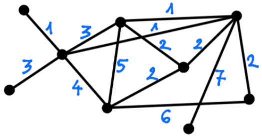
Random resistor networks on simple point processes and Mott's law

Alessandra Faggionato

Sapienza University of Rome

Conduction in disordered media

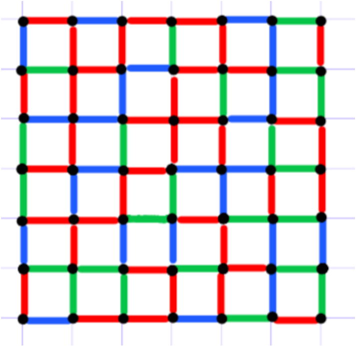
- Random resistor networks (**RNs**) have been used for decades in Physics to study conduction in disordered media.
- Random RNs have been much studied also by probabilists, especially in Percolation Theory (e.g. G. Grimmett, H. Kesten, L.T Chayes, L. Chayes, Y. Peres, R. Lyons,....) .
- Mathematically, a RN is a weighted undirected graph. Edges correspond to electrical filaments, weights to conductances.



Some popular RNs

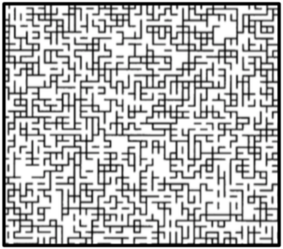
See e.g. Percolation and Conduction, Rev. Mod. Phys. 45 (1973), S. Kirkpatrick

- **Box in $\mathbb{Z}^d$ with random conductances**

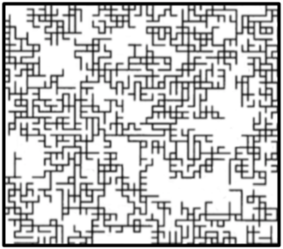


Some popular RNs

- **RN built on supercritical bond percolation on $\mathbb{Z}^d$** (with random conductances)
  Mixtures of insulating and conducting materials



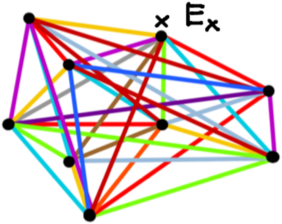
Supercritical bond percolation on Z^d



Infinite percolation cluster

Some popular RNs

- **Miller-Abrahams resistor network on $\mathbb{R}^d$**
  Amorphous solids as doped semiconductors

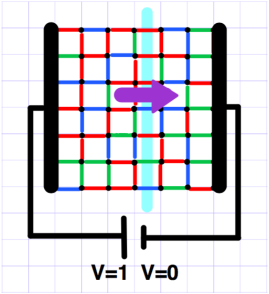


Each node x has an energy mark $E_x \in \mathbb{R}$
Between nodes $x \neq y$ there is an electrical filament with conductance

$$c_{x,y} := \exp\Big\{ - |x-y| - \beta(|E_x| + |E_y| + |E_x - E_y|)\Big\}$$

β: inverse temperature

Directional conductivity

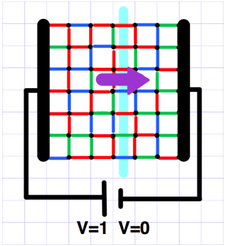


Box of size ℓ

σ_ℓ:= Directional conductivity

:=current along 1st direction under unit potential difference.

Scaling limit of directional conductivity



Box of size ℓ

Fact (S.M. Kozlov, Math. USSR Sbornik 57, 1987)

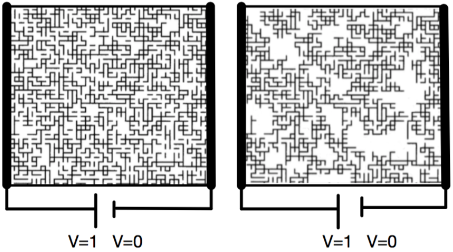
Consider $\mathbb{Z}^2$. *If the conductivities* $c_{x,y}$ *are i.i.d. with values in* $[a, b]$ *with* $0 < a < b < +\infty$, *then*

$$\lim_{\ell \to +\infty} \sigma_\ell = \sigma \qquad a.s.$$

where σ *is non-random and* $\sigma\mathbb{I}$ *is the effective homogenized matrix.*

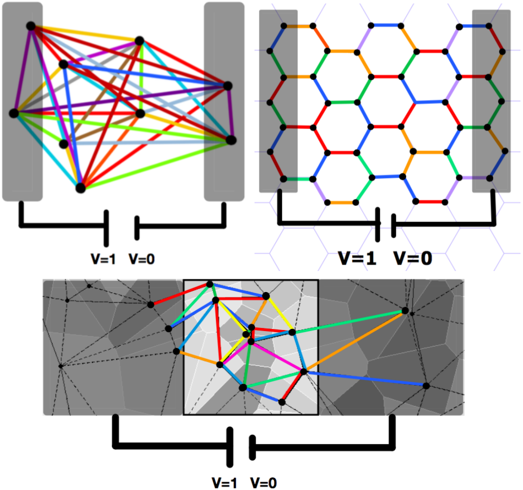
General dimension d: $\lim_{\ell \to +\infty} \ell^{2-d} \sigma_\ell = \sigma$ a.s.

Scaling limit of directional conductivity: what about other RNs?



It was an open problem
(the solution, soon)
See Problem 1.18 in
Recent progress on the Random Conductance Model
M. Biskup,
Probability Surveys, 2011.

Scaling limit of directional conductivity: what about other RNs?



- All the above RNs, and many more, can be thought of as RNs on simple point processes.
- We will describe a "universal" result for the scaling limit of the directional conductivity

Environment ω and random weighted graph $\mathcal{G}(\omega)$

- $(\Omega, \mathcal{F}, \mathcal{P})$ probability space
- $\omega \in \Omega$: environment, modeling the disorder and describing all sources of microscopic randomness
- Our finite RNs will be built from an infinite resistor network $\mathcal{G}(\omega)$ (=infinite weighted undirected graph)

Simple point process $\hat{\omega}$

We fix a **simple point process**, i.e.

$$\Omega \ni \omega \mapsto \hat{\omega} \in \{ \text{ locally finite subsets of } \mathbb{R}^d \, \}$$

Conductance field

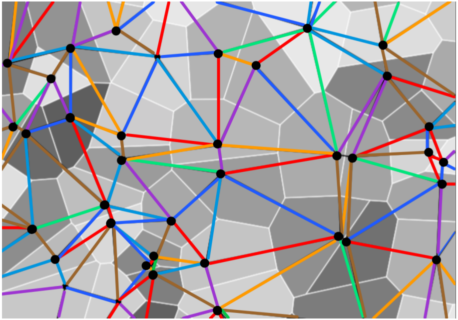
We fix a **conductance field**

$$c : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \ni (\omega, x, y) \mapsto c_{x,y}(\omega) \in [0, +\infty)$$

- $\triangleright$ $c_{x,y}(\omega) = c_{y,x}(\omega)$
- Relevant values are for $x \neq y$ in $\hat{\omega}$

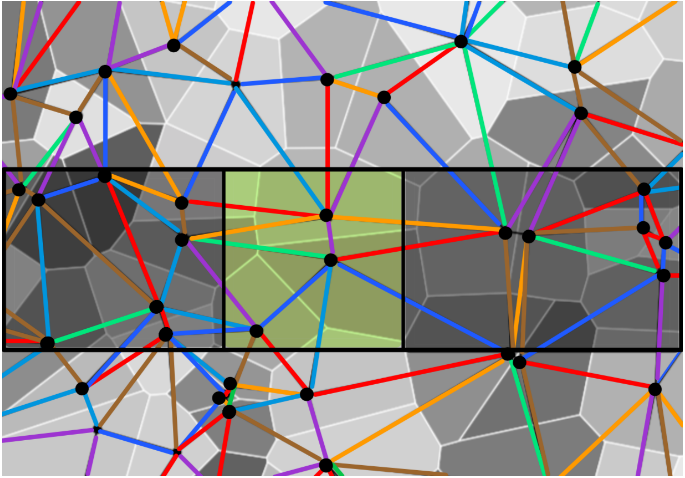
Infinite resistor network $\mathcal{G}(\omega)$

- $\boxed{\{ \text{ Nodes of } \mathcal{G}(\omega) \ \} := \hat{\omega}}$
- $\boxed{\{\text{filaments of } \mathcal{G}(\omega)\ \} := \{ \ \{x, y\} \ : \ x \neq y \text{ in } \hat{\omega}\,, \ c_{x,y}(\omega) > 0 \ \}}$
- $\boxed{\text{Conductance of } \{x, y\} := c_{x,y}(\omega)}$



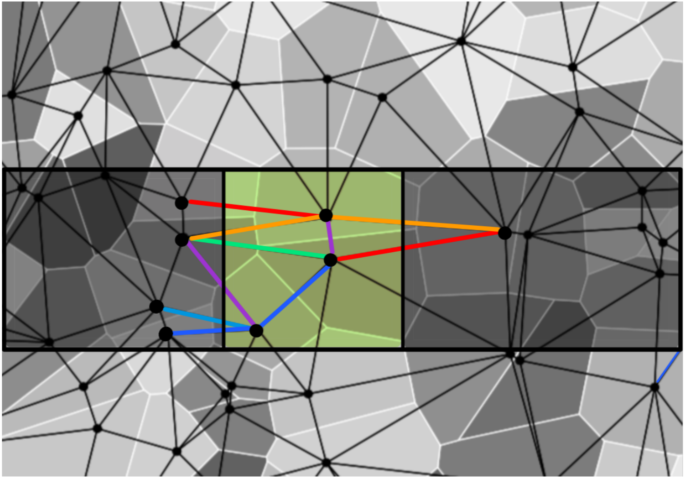
From $\mathcal{G}(\omega)$ to finite RNs

$\ell > 0$, $\Lambda_\ell := [-\ell/2, \ell/2]^d = \blacksquare$



From $\mathcal{G}(\omega)$ to finite RNs

Keep only filaments in the stripe with at least one node in Λ_ℓ



Statistical homogeneity and ergodicity of the medium

- We deal with media which are

  disordered at **microscopic level**,

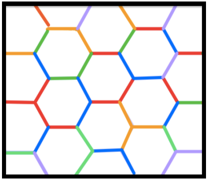
  homogeneous at **macroscopic level**.

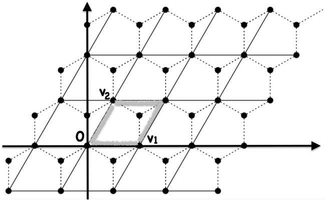
- To formalize that, we need another **MAIN INGREDIENT**: **Group** $\mathbb{G} = \mathbb{R}^d, \mathbb{Z}^d$ acting on
  - the Euclidean space $\mathbb{R}^d$
  - the probability space $(\Omega, \mathcal{F}, \mathcal{P})$

Action of $\mathbb{G}$ on the Euclidean space $\mathbb{R}^d$

- $(\tau_g)_{g\in\mathbb{G}}$, $\tau_g : \mathbb{R}^d \to \mathbb{R}^d$ translation
- Just for simplicity, here: $\tau_g x = x + g$
- In general, $\tau_g x = x + Vg$ with V invertible $d \times d$ matrix

General case with V and $\mathbb{G} = \mathbb{Z}^d$: relevant for graphs built on crystal lattices





Action of $\mathbb{G}$ on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$

- Action of $\mathbb{G}$ on the probability space: $(\theta_g)_{g\in\mathbb{G}}$,

$$\theta_g : \Omega \to \Omega\,, \quad \theta_0 = \mathbb{1}\,, \quad \theta_g \circ \theta_{g'} = \theta_{g+g'} \text{ for all } g, g' \in \mathbb{G}$$

- $\theta_g\omega$ describes the updated environment when we perform a translation τ_{-g} on the Euclidean space.
- This is formalized by assuming some simple **covariant relations between the two $\mathbb{G}$–actions**:

For all $\omega \in \Omega$, $g \in \mathbb{G}$ and $x, y \in \widehat{\theta_g\omega}$, it holds

$$\widehat{\theta_g\omega} = \tau_{-g}(\hat{\omega})\,,$$
$$c_{x,y}(\theta_g\omega) = c_{\tau_g x, \tau_g y}(\omega)\,.$$

Assumptions

▷ $\mathcal{P}_0$:=Palm distribution. Roughly, $\mathcal{P}_0 := \mathcal{P}(\cdot|0 \in \hat{\omega})$

Assumptions:

(A1) $\mathcal{P}$ is stationary and ergodic w.r.t. the action of $\mathbb{G}$ on Ω;

(A2) the intensity m of $\hat{\omega}$ (i.e. mean density) is finite and positive;

(A3) covariant relations for the two $\mathbb{G}$–actions;

(A4) for $\mathcal{P}$–a.a. ω the weighted undirected graph $\mathcal{G}(\omega)$ is connected [it can be relaxed];

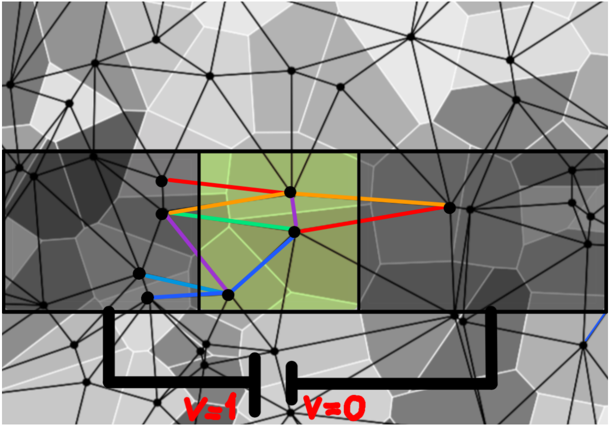
(A5) $\lambda_0, \lambda_2 \in L^1(\mathcal{P}_0)$, where $\lambda_k(\omega) := \sum_{x \in \hat{\omega}: x \neq 0} c_{0,x}(\omega)|x|^k$

(A6) $L^2(\mathcal{P}_0)$ is separable.

▷ Consequence: $\mathcal{P}(|\hat{\omega}| = +\infty) = 1$.

Directional conductivity

Connect the RN at scale ℓ to a battery with a unit potential difference



σ_ℓ:=current flowing along the 1st direction.

Scaling limit of directional conductivity

D:=effective homogenized matrix

Theorem (AF, arXiv:2108.11258)

Under the previous general assumptions:
Suppose that e_1 *is an eigenvector of* D. *Then for* $\mathcal{P}$*–a.a.* ω *it holds*

$$\sigma := \lim_{\ell \to +\infty} \ell^{2-d} \sigma_\ell(\omega) = m D_{1,1} = m\, e_1 \cdot D e_1 \,.$$

- Extensions: $e_1 \in \mathrm{Ker}(D)^\perp$, replace Λ_ℓ by suitable parallelepiped
- Kirchhoff's laws $\Longrightarrow$ Electrical potential V_ℓ^ω satisfies a discrete elliptic equation

  $$\nabla \cdot (D_\ell(\omega) \nabla V_\ell^\omega) = 0$$

  with suitable b.c.
- Stochastic homogenization

Effective homogenized matrix

To simplify, here, we take $\hat{\omega} \subset \mathbb{G}$

Definition

We define the **effective homogenized matrix** D as the $d \times d$ nonnegative symmetric matrix such that, for all $a \in \mathbb{R}^d$,

$$a \cdot Da = \inf_{f \in L^\infty(\mathcal{P}_0)} \frac{1}{2} \int d\mathcal{P}_0(\omega) \sum_{x \in \hat{\omega}} c_{0,x}(\omega) \left(a \cdot x - \nabla f(\omega, x)\right)^2 ,$$

where $\nabla f(\omega, x) := f(\theta_x \omega) - f(\omega)$.

Effective homogenized matrix

- D can be degenerate and non zero (A.F, arXiv:2108.11258).
- The continuous time random walk on $\hat{\omega}$ with probability rate $c_{x,y}(\omega)$ for a jump from x to y has asymptotic diffusion matrix equal to $2D$ (A.F. arXiv:2009.08258 (2020), to appear in AIHP)
- D can be computed explicitly in few cases (linear resistor networks)

Miller-Abrahams random resistor network

Effective model of **Mott's variable range hopping**:

- phonon–assisted electron hopping in amorphous solids in the regime of strong Anderson localization
- introduced by N. Mott to explain anomalous decay of conductivity at low temperature

The environment ω is the realization of a marked simple point process, $\omega = \{(x, E_x)\}$, $\hat{\omega} = \{x\}$.
$\mathcal{G}(\omega)$: complete graph on $\hat{\omega}$ with weights

$$c_{x,y}(\omega) := \exp\Big\{ -|x-y| - \beta(|E_x| + |E_y| + |E_x - E_y|)\Big\}$$

Bounds

General Assumptions read:
$\mathcal{P}$ stationary, ergodic, $\mathbb{E}\big[\big(\sharp\hat{\omega}\cap[0,1]^d\big)^2\big] < +\infty$.

- $\sigma(\beta) = mD(\beta)_{1,1}$
- $D(\beta)_{1,1} = \inf_{f\in L^\infty(\mathcal{P}_0)} \mathcal{F}(f)$
  $\Longrightarrow$ for each test function f we have an upper bound on $D(\beta)_{1,1}$
- $\sigma(\beta)$ is the scaling limit of directional conductivity:
  $\Longrightarrow$ we get lower bounds on $\sigma(\beta)$ by Rayleigh's monotonicity law for resistor networks

Mott's law

- Physics law
- Energy marks: i.i.d. random variables with law ν, $\nu(dE) = c\,|E|^\alpha dE$ and $\alpha \geq 0$, on a finite interval $[-A, A]$
- $d \geq 2$
- Heuristic derivation. Several attempts in the Physics literature to make it more robust

Mott's law:

$$\boxed{\sigma(\beta) \approx a(\beta) \exp\left\{-\kappa\, \beta^{\frac{\alpha+1}{\alpha+1+d}}\right\} \text{ for } \beta \text{ large}}$$

$a(\beta)$ exhibits a negligible β–dependence.

Rich debate: κ?

Critical conductance

$\mathcal{G}(\omega)$: complete graph on $\hat{\omega}$ with weights

$$c_{x,y}(\omega) := \exp\Big\{ - |x-y| - \beta(|E_x| + |E_y| + |E_x - E_y|)\Big\}$$

Definition (Critical conductance $c_c(\beta)$)

Given $\xi > 0$ keep in $\mathcal{G}(\omega)$ only filaments with $c_{x,y}(\omega) \geq \xi$. Then

$$\begin{cases} \text{for } \xi > c_c(\beta) \text{ the resulting graph a.s. does not percolate,} \\ \text{for } \xi < c_c(\beta) \text{ the resulting graph a.s. percolates.} \end{cases}$$

Recall Mott's law: $\sigma(\beta) \approx a(\beta) \exp\{-\kappa\, \beta^{\frac{\alpha+1}{\alpha+1+d}}\}$

Physics heuristics:

- $c_c(\beta) \approx a'(\beta) \exp\{-\kappa' \beta^{\frac{\alpha+1}{\alpha+1+d}}\}$
- Conjecture: $\kappa = \kappa'$
- Debate: κ'?

Assumptions

- Nodes $\{x\}$: homogeneous Poisson point process with intensity m
- I.I.D. energy marks with common distribution ν such that, near the origin, ν has form

$$\nu(dE) = \begin{cases} C_0 E^\alpha \mathbb{1}(E \geq 0) dE & \text{Case A} \\ C_0 |E|^\alpha dE & \text{Case B} \end{cases}$$

Mott's law for the critical conductance

Theorem (A.F. arXiv:2301.06318)

$$\kappa' := \lim_{\beta\to+\infty} \beta^{-\frac{\alpha+1}{\alpha+1+d}} \ln c_c(\beta) = -\big(\lambda_c C_0^{\alpha+1}/m\big)^{\frac{1}{\alpha+1+d}}$$

where λ_c *is a constant of percolation-type depending only from* α, d *and Case A or B.*

Hence we have proved that

$$c_c(\beta) \approx a'(\beta)\exp\{-\kappa'\beta^{\frac{\alpha+1}{\alpha+1+d}}\}\,, \qquad \kappa' = -\big(\lambda_c C_0^{\alpha+1}/m\big)^{\frac{1}{\alpha+1+d}}$$

Mott's law

Recall: $\kappa' = -\left(\lambda_c C_0^{\alpha+1}/m\right)^{\frac{1}{\alpha+1+d}}$

Theorem (A.F.)

In Case (A): $\kappa := \lim_{\beta\to+\infty} \beta^{-\frac{1+\alpha}{d+1+\alpha}} \ln \sigma(\beta) = \kappa'$. *Hence,*

$$\sigma(\beta) \approx a(\beta) \exp\left\{-\kappa\, \beta^{\frac{\alpha+1}{\alpha+1+d}}\right\}, \qquad \kappa = \kappa' .$$

In case (B): same result under a suitable Ansatz on left-right crossings (up to now rigorously proved only in Case A, where FKG inequality holds)

Poisson point processes are the right object:
basic mechanism of thinning at low temperature leads to a Poisson point process after space rescaling (see [AF, arXiv:2301.06318] for more results on **universality** behind Mott's law)

Bibliography

The discussed results are taken from:

1. A.F.; *Scaling limit of the conductivity of random resistor networks on simple point processes.* arXiv:2108.11258
2. A.F.; *Mott's law for the Miller-Abrahams random resistor network and for Mott's random walk.* arXiv:2301.06318

Percolation properties in the Miller–Abrahams random resistor network:

- A.F., H.A. Mimun; *Connection probabilities in the Poisson Miller–Abrahams random resistor network and other Poisson random graphs with bounded edges.* ALEA, 2019
- A.F., H.A. Mimun; *Left-right crossings in the Miller-Abrahams random resistor network and in generalized Boolean models.* SPA, 2021.

Many references therein...