The Quantum SSEP
&
the emergence of free probability in noisy many-body systems

How to characterise fluctuations in diffusive out-of-equilibrium many-body quantum systems ?

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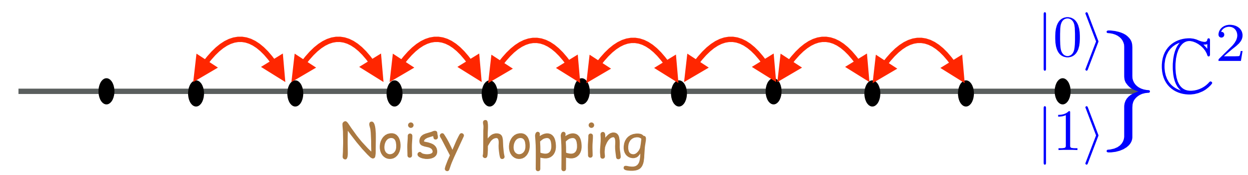
The Quantum SSEP & Motivations :

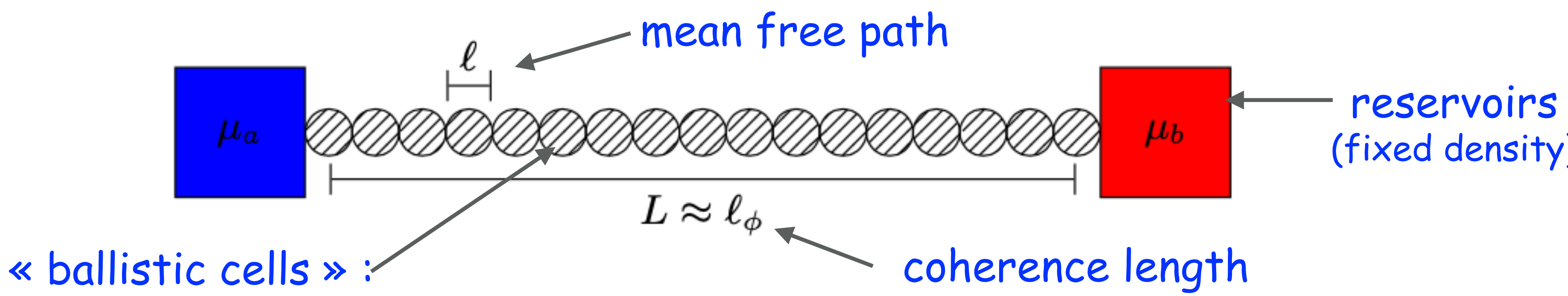
— Fermions hopping on a lattice with **Brownian amplitudes**



$$dH_t = \sqrt{D}\, \sum_j \left(c^\dagger_{j+1} c_j \, dW^j_t + c^\dagger_j c_{j+1} \, d\overline{W^j_t}\right)$$

+ boundary terms…
+ injection/extraction…

— Schematic representation of « **mesoscopic** » systems coupled to reservoirs



small scale / fast d.o.f. —> **noise**

A simple model of **quantum fluctuating diffusion**

- closed —> equilibrium
- open —> driven out of equilibrium

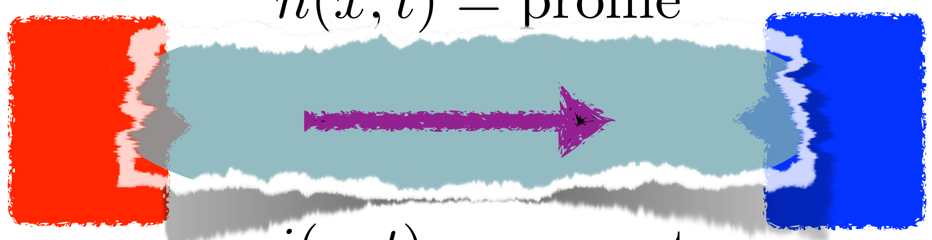
→ <u>**Aims**</u>: **Describing fluctuations of coherent effects at mesoscopic scale (out-of-equilibrium).**

The (classical) Macroscopic Fluctuation Theory

[Bertini, Sole, Gabrielli, Jona-Lasinio, Landim, …]

— MFT describes **fluctuations in out-of-equilibrium « classical » systems**

—> Statistics on profiles, currents, transport, and their fluctuations



$$\begin{cases} \partial_t n(x,t) + \partial_x j(x,t) = 0 \\ j(x,t) = -D(n)\partial_x n(x,t) + \sqrt{L^{-1}\sigma(n)}\,\xi(x,t) \end{cases}$$

(noisy Fourier-Fick's law)

Large deviation functions

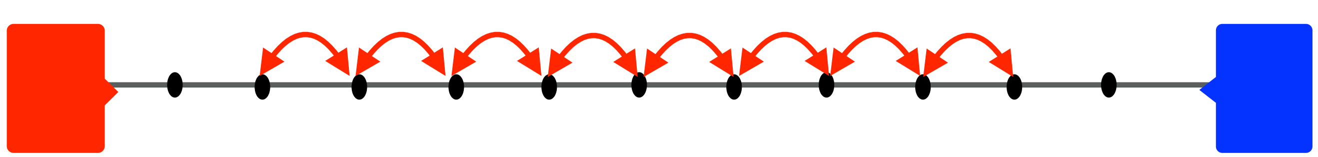
$$\mathbb{P}\mathrm{rob}\Big[\text{profile} = n(\cdot)\Big] \asymp e^{-(L/a_{uv})\,F[n(\cdot)]}$$

⟵ « analogue » of free energy out-of-equilibrium (non-local)

— MFT defines as an effective theory for a noisy Fourier-Fick's law

MFT emerged from studies of (stochastic) lattice models…

[Kipnis, Landim, Liggett, Spohn, Derrida, Mallick, Evans, et al …]



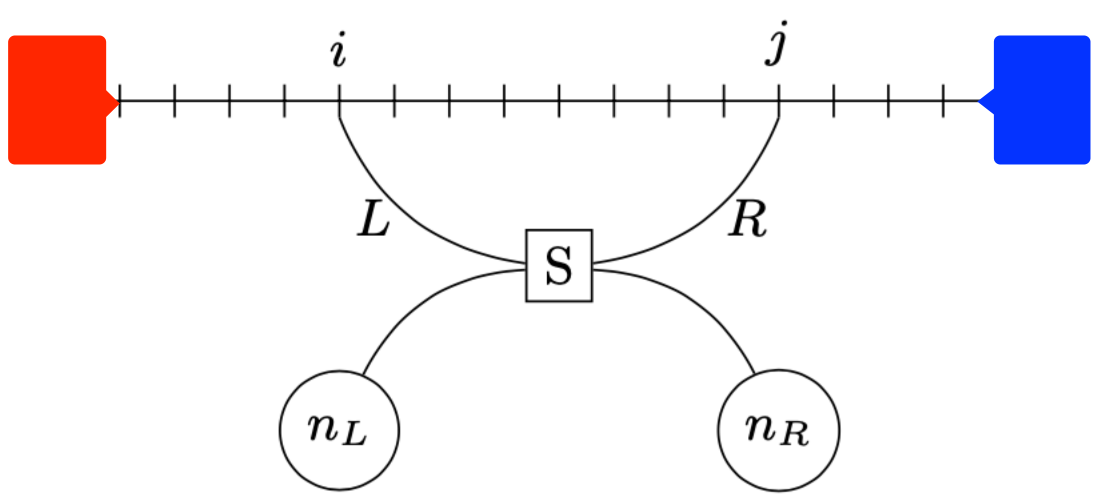
e.g. SSEP (Markov chain)

Towards a « Quantum Mesoscopic Fluctuation Theory »

— **Is there universality in the fluctuations of quantum coherent effects in out-of-equilibrium diffusive many-body systems ?...**

→ Get quantitative description of **transport**, **interferences**, **coherent processes**, **entanglement**, **monitoring**, **etc**, **and their fluctuations** in those systems ...

— Experimental measurability of these fluctuating coherent effects



→ **How sites i & j are entangled ?**

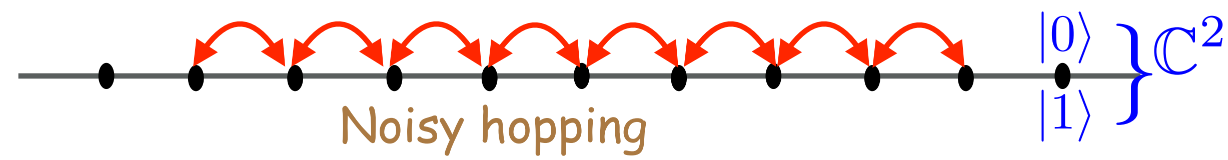
Measurement via interference patterns of charge/current extracted from two different positions and their statistics

$\to$ coherences G_{ij} [Huse, Gullans]

— **Nice, rich, quantum stochastic processes**, generalising well known classical processes (say SSEP, ASEP) with **unexpected connexions** (probability & group theory, combinatorics, ...)

The Quantum SSEP (structure & basics) :

— Quadratic but noisy model :



$dH_t = \sqrt{D} \sum_j \left(c^\dagger_{j+1} c_j \, dW^j_t + c^\dagger_j c_{j+1} \, d\overline{W^j_t} \right)$ + boundary (injection/extraction) terms…

— **Stochastic many-body quantum system** :

The evolution is stochastic, so is the quantum state and hence the 2-point functions G

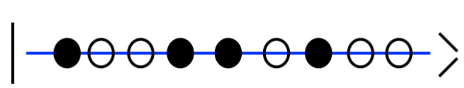
—> **Stochastic process on coherences** $G_{ij} = \langle c^\dagger_j c_i \rangle_t = \mathrm{Tr}(\rho_t \, c^\dagger_j c_i)$

$$G_{t+dt} = e^{-idh_t} G_t e^{+idh_t}$$

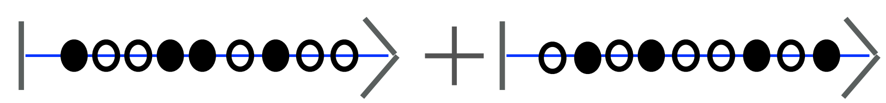
(with dh the one-particle hamiltonian)

— Q-SSEP includes SSEP (in **mean**) but codes for **quantum coherent effects.**

classical



quantum



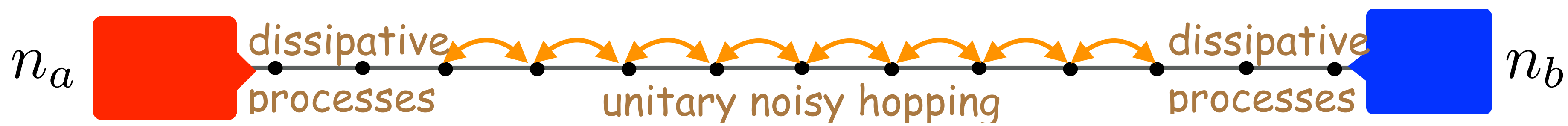
—> More than in the classical analogue (due to coherent/interference effects)

→ **Look for fluctuations of the coherences and higher moments**

$$\mathbb{E}[G_{i_1 i_n} \cdots G_{i_2, i_1}]$$

Non-equilibrium coherent fluctuations

— **Open boundary condition** —> system is driven **out-of-equilibrium**



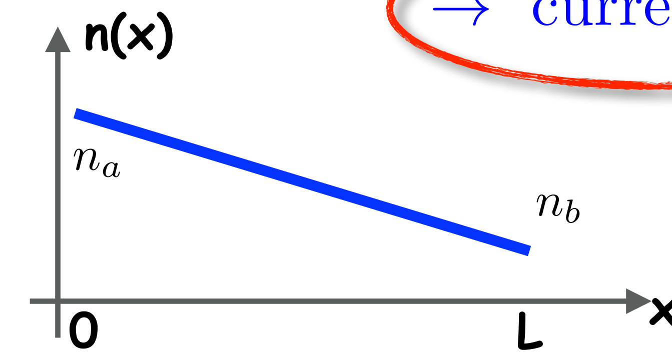
— Steady **mean profile** —> out-of-equilibrium:

$$[n_j] := \mathbb{E}[\langle c_j^\dagger c_j \rangle] = n_a + x(n_b - n_a)$$

(x=j/N, at large system size L=Na)

$\mathbb{E}[G_{ij}] = 0$ —> decoherence (in mean)

$\to$ current $j \neq 0$



— **Fluctuations & coherences** —> Steady statistics of (coherent) fluctuations.

$$\mathbb{E}[G_{ij}G_{ji}] = \frac{1}{N}(\Delta n)^2 x(1-y) + O(N^{-2})$$ for $G_{ij} = \langle c_j^\dagger c_i \rangle$

—> long range (multi-point) correlations.

Sub-leading (in system size) **fluctuating coherences** (beyond mean decoherence)
… with some (hidden) patterns (—> **free probability**…)

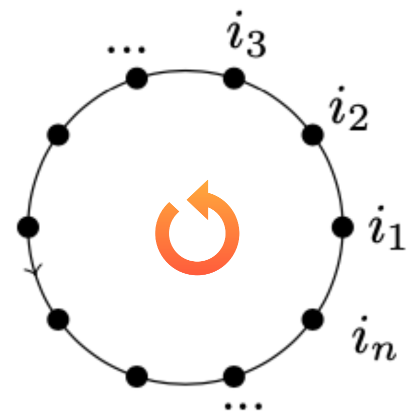
Fluctuations of coherences (large deviation)

— **Statistics of the coherences** : higher moments... $G_{ij} = \langle c_j^\dagger c_i \rangle_t = \mathrm{Tr}(\rho_t \, c_j^\dagger c_i)$

$$\mathbb{E}[G_{i_1 i_n} \cdots G_{i_3 i_2} G_{i_2 i_1}]^c = (\frac{a_{uv}}{L})^{n-1} \; g_n(x_1, \cdots, x_n; \tau) + O((\frac{a_{uv}}{L})^n)$$

n-th cumulants

recursively known



in the scaling limit $x_k = i_k/N, \; \tau = t/N^2, \; N = L/a_{uv}$

—> (formal) existence of **large deviation function**

— Existence of **scaling limit** : numerical check.

one-point function

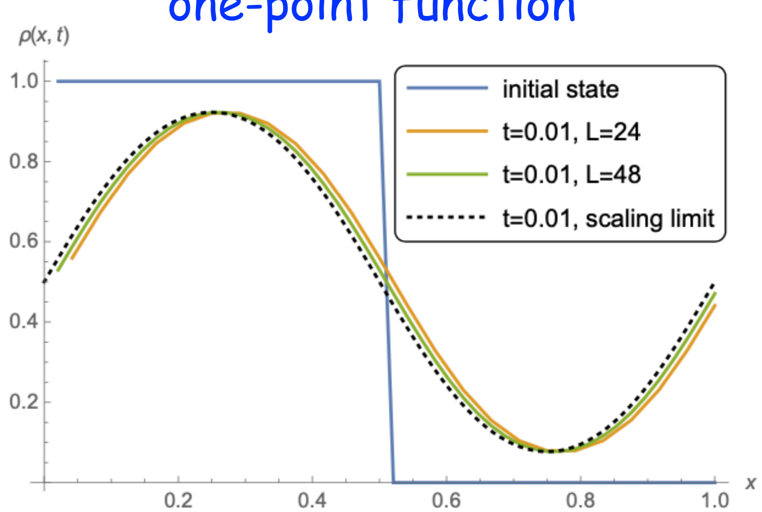


FIG. 2. The discrete fermion density $n_{Lx}(L^2 t)$ for system sizes $L = 24$ and $L = 48$ together with the scaling limit $\rho(x,t)$ at $t = 0.01$ as a function of space $x = i/L$. The extraction and injection rates are $\alpha_1 = \beta_1 = \alpha_L = \beta_L = 1$ and don't fit the initial conditions. The agreement is very good.

two-point function

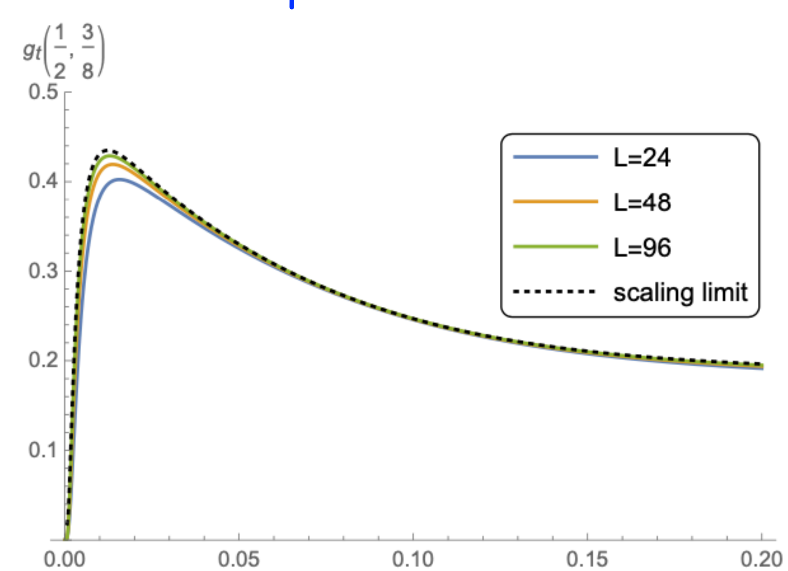
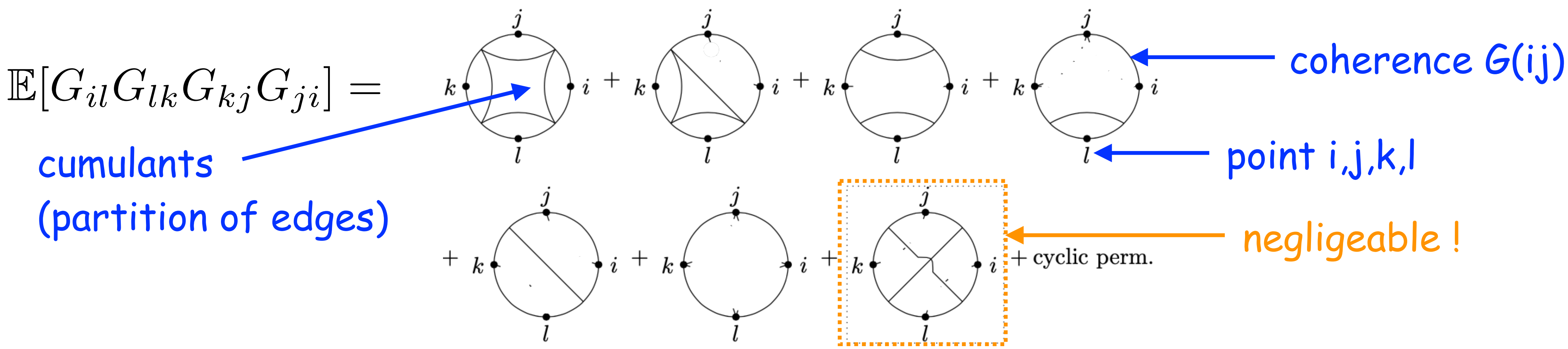


FIG. 3. Boundary conditions that fit the initial domain wall state ($n_a = 1, \, n_b = 0$).

Fluctuations: the role of free probability

— Look at the moments of the coherences $\mathbb{E}[G_{i_1 i_n} \cdots G_{i_2,i_1}]$
in terms of their **cumulants** $\mathbb{E}[G_{i_1 i_n} \cdots G_{i_2,i_1}]^c$ (—> via sum on partitions)



Only the **non-crossing partitions** contribute.
Generalisation to higher order expectations

$\Longrightarrow$ **Free probability !**

— **Universality** : it relies only three conditions.

— U(1) invariance : $G_{jk} \equiv_{\text{in law}} e^{i\theta_j} G_{jk} e^{-i\theta_k}$

— Scaling of the loop expectation values : $\mathbb{E}[G_{j_1 j_n} \cdots G_{j_3 j_2} G_{j_2 j_1}] \sim N^{1-n}$

— Factorisation of product of loops: $\mathbb{E}[G_{j_1 j_n} \cdots G_{j_2 j_1} \cdot G_{i_1 i_p} \cdots G_{i_2 i_1}] =$
$\mathbb{E}[G_{j_1 j_n} \cdots G_{j_2 j_1}]\, \mathbb{E}[G_{i_1 i_p} \cdots G_{i_2 i_1}]$

Fluctuations: the role of free probability

— Application (i) : **Steady measure and free cumulants** [(B. J)…. Ph. Biane… (then B. H.)]

Theorem:
Let φ the Lebesgue measure on $[0,1]$,
Let $I_x := \mathbb{I}_{[0,x]}$ with $\varphi(I_{x_1}, \cdots, I_{x_n}) = \min(x_1, \cdots, x_n)$,
Then: $g_n(x_1, \cdots, x_n; \tau = \infty) = \kappa_n(I_{x_1}, \cdots, I_{x_n})$ ⟵ free cumulants

—> Higher Q-SSEP steady cumulants are free cumulants of commuting variables

A few exemples : (for $0 < x_1 < x_2 < x_3 < x_4 < 1$)

$$\kappa_2(x,y) = x(1-y) \ , \ \kappa_3(x,y) = x(1-2y)(1-z)$$

$$\kappa_4(x_1,x_2,x_3,x_4) = x_1(1-3x_2-2x_3+5x_2x_3)(1-x_4)$$

$$\kappa_4(x_1,x_3,x_2,x_4) = x_1(1-4x_2-x_3+5x_2x_3)(1-x_4)$$

⟵ depend on the ordering of the points on the loop

Combinatorial aspect —> associahedron ;
Q-SSEP —> free probability for classical SSEP.

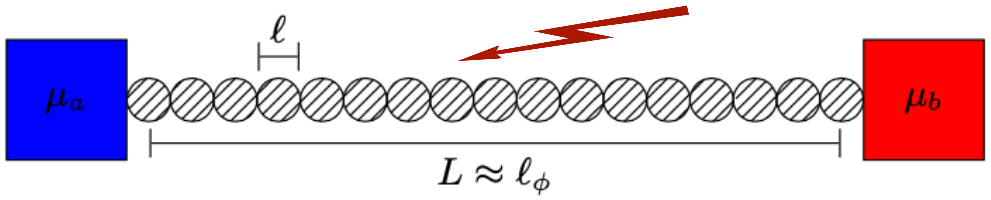
— Application (ii) : **Dynamics** —> The free cumulants of the expectations of coherences satisfy simple scaling hydrodynamic equations.

Emergence of free probability

in noisy mesoscopic systems

— **Coarse-grained description** (at mesoscopic scale)

(i) separation of time scales :
fast, closed dynamics within ballistic cells for $t < t_\ell$
—> unitary dynamics within each cells for $t < t_\ell$



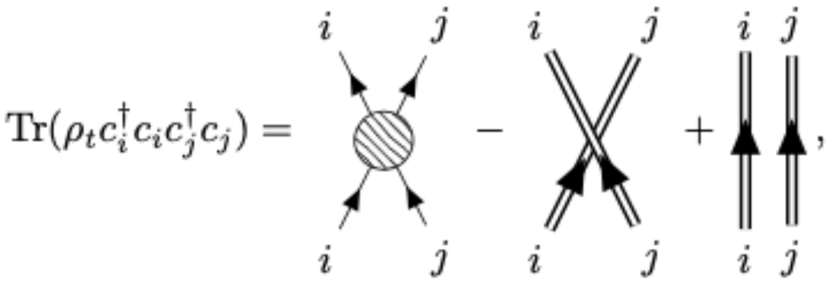
(ii) ergodicity of the fast dynamics (—> noise) :

$$\mathbb{E}_t[G_{jk}] := \frac{1}{t_\ell}\int_t^{t+t_\ell} dt'\, G_{jk}(t') = \mathrm{Tr}(\rho_t\, [c_i^\dagger c_j]_U)$$

(local) Haar measure

— **Validity of U(1) sym. + MFT in mean —> the « universality three conditions »**

- U(1) invariance : $G_{jk} \equiv_{\text{in law}} e^{i\theta_j} G_{jk} e^{-i\theta_k}$ <— local conservation (transport) and closed unitary dynamics at short time
- Scaling of the loop expectation values : <— If mean densities satisfy MFT
  $\mathbb{E}[G_{j_1 j_n} \cdots G_{j_3 j_2} G_{j_2 j_1}] \sim N^{1-n}$
  If some perturbation theory is valid $(H = H_0 + V)$
  (if no cancelation between the two first diagrams)



- Factorisation of loop expectations : <— closed fast dynamics / cells independence

— **Validation / Violation of these three conditions …**

Conclusion :

Many open questions related to quantum stochastic processes and to constructing a « Quantum Mesoscopic Fluctuation Theory » but Q-SSEP already provides a few hints

… … …

Conjectural ubiquitous/universal role of « free probability » in fluctuating (at- or out-of-equilibrium) many body quantum systems

… … …

- non-crossing partitions <— scaling of cumulants and U(1) invariance.
- as large matrices <— associated to ballistic cells d.o.f.'s
- similar occurence in ETH [see S. Pappalardi, L. Fioni, J. Kurchan]
  (locality/coarse-graining in energy versus locality in space)
- L. Hurla & D. B. , arXiv:2204.11680

Thank you !!!

Hidden free probability in classical SSEP :

[M. Bauer, D.B. Ph. Biane]

— **Large deviation Q-SSEP**

$$\mathbb{P}\left[\mathfrak{n}(\cdot) \approx n(\cdot)\right] \asymp_{N\to\infty} e^{-N\, I_{\rm ssep}[n]},$$

—> In terms of free probability data

$$I_{\rm ssep}[n] = \max_{g(\cdot),q(\cdot)} \Big(\int_0^1 dx \Big[n(x) \log\big(\frac{n(x)}{g(x)}\big) + (1-n(x)) \log\big(\frac{1-n(x)}{1-g(x)}\big) + q(x)g(x) \Big] - F_0^{\rm ssep}[q] \Big)$$

with $$F_0^{\rm ssep}[a] = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} R_n(\mathbb{I}_{[a]}),$$

n-th free cumulants of $\mathbb{I}_{[a]}(x) = \int_x^1 dy\, a(y)$ w.r.t. to the Lebesgue measure on [0,1]

— **Three steps** :

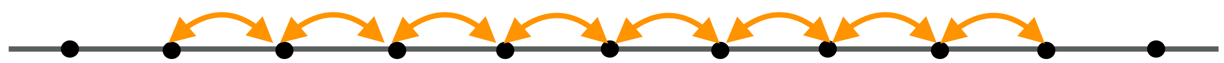
(i) : Q-SSEP /classical SSEP relation

(ii) : Free probability description of the Q-SSEP steady measure

(iii) : A relation between cumulants and « non-coincidants » cumulants for Bernouilli variables

Closed model : Fluctuations at Equilibrium

— **Periodic boundary condition** —> Equilibrium state (at large time) + fluctuations



(on a periodic loop, with N sites)

— **Steady/Invariant measure** : $G_{ij} = \langle c_j^\dagger c_i \rangle_t = \mathrm{Tr}(\rho_t \, c_j^\dagger c_i)$

- 1-point functions (mean decoherence)

$$\mathbb{E}[G_{ii}] = M_1/N = \bar{m} \quad \text{(equilibrium)}$$
$$\mathbb{E}[G_{ij}] = 0 \quad (i \neq j) \quad \text{(decoherence)}$$

$M_k := \mathrm{tr}(G^k) =: N m_k$

$(\Delta \bar{m}) \equiv$ initial density variance

- 2-point functions (fluctuating coherence)

$$\mathbb{E}[|G_{ij}|^2] = \frac{N M_2 - M_1^2}{N(N^2-1)} \equiv \frac{(\Delta \bar{m})^2}{N} \quad \text{i.e.} \quad \langle c_j^\dagger c_i \rangle \simeq O(1/\sqrt{N})$$

—> Fluctuating coherence (generated from initial inhomogeneities)

— Steady state = (non random) equilibrium state + fluctuations

$$\lim_{t\to\infty} \rho_t = \rho_{\mathrm{eq}} + \delta\rho$$

Fluctuating deviation from equilibrium
(cf. higher moments, large deviation function…)

Reminder on Free Probability

For a collection of random variables X_j

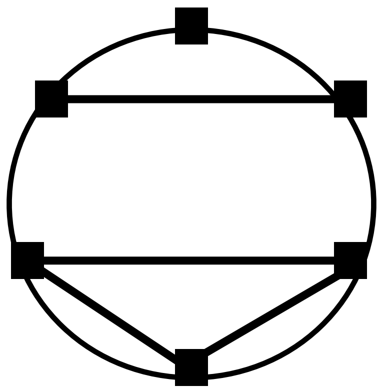
— (Standard) cumulants (—> sum over partitions {1,...,m}) :

$$\mathbb{E}[X_1 \cdots X_m] = \sum_{\pi \in \mathcal{P}_m} C_\pi(X)$$

$$m_1 = c_1,$$
$$m_2 = c_2 + c_1^2,$$
$$m_3 = c_3 + 3c_2c_1 + c_1^3,$$
$$m_4 = c_4 + 4c_1c_3 + 3c_2^2 + 6c_1^2c_2 + c_1^4.$$

— Free cumulants (—> sum over non-crossing partitions {1,...,m}) :

$$\mathbb{E}[X_1 \cdots X_m] = \sum_{\pi \in NC_m} C_\pi(X)$$



$$m_1 = \kappa_1,$$
$$m_2 = \kappa_2 + \kappa_1^2,$$
$$m_3 = \kappa_3 + 3\kappa_2\kappa_1 + \kappa_1^3,$$
$$m_4 = \kappa_4 + 4\kappa_1\kappa_3 + 2\kappa_2^2 + 6\kappa_1^2\kappa_2 + \kappa_1^4.$$

(Un-reasonable) connexions to combinatorics

- Recall $\left[G_{i_1 i_N} \cdots G_{i_3 i_2} G_{i_2 i_1}\right]^c = \frac{1}{L^{N-1}}\, g_N(x_1, \cdots, x_N) + O(L^{-N})$

- Take all points to be equal : $x_j = -t$

Then : $g_N(\mathbf{x})|_{x=-t} = t(1+t)\Phi_{N-1}(t)$

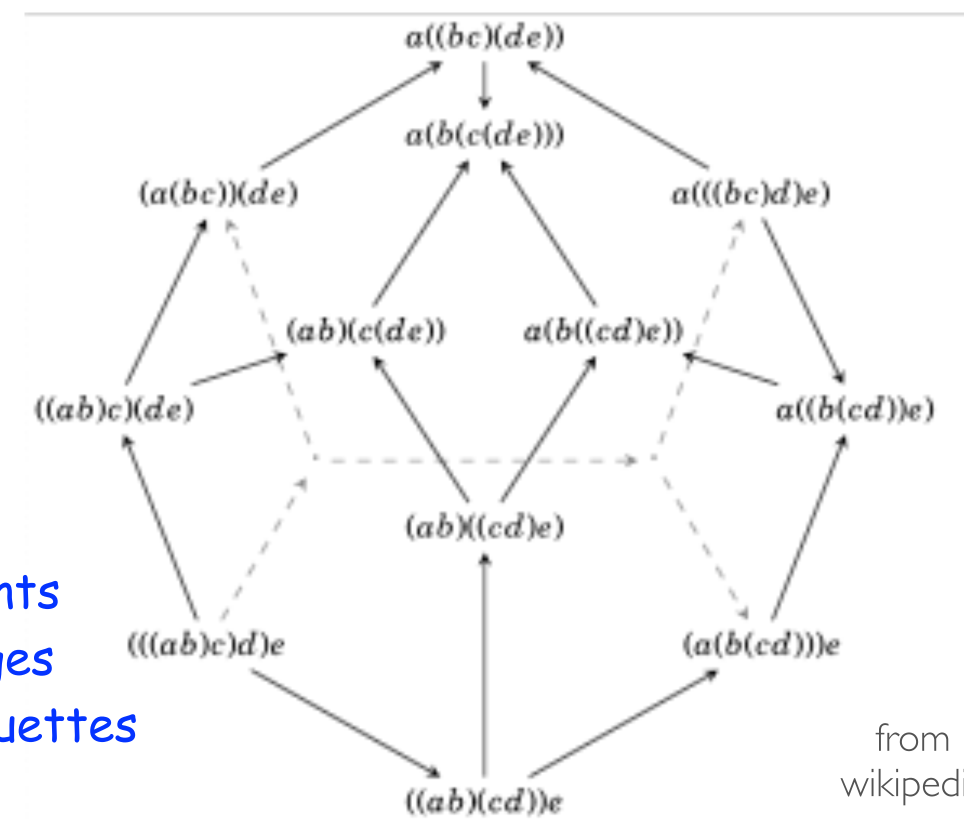
$$\begin{aligned}
\Phi_2(t) &= 1+2t, \\
\Phi_3(t) &= 1+5t+5t^2, \\
\Phi_4(t) &= 1+9t+21t^2+14t^3, \\
\Phi_5(t) &= 1+14t+56t^2+84t^3+42t^4.
\end{aligned}$$

—> These are the generating functions counting the number of (n-k) dimensional faces in the **associahedron** of order n.

- vertices = different parenthesis
- edges = associativity rule

$$(a(bc)) \longleftrightarrow ((ab)c)$$

- these graphs are polytopes.



14 points
21 edges
9 plaquettes

from wikipedia