

- $\mathcal{V} \vee \mathcal{W}$ is the join of the two set partitions $\mathcal{V}$ and $\mathcal{W}$
- $\pi\sigma$ is the product of the two permutations π and σ

Product of surfaced permutations

Definition

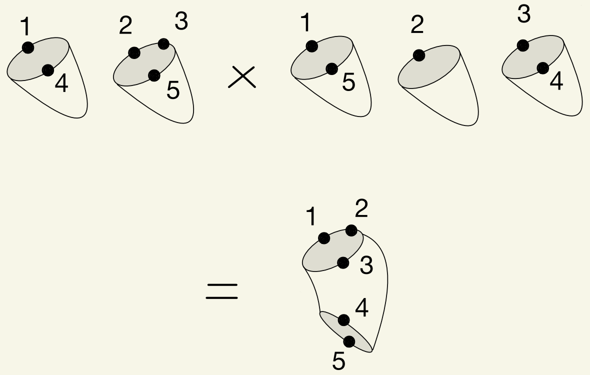
Let $(\mathcal{V}, \pi, g)$ and $(\mathcal{W}, \sigma, h)$ be surfaced permutations. We define their product to be

$$(\mathcal{V}, \pi, g) \odot (\mathcal{W}, \sigma, h) = (\mathcal{V} \vee \mathcal{W}, \pi\sigma, k)$$

where

- $\mathcal{V} \vee \mathcal{W}$ is the join of the two set partitions $\mathcal{V}$ and $\mathcal{W}$
- $\pi\sigma$ is the product of the two permutations π and σ
- k is given in terms of g and h and a "genus defect" coming from the multiplication
  (genus can be created, but not destroyed by multiplication)

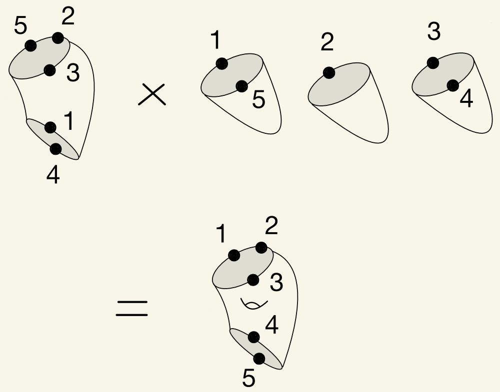
Example $((n=1,g=0)\times(n=1,g=0)=(n=2,g=0))$



no creation of genus

$$\{(1,4)\}\{(2,3,5)\}\times\{(1,5)\}\{(2)\}\{(3,4)\}=\{(123)(45)\}.$$

Example $((n=2, g=0) \times (n=1, g=0) = (n=2, g=1))$



creation of genus

$$\{(1,4)(2,3,5)\} \times \{(1,5)\}\{(2)\}\{(3,4)\} = \{(123)(45)\}.$$

Functions on $\mathbb{PS}$ and their convolution

Definition

Let $f_1, f_2 \colon \mathbb{PS} \to \mathbb{C}$ be functions, we define their convolution by

$$f_1 \circledast f_2(\mathcal{U}, \gamma, k) = \sum_{(\mathcal{V},\pi,g)\odot(\mathcal{W},\sigma,h)=(\mathcal{U},\gamma,k)} f_1(\mathcal{V}, \pi, g) f_2(\mathcal{W}, \sigma, h).$$

Functions on $\mathbb{PS}$ and their convolution

Definition

Let $f_1, f_2 \colon \mathbb{PS} \to \mathbb{C}$ be functions, we define their convolution by

$$f_1 \circledast f_2(\mathcal{U}, \gamma, k) = \sum_{(\mathcal{V},\pi,g)\odot(\mathcal{W},\sigma,h)=(\mathcal{U},\gamma,k)} f_1(\mathcal{V}, \pi, g) f_2(\mathcal{W}, \sigma, h).$$

Definition

A function $f \colon \mathbb{PS} \to \mathbb{C}$ is called multiplicative if

$$f(\mathcal{V}, \pi, g) = \prod_{B \in \mathcal{V}} f(B, \pi|_B, g|_B)$$

and if it is invariant under conjugation of π.

Functions on $\mathbb{PS}$ and their convolution

Definition

Let $f_1, f_2 \colon \mathbb{PS} \to \mathbb{C}$ be functions, we define their convolution by

$$f_1 \circledast f_2(\mathcal{U}, \gamma, k) = \sum_{(\mathcal{V},\pi,g)\odot(\mathcal{W},\sigma,h)=(\mathcal{U},\gamma,k)} f_1(\mathcal{V}, \pi, g) f_2(\mathcal{W}, \sigma, h).$$

Definition

A function $f \colon \mathbb{PS} \to \mathbb{C}$ is called multiplicative if

$$f(\mathcal{V}, \pi, g) = \prod_{B \in \mathcal{V}} f(B, \pi|_B, g|_B)$$

and if it is invariant under conjugation of π.

Fact

The convolution of two multiplicative functions is multiplicative.

Delta, Zeta, and Möbius function on $\mathbb{PS}$

Delta function

The unit element w.r.t. the convolution is given by the multiplicative delta function

$$\delta(\mathcal{V}, \pi, g) = \begin{cases} 1 & \text{if } (0_e, e, 0) \\ 0 & \text{otherwise} \end{cases},$$

Delta, Zeta, and Möbius function on $\mathbb{PS}$

Delta function

The unit element w.r.t. the convolution is given by the multiplicative delta function

$$\delta(\mathcal{V}, \pi, g) = \begin{cases} 1 & \text{if } (0_e, e, 0) \\ 0 & \text{otherwise} \end{cases},$$

Zeta function

The zeta function ζ on $\mathbb{PS}$ is the multiplicative function given by

$$\zeta(\mathcal{V}, \pi, g) = \begin{cases} 1 & \text{if } \mathcal{V} = 0_\pi \text{ and } g = 0 \\ 0 & \text{otherwise} \end{cases}$$

Delta, Zeta, and Möbius function on $\mathbb{PS}$

Delta function

The unit element w.r.t. the convolution is given by the multiplicative delta function

$$\delta(\mathcal{V}, \pi, g) = \begin{cases} 1 & \text{if } (0_e, e, 0) \\ 0 & \text{otherwise} \end{cases},$$

Zeta function

The zeta function ζ on $\mathbb{PS}$ is the multiplicative function given by

$$\zeta(\mathcal{V}, \pi, g) = \begin{cases} 1 & \text{if } \mathcal{V} = 0_\pi \text{ and } g = 0 \\ 0 & \text{otherwise} \end{cases}$$

Möbius function

The zeta function has an inverse with respect to convolution. This is called Möbius function and denoted by μ. It is also multiplicative.

$$\zeta \circledast \mu = \delta, \qquad \mu \circledast \zeta = \delta.$$

Restriction to planar ($g = 0$) sector

Fact

This all restricts consistently to the genus $= 0$ sector:

Restriction to planar ($g = 0$) sector

Fact

This all restricts consistently to the genus = 0 sector: if we consider only $k = 0$, then the factorizations in

$$f_1 \circledast f_2(\mathcal{U}, \gamma, k) = \sum_{(\mathcal{V},\pi,g)\odot(\mathcal{W},\sigma,h)=(\mathcal{U},\gamma,k)} f_1(\mathcal{V}, \pi, g) f_2(\mathcal{W}, \sigma, h).$$

run automatically only over $g = 0$ and $h = 0$.

Restriction to planar ($g = 0$) sector

Fact

This all restricts consistently to the genus = 0 sector: if we consider only $k = 0$, then the factorizations in

$$f_1 \circledast f_2(\mathcal{U}, \gamma, k) = \sum_{(\mathcal{V},\pi,g)\odot(\mathcal{W},\sigma,h)=(\mathcal{U},\gamma,k)} f_1(\mathcal{V}, \pi, g) f_2(\mathcal{W}, \sigma, h).$$

run automatically only over $g = 0$ and $h = 0$.

This corresponds then to the combinatorics of higher order free probability theory.

Higher order free probability theory

Moment-cumulant formulas

The above theory restricted to the planar sector ($g = 0$) yields the combinatorial theory of (higher order) free probability theory.

Higher order free probability theory

Moment-cumulant formulas

The above theory restricted to the planar sector ($g = 0$) yields the combinatorial theory of (higher order) free probability theory.

- the relevant objects are
  - the multiplicative moment function φ
  - the multiplicative cumulant function κ

Higher order free probability theory

Moment-cumulant formulas

The above theory restricted to the planar sector ($g = 0$) yields the combinatorial theory of (higher order) free probability theory.

- the relevant objects are
  - the multiplicative moment function φ
  - the multiplicative cumulant function κ
- the relation between them is given by the moment-cumulant formula

$$\varphi = \kappa \circledast \zeta, \qquad \kappa = \varphi \circledast \mu \qquad \text{restricted to } g = 0.$$

Higher order free probability theory

Moment-cumulant formulas

The above theory restricted to the planar sector ($g = 0$) yields the combinatorial theory of (higher order) free probability theory.

- the relevant objects are
  - the multiplicative moment function φ
  - the multiplicative cumulant function κ
- the relation between them is given by the moment-cumulant formula

$$\varphi = \kappa \circledast \zeta, \qquad \kappa = \varphi \circledast \mu \qquad \text{restricted to } g = 0.$$

- in this context the surfaced permutations with $g = 0$ are called partitioned permutations

Higher order free probability theory

Moment-cumulant formulas

The above theory restricted to the planar sector ($g = 0$) yields the combinatorial theory of (higher order) free probability theory.

- the relevant objects are
  - the multiplicative moment function φ
  - the multiplicative cumulant function κ
- the relation between them is given by the moment-cumulant formula

$$\varphi = \kappa \circledast \zeta, \qquad \kappa = \varphi \circledast \mu \qquad \text{restricted to } g = 0.$$

- in this context the surfaced permutations with $g = 0$ are called partitioned permutations
- in particular, for $n = 1$, partitioned permutations can be identified with non-crossing partitions and everything reduces to ordinary free probability

Our questions on higher order free probability theory

Main questions

- Are there extensions of the planar free probability theory to general genus?

Our questions on higher order free probability theory

Main questions

- Are there extensions of the planar free probability theory to general genus?
- Are there reformulations of the combinatorial moment-cumulant relations in terms of generating power series?

Our questions on higher order free probability theory

Main questions

- Are there extensions of the planar free probability theory to general genus?
  - yes, according to the extension of the theory of multiplicative functions from partitioned permutations to surfaced permutations, in [BCGLS]
- Are there reformulations of the combinatorial moment-cumulant relations in terms of generating power series?

[BCGLS]

"Analytic theory of higher order free cumulants" (arxiv.2112.12184)
by G. Borot, S. Charbonnier, E. Garcia-Failde, F. Leid and S. Shadrin

Our questions on higher order free probability theory

Main questions

- Are there extensions of the planar free probability theory to general genus?
  - yes, according to the extension of the theory of multiplicative functions from partitioned permutations to surfaced permutations, in [BCGLS]
- Are there reformulations of the combinatorial moment-cumulant relations in terms of generating power series?
  - yes, according to [BCGLS]

[BCGLS]

"Analytic theory of higher order free cumulants" (arxiv.2112.12184)
by G. Borot, S. Charbonnier, E. Garcia-Failde, F. Leid and S. Shadrin

Generating power series formulas for $g = 0$: $n = 1, 2$

Voiculescu 1986, Speicher 1994;

Consider the generating series

$$M_1(x) = 1 + \sum_{l \in \mathbb{N}} \varphi_l x^l, \quad C_1(x) = 1 + \sum_{l \in \mathbb{N}} \kappa_l x^l$$

then

$$M_1(x) = C_1(xM_1(x)),$$

Generating power series formulas for $g = 0$: $n = 1, 2$

Collins, Mingo, Sniady, Speicher 2008

Consider the generating series

$$M_2(x_1, x_2) = \sum_{l_1, l_2 \in \mathbb{N}} \varphi_{l_1, l_2} x_1^{l_1} x_2^{l_2}, \quad C_2(x_1, x_2) = \sum_{l_1, l_2 \in \mathbb{N}} \kappa_{l_1, l_2} x_1^{l_1} x_2^{l_2}$$

then

$$M_2(x_1, x_2) + \frac{x_1 x_2}{(x_1 - x_2)^2} = \frac{\mathrm{d} \ln y_1}{\mathrm{d} \ln x_1} \frac{\mathrm{d} \ln y_2}{\mathrm{d} \ln x_2} \left(C_2(y_1, y_2) + \frac{y_1 y_2}{(y_1 - y_2)^2} \right),$$

where $y_i = x_i M_1(x_i)$.

Generating power series formulas for $g = 0$: $n = 1, 2$

Voiculescu 1986, Speicher 1994; Collins,Mingo,Sniady,Speicher 2008

Consider the generating series

$$M_1(x) = 1 + \sum_{l \in \mathbb{N}} \varphi_l x^l, \quad C_1(x) = 1 + \sum_{l \in \mathbb{N}} \kappa_l x^l$$

$$M_2(x_1, x_2) = \sum_{l_1, l_2 \in \mathbb{N}} \varphi_{l_1, l_2} x_1^{l_1} x_2^{l_2}, \quad C_2(x_1, x_2) = \sum_{l_1, l_2 \in \mathbb{N}} \kappa_{l_1, l_2} x_1^{l_1} x_2^{l_2}$$

then

$$M_1(x) = C_1(x M_1(x)),$$

$$M_2(x_1, x_2) + \frac{x_1 x_2}{(x_1 - x_2)^2} = \frac{\mathrm{d} \ln y_1}{\mathrm{d} \ln x_1} \frac{\mathrm{d} \ln y_2}{\mathrm{d} \ln x_2} \left(C_2(y_1, y_2) + \frac{y_1 y_2}{(y_1 - y_2)^2} \right),$$

where $y_i = x_i M_1(x_i)$.

Generating power series formulas for $g = 0$: $n > 2$

Notation

For $n \in \mathbb{N}$ we denote

$$M_n(x_1, \ldots, x_n) = \delta_{n,1} + \sum_{l_1,\ldots,l_n \in \mathbb{N}} \varphi_{l_1 \ldots l_n} x_1^{l_1} \ldots x_n^{l_n},$$

$$C_n(x_1, \ldots, x_n) = \delta_{n,1} + \sum_{l_1,\ldots,l_n \in \mathbb{N}} \kappa_{l_1 \ldots l_k} x_1^{l_1} \ldots x_n^{l_n}.$$

Borot, Charbonnier, Garcia-Failde, Leid, Shadrin 2021

We have

$$M_n(x_1, \ldots, x_n) = \sum_{r_1,\ldots,r_n \in \mathbb{N}} \sum_{T \in \mathcal{G}_{0,n}(\mathbf{r}+\mathbf{1})} O^{\vee}_{r_i}(y_i) \prod_{I \in \mathcal{I}(T)}{}^{\prime} C_{\#I}(y_I).$$

Generating power series formulas for $g = 0$: $n > 2$

Borot, Charbonnier, Garcia-Failde, Leid, Shadrin 2021

$$M_n(x_1, \ldots, x_n) = \sum_{r_1, \ldots, r_n \in \mathbb{N}} \sum_{T \in \mathcal{G}_{0,n}(\mathbf{r}+\mathbf{1})} O^\vee_{r_i}(y_i) \prod_{I \in \mathcal{I}(T)}{}' C_{\#I}(y_I),$$

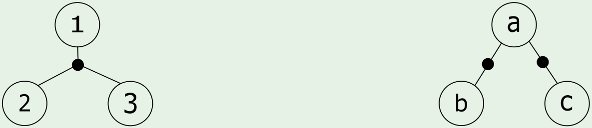
where $y_i = x_i M_1(x_i)$, $\mathbf{r} + \mathbf{1} = (r_1 + 1, \ldots, r_n + 1)$,

- $\mathcal{G}_{0,n}(\mathbf{r} + 1)$ is set of bicolored trees, $\mathcal{I}(T)$ the set of black vertices identified with its adjacent white vertices,
- $$O^\vee_r(y) = \sum_{m \geq 0} \left(\frac{\mathrm{d} \ln y}{\mathrm{d} \ln x} y \partial_y \right)^m \frac{\mathrm{d} \ln y}{\mathrm{d} \ln x} [v^m] \left(\partial_w + \frac{v}{w} \right)^r \cdot 1 \Big|_{w = C_1(y)},$$
- $\prod'$ means $C_2(y_i, y_j)$ is replaced by
$$C_2(y_i, y_j) + \frac{y_i y_j}{(y_i - y_j)^2}.$$

The case k=3

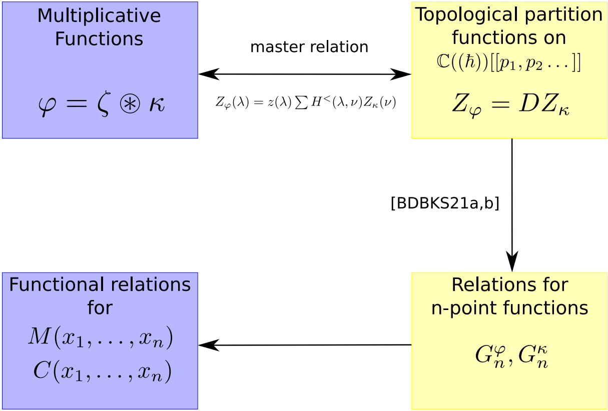
Example

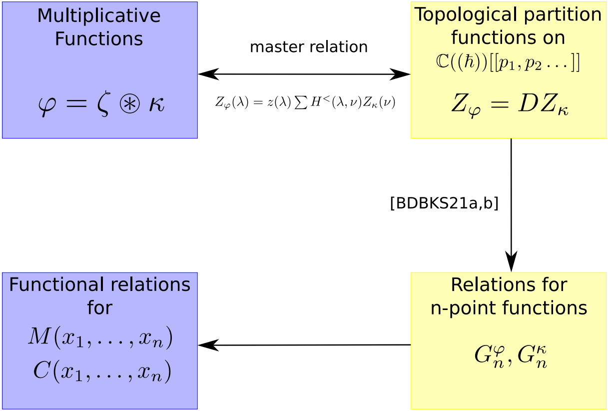
The only types of trees that contribute to $M_3(x_1, x_2, x_3)$ are the following



$$M_3(x_1,x_2,x_3) =$$

$$\frac{C_3(x_1,x_2,x_3)}{\prod_{i=1}^3 C_1(y_i)x'(y_i)} + \frac{y_1}{\prod_{i=1}^3 C_1(y_i)x'(y_i)} \frac{\partial}{\partial y_1} \frac{\tilde{C}_2(y_1,y_2)\tilde{C}_2(y_1,y_3)}{C_1(y_1)x'(y_1)} + \dots$$


Multiplicative Functions
$\varphi = \zeta \circledast \kappa$
master relation
$Z_\varphi(\lambda) = z(\lambda) \sum H^{<}(\lambda,\nu) Z_\kappa(\nu)$
Topological partition functions on $\mathbb{C}((\hbar))[[p_1, p_2 \dots]]$
$Z_\varphi = DZ_\kappa$
[BDBKS21a,b]
Functional relations for
$M(x_1, \dots, x_n)$
$C(x_1, \dots, x_n)$
Relations for n-point functions
G_n^φ, G_n^κ



Bychkov, Dunin-Barkowski, Kazarian, Shadrin:

- Explicit closed algebraic formulas for Orlov– Scherbin n-point functions
- Generalised ordinary vs fully simple duality for n-point functions and a proof of the Borot–Garcia-Failde conjecture

Some more questions

- give direct combinatorial proof of formula for generating power series

Some more questions

- give direct combinatorial proof of formula for generating power series
- understand relation with symplectic invariance of topological recursion

Some more questions

- give direct combinatorial proof of formula for generating power series
- understand relation with symplectic invariance of topological recursion
- is there some universality behind these constructions?

Some more questions and partial answers:

- give direct combinatorial proof of formula for generating power series
  - Lionni: From higher order free cumulants to non-separable hypermaps; arXiv:2212.14885
- understand relation with symplectic invariance of topological recursion
  - Hock: A simple formula for the $x - y$ symplectic transformation in topological recursion; arXiv:2211.08917
  - Alexandrov, Bychkov, Dunn-Barkowski, Kazarian, Shadrin: A universal formula for the $x - y$ swap in topological recursion; arXiv:2212.00320
- is there some universality behind these constructions?

Some more questions and partial answers:

- give direct combinatorial proof of formula for generating power series
  - Lionni: From higher order free cumulants to non-separable hypermaps; arXiv:2212.14885
- understand relation with symplectic invariance of topological recursion
  - Hock: A simple formula for the $x-y$ symplectic transformation in topological recursion; arXiv:2211.08917
  - Alexandrov, Bychkov, Dunn-Barkowski, Kazarian, Shadrin: A universal formula for the $x-y$ swap in topological recursion; arXiv:2212.00320
- **is there some universality behind these constructions?**
  - **????????????**

Some more questions and partial answers:

- give direct combinatorial proof of formula for generating power series
  - Lionni: From higher order free cumulants to non-separable hypermaps; arXiv:2212.14885
- understand relation with symplectic invariance of topological recursion
  - Hock: A simple formula for the $x - y$ symplectic transformation in topological recursion; arXiv:2211.08917
  - Alexandrov, Bychkov, Dunn-Barkowski, Kazarian, Shadrin: A universal formula for the $x - y$ swap in topological recursion; arXiv:2212.00320
- **is there some universality behind these constructions?**
  - **????????????**

Thank you for your attention!