COVARIANCE STABILITY & EIGENVECTOR OVERLAPS: A (FREE) RMT APPROACH

J.-Ph. Bouchaud (CFM/X-CFM)

(Joint work with Romain Allez, Joel Bun, Iacopo Mastromatteo, Marc Potters, Pierre-Alain Reigneron and Konstantin Tikhonov, 2014 – 2022)

$$\mathbf{M} = \mathbf{C} + \mathbf{O}\mathbf{B}\mathbf{O}^{\dagger}$$

Randomly Perturbed Matrices

Questions in this talk:

- ➢ How similar are
- • the eigenvectors of a « pure » matrix **C** and those of a noisy observation of **C**? (eigenvalues are well known)
- • the eigenvectors of two independent noisy observations of **C**?
- ➢ So what?

Models of Randomly Perturbed Matrices

(Free) Additive noise

$$\mathbf{M} = \mathbf{C} + \mathbf{O}\mathbf{B}\mathbf{O}^\dagger$$

« Pure system »
« Signal »

« Noise »
B diagonal
O random rotation

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(Free) Multiplicative noise

$$\mathbf{M} = \sqrt{\mathbf{C}}\mathbf{O}\mathbf{B}\mathbf{O}^\dagger\sqrt{\mathbf{C}}$$

« Pure system »
« Signal »

« Noise »
B diagonal
O random rotation

3

Models of Randomly Perturbed Matrices

Additive noise

$$\mathbf{M} = \mathbf{C} + \mathbf{OBO}^\dagger$$

Multiplicative noise

$$\mathbf{M} = \sqrt{\mathbf{C}}\mathbf{OBO}^\dagger\sqrt{\mathbf{C}}$$

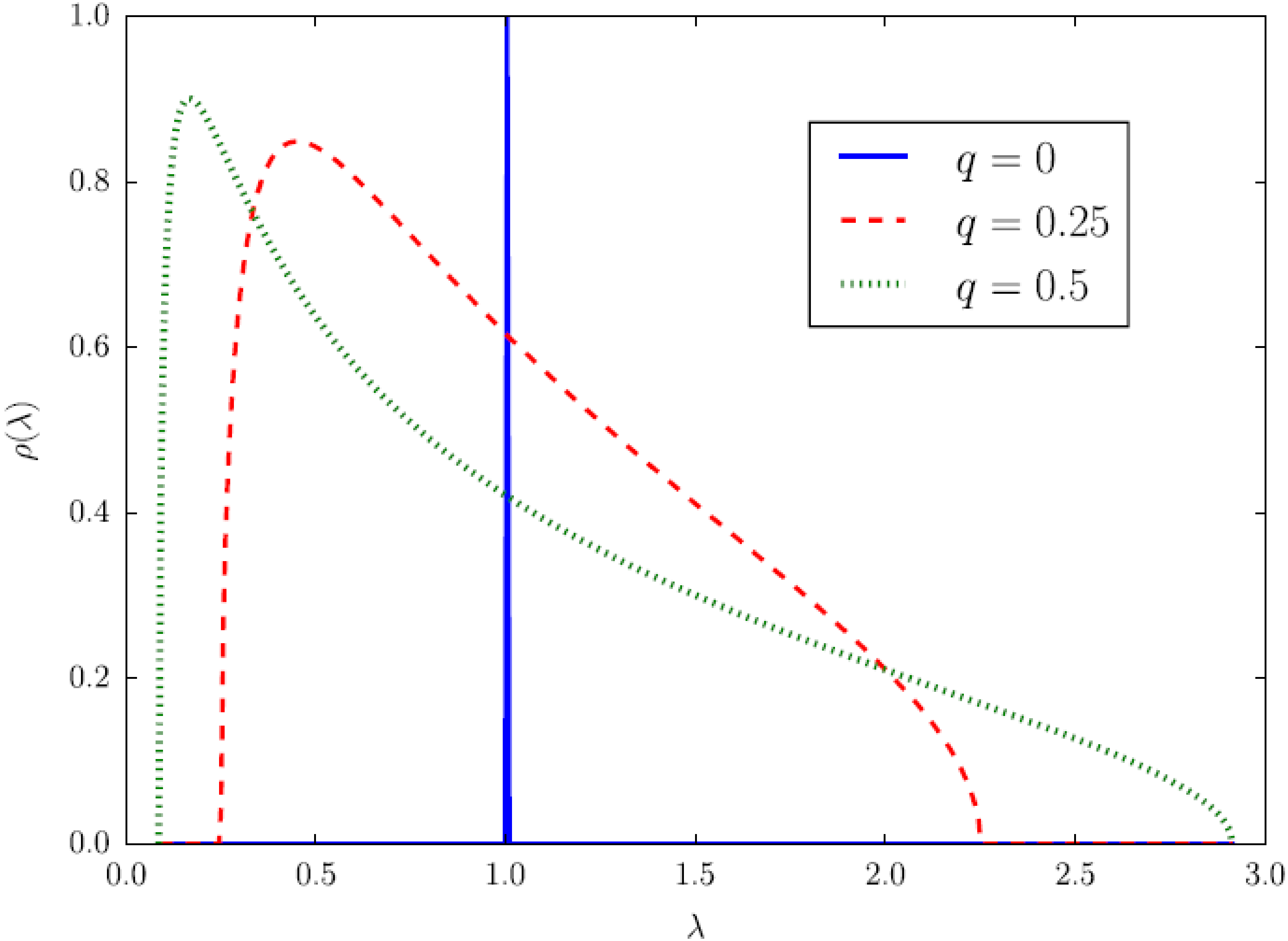
➢ Additive examples:

- Inference of **C** given **M** + an observation noise model, e.g. **B** = **W**(igner)
- Quantum mechanics with a time dependent random perturbation
- Dyson Brownian motion: $\mathbf{OBO}^\dagger$ = **W**(t) Brownian noise
  → stochastic evolution of eigenvalues & eigenvectors

➢ Multiplicative example:

- Empirical **M** vs. « True » covariance matrix **C**;
  $\mathbf{OBO}^t = \mathbf{XX}^t$ = **W**(ishart), where **X** is a N x T white noise matrix

→ The Marcenko-Pastur distribution



$\rho(\lambda)$
λ
$q = 0$
$q = 0.25$
$q = 0.5$
0.0
0.2
0.4
0.6
0.8
1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Object of interest: Overlaps

$$\Phi\left(\lambda_i, c_j\right) := N\mathbb{E}\left[\langle \mathbf{u}_i | \mathbf{v}_j \rangle^2\right]$$

« Overlap »

Eigenvector of **M**

Eigenvector of **C**

Note:

- *N* = size of the matrices, *N* >> 1 in the sequel
- **E**[..]: average over small intervals of λ, of width >> 1/*N*
- The overlaps are quickly of order 1/*N* as a function of the perturbation (« fast » local equilibrium) – but with some remaining structure!

$$d|\psi_i^t\rangle = -\frac{1}{2N}\sum_{j\neq i}\frac{dt}{(\lambda_i(t)-\lambda_j(t))^2}|\psi_i^t\rangle + \frac{1}{\sqrt{N}}\sum_{j\neq i}\frac{dw_{ij}(t)}{\lambda_i(t)-\lambda_j(t)}|\psi_j^t\rangle$$

(Dyson Brownian motion for eigenvectors)

Basic tools

Resolvent:

$$\mathbf{G}_{\mathbf{M}}(z) := (z\mathbf{I}_N - \mathbf{M})^{-1}$$

Stieltjes transform and spectral density (or eigenvalue distribution)

$$\mathrm{Im}\, \mathfrak{g}_{\mathbf{M}}(\lambda - i\eta) \equiv \mathrm{Im}\, \frac{1}{N} \mathrm{Tr}\left[\mathbf{G}_{\mathbf{M}}(\lambda - i\eta)\right] = \pi\, \rho_{\mathbf{M}}(\lambda)$$

Overlaps:

$$\langle \mathbf{v}_i | \mathrm{Im}\, \mathbf{G}_{\mathbf{M}}(\lambda - i\eta) | \mathbf{v}_i \rangle \approx \pi \rho_{\mathbf{M}}(\lambda) \Phi(\lambda, c_i)$$

Note: everywhere the « resolution » $\eta \to 0$ but $\gg 1/N$

Basic tools

R-Transform

$$\mathcal{B}_{\mathbf{M}}(\mathfrak{g}_{\mathbf{M}}(z)) = z. \qquad \mathcal{R}_{\mathbf{M}}(z) := \mathcal{B}_{\mathbf{M}}(z) - \frac{1}{z}$$

e.g. the ***R***-transform of a Wigner matrix is ***R***(z)=σ^2 z

S-Transform

$$\mathcal{T}_{\mathbf{M}}(z) = z\mathfrak{g}_{\mathbf{M}}(z) - 1, \qquad \mathcal{S}_{\mathbf{M}}(z) := \frac{z+1}{z\mathcal{T}_{\mathbf{M}}^{-1}(z)}$$

e.g. the ***S***-transform of a Wishart matrix is ***S***(z)=1/(1+qz) with: **q=N/T**

A Matrix Subordination Law (Allez, Bun, Bouchaud, Potters)

Additive noise

$$\langle \mathbf{G}_{\mathbf{M}}(z) \rangle = \mathbf{G}_{\mathbf{C}}(Z(z))$$

$$Z(z) = z - \mathcal{R}_{\mathbf{B}}(\mathfrak{g}_{\mathbf{M}}(z))$$

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Multiplicative noise

$$z\langle \mathbf{G}_{\mathbf{M}}(z) \rangle = Z(z)\mathbf{G}_{\mathbf{C}}(Z(z))$$

$$Z(z) = z\mathcal{S}_{\mathbf{B}}(z\mathfrak{g}_{\mathbf{M}}(z) - 1)$$

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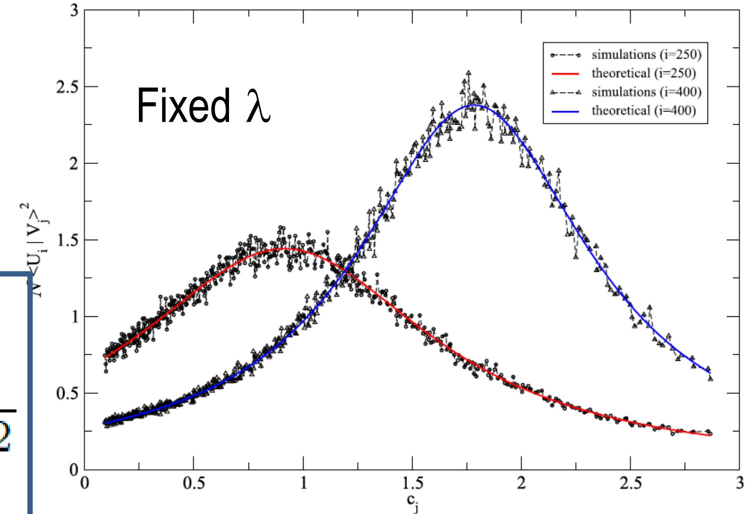
Notes:

- Results obtained using a replica representation of the resolvent + low rank HCIZ
- Taking the trace of these matrix equalities recovers the « free » convolution rules and the corresponding spectra of eigenvalues:

$$\mathcal{R}_{\mathbf{M}}(z) = \mathcal{R}_{\mathbf{C}}(z) + \mathcal{R}_{\mathbf{B}}(z)$$

$$\mathcal{S}_{\mathbf{M}}(u) = \mathcal{S}_{\mathbf{C}}(u)\mathcal{S}_{\mathbf{B}}(u)$$

Overlaps: simplified results (bulk)



Additive noise when **B=Wigner** (cf. Wilkinson)

$$\Phi(\lambda, c) = \frac{\sigma^2}{(c - \lambda + \sigma^2 \mathfrak{h}_{\mathbf{M}}(\lambda))^2 + \sigma^4 \pi^2 \rho_{\mathbf{M}}(\lambda)^2}$$

Notes:

- Tends to a delta function when σ=0 (no noise)
- Cauchy-like formula with a power-law tail for large $|c – \lambda|$ → « Lévy flight »
- Note: True for all « Wigner-like » matrices (not necessarily Gaussian)

Empirical covariance matrices (multiplicative noise)

$$\Phi(\lambda, c) = \frac{qc\lambda}{(c(1-q) - \lambda + qc\lambda \mathfrak{h}_{\mathbf{M}}(\lambda))^2 + q^2 \lambda^2 c^2 \pi^2 \rho_{\mathbf{M}}(\lambda)^2}$$

Notes:

- First obtained by Ledoit & Péché, can be generalized to a broader class of noise
- Tends to a delta function when q=0 (infinite T for a fixed N)

From Overlaps to Rotationally Invariant Estimators

- Assume one has no prior about $\mathbb{C}$
- What is the best L_2 estimator $\boldsymbol{\Xi}(\mathbf{M})$ of $\mathbb{C}$ knowing **M**?
- Without any indication about the directions of the eigenvectors of $\mathbb{C}$, one is stuck with those of **M:**

$$\boldsymbol{\Xi}(\mathbf{M}) = \sum_{i=1}^{N} \xi_i \, |\mathbf{u}_i\rangle \, \langle \mathbf{u}_i|$$

- The L_2 –optimal ξ are in principle given by:

$$\widehat{\xi}_i = \sum_{j=1}^{N} \langle \mathbf{u}_i | \mathbf{v}_j \rangle^2 c_j$$

- Looks silly: the c's and **v**'s are assumed to be unknown!

From Overlaps to Rotationally Invariant Estimators

$$\widehat{\xi}_i = \sum_{j=1}^{N} \langle \mathbf{u}_i | \mathbf{v}_j \rangle^2 c_j$$

- The high dimensional « miracle »

$$\widehat{\xi}_i \underset{N\to\infty}{=} \int c\,\rho_{\mathbf{C}}(c)\,\Phi(\lambda_i, c)\,\mathrm{d}c.$$
$$= \frac{1}{N\pi\rho_{\mathbf{M}}(\lambda_i)} \lim_{z\to\lambda_i - i0+} \operatorname{Im}\operatorname{Tr}\left[\mathbf{G}_{\mathbf{M}}(z)\mathbf{C}\right]$$

- Note : result only depends on the observable **M** !
- Exemple: Wishart
  (Ledoit-Péché)

$$F_2(\lambda) = \frac{\lambda}{(1 - q + q\lambda\mathfrak{h}_{\mathbf{M}}(\lambda))^2 + q^2\lambda^2\pi^2\rho_{\mathbf{M}}^2(\lambda)}$$

- Note : F_2 becomes linear if $\mathbb{C}$ is assumed to be an Inverse-Wishart matrix (conjugate prior) → « Linear shrinkage »

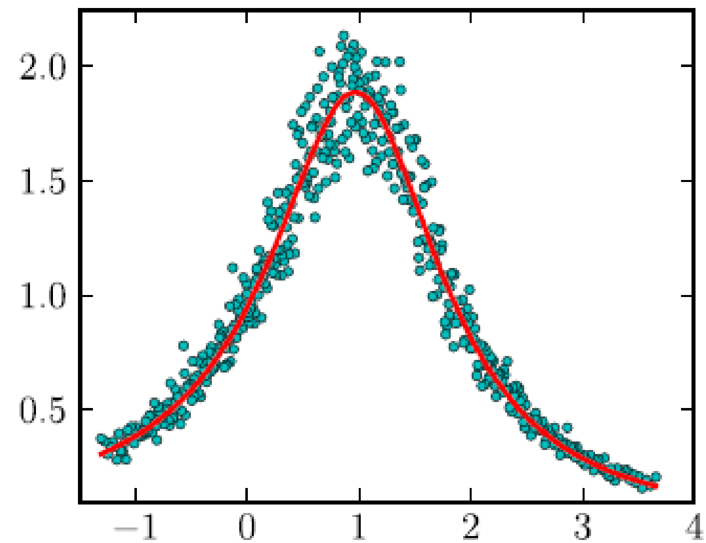
Overlaps between independent realisations

- Extending the above tricks allows us to compute the overlap

$$\Phi(\lambda, \tilde{\lambda}) := N\mathbb{E}\left[\langle \mathbf{u}_\lambda , \tilde{\mathbf{u}}_{\tilde{\lambda}} \rangle^2\right]$$

for *two independent* realisations, e.g. **M** = $\mathbb{C}$ + **W** and $\tilde{\mathbf{M}}$ = $\mathbb{C}$ + $\tilde{\mathbf{W}}$

- The result is cumbersome but <u>explicit</u>, *both for the multiplicative & additive cases* (Bun, Bouchaud, Potters)

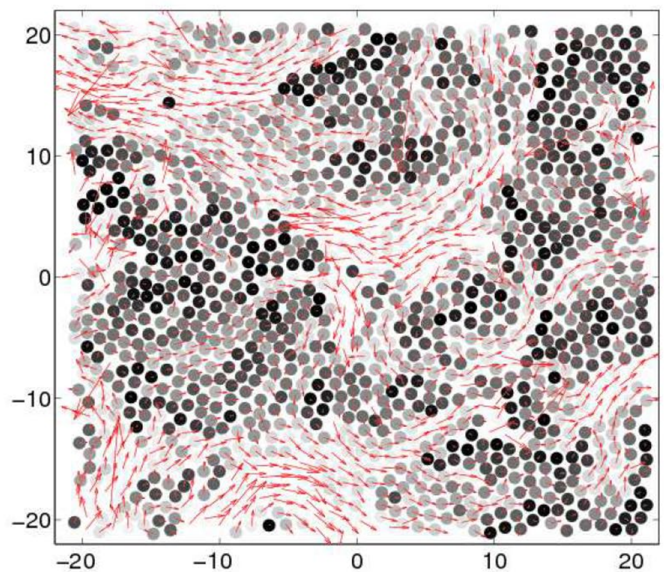


Overlap for a fixed $\tilde{\lambda}$ as a function of λ

- The formula again does **not** depend explicitly on the (possibly unknown) $\mathbb{C}$
- It can be used to test whether **M** and $\tilde{\mathbf{M}}$ originate from the same (unknown) $\mathbb{C}$
- Again, universal within the whole class of Wigner/Wishart like matrices

Overlaps between independent realisations

- The covariance matrix in non-stationary environment
- The Hessian matrix of (slowly) evolving glassy configurations

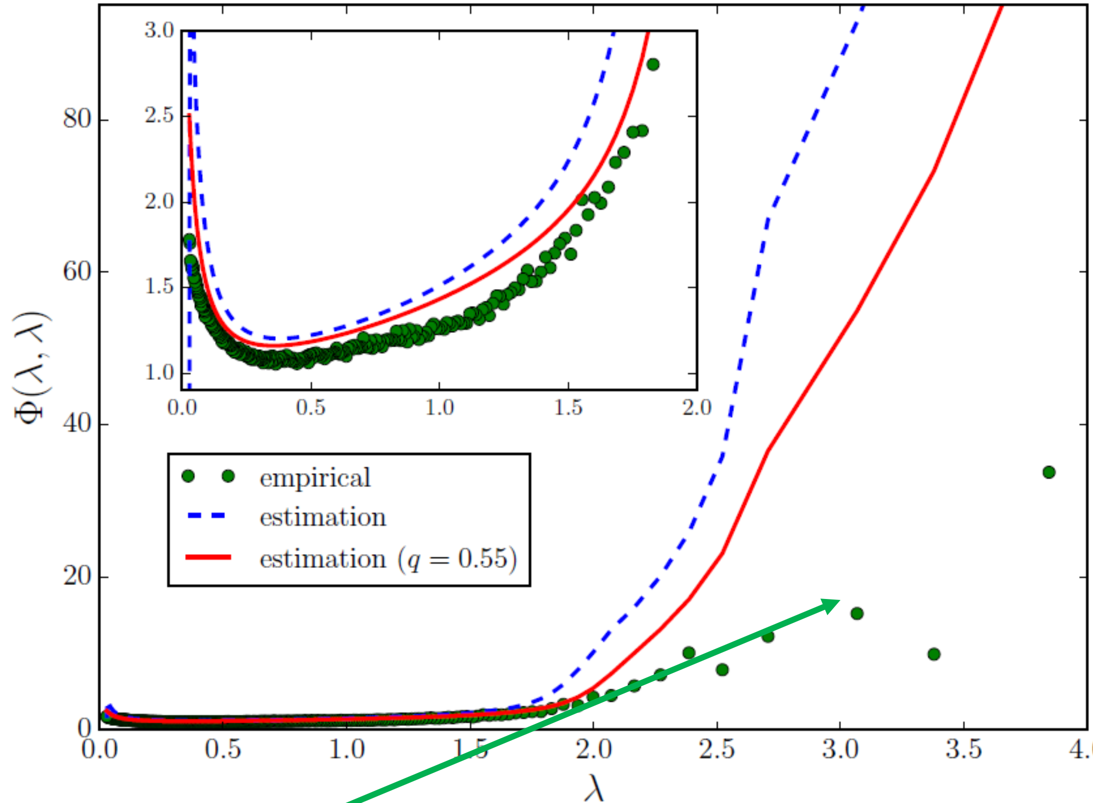


with F. Lechenault, O. Dauchot, G. Biroli

Overlaps between independent realisations

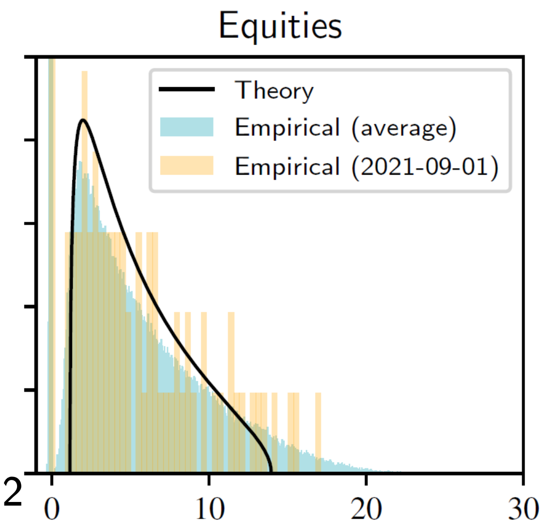
- The case of financial covariance matrices: is the « true » underlying correlation structure stable in time?

(Non overlapping time periods)



- Large eigenvectors are **unstable** (cf. R Allez, JPB and J. Bun, A. Knowles)
- Important for portfolio optimisation (uncontrolled risk exposure to large modes)
- « Eyeballing » test: should be turned into a true statistical test

A simpler, global test: « fleeting modes »



- Is the « true » underlying correlation structure stable in time?

- Consider the N x N matrix $\mathbb{D} = (\mathbb{E}_{\text{in}})^{-1/2}\,\mathbb{E}_{\text{out}}\,(\mathbb{E}_{\text{in}})^{-1/2}$

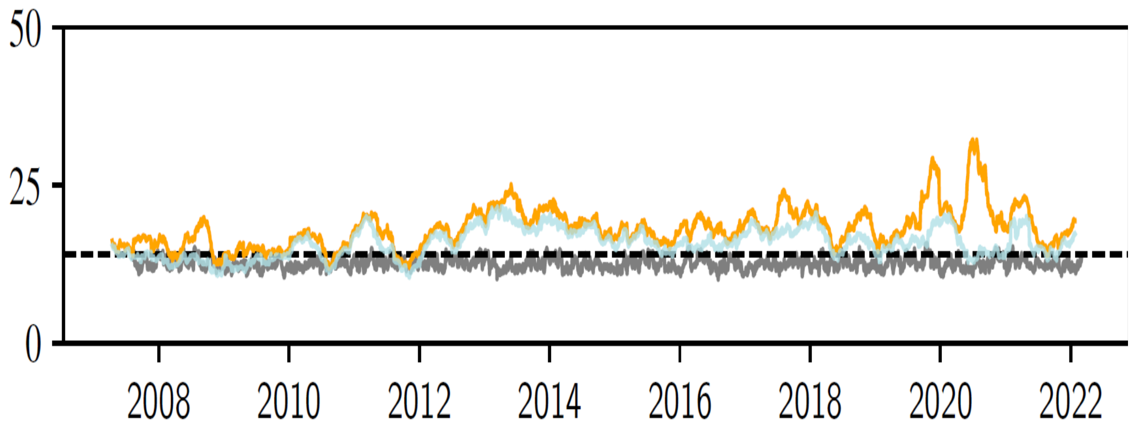
- Where $\mathbb{E}_{\text{in}}$ is the in-sample empirical covariance matrices, defining unit-risk, decorrelated in-sample portfolios

- The eigenvalues/eigenvectors of $\mathbb{D}$ contain relevant information, with max λ's corresponding to maximally over-realizing directions

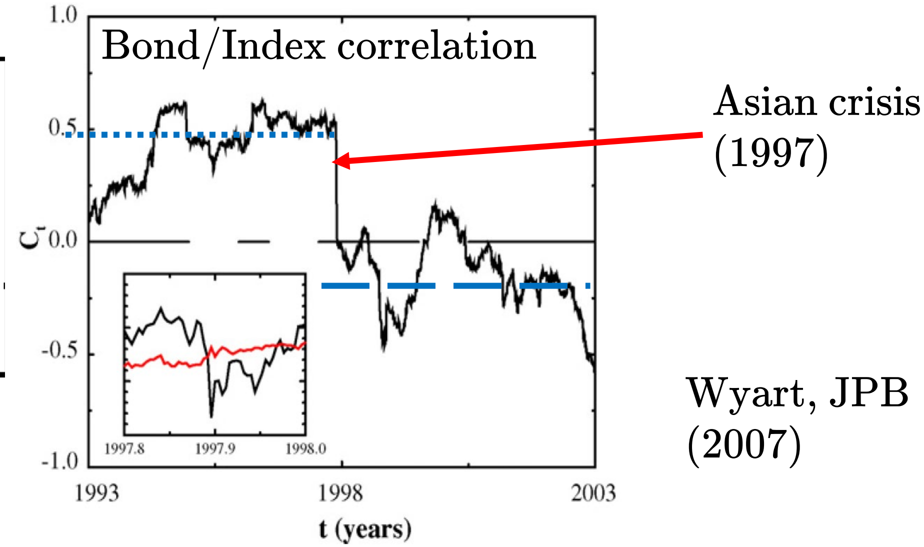
- Null-hypothesis <u>independent</u> of the true covariance matrix $\mathbb{C}$, related to the Jacobi ensemble and only dependent on q_{in} and q_{out}

$$\rho(\lambda) = \frac{1-q_{\text{in}}}{2\pi}\frac{\sqrt{[(\lambda_{\max}-\lambda)(\lambda-\lambda_{\min})]^+}}{\lambda(q_{\text{in}}\lambda + q_{\text{out}})} + \left[1-q_{\text{out}}^{-1}\right]^+\delta(\lambda)$$

IM, MP, KT, JPB



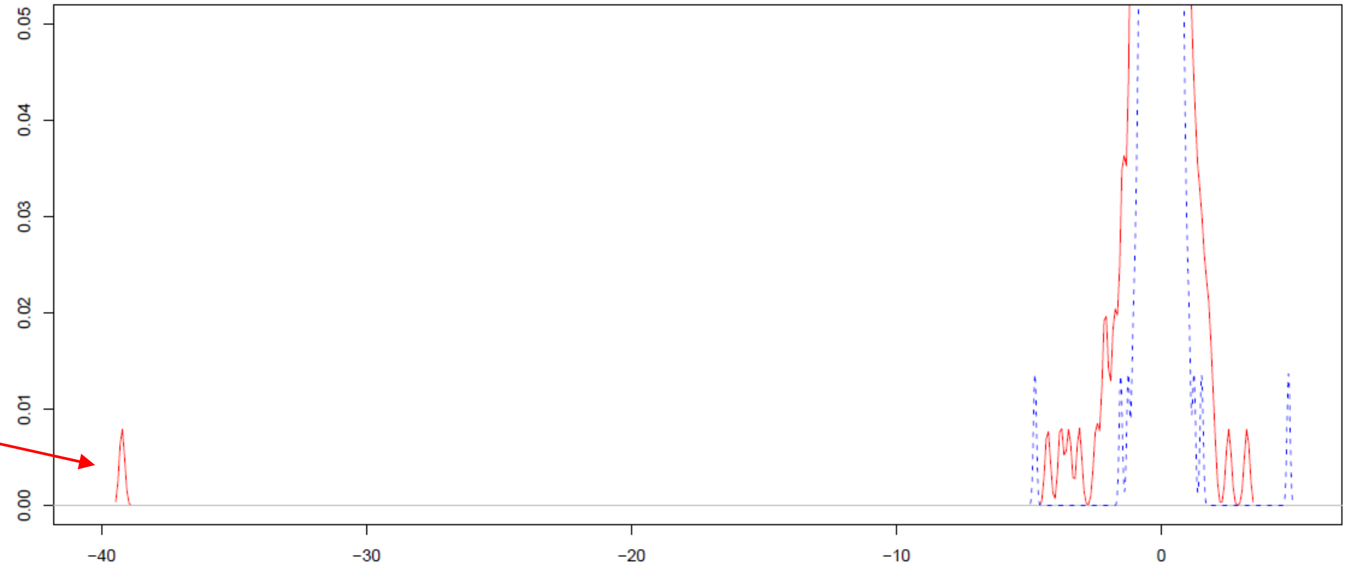
max λ (equities)



Correlations are time dependent

- ➢ What is driving such time dependence?
- Long term evolutions: new firms, evolving business models, macroeconomic effects (e.g. Bond/Index correlation)
- Trading impacts prices → « fleeting modes » reflect traded portfolios (e.g. momentum)
- Behavioural effects, e.g. index I(t) down drives correlations up

Signal: correlations rotate towards (1,1,…1) in down markets



Correlations are time dependent

- ➢ Determining the impact of some macro-variables on correlations
- « Principal Regression Analysis »

$$R_i(t)\, R_j(t) = \mathbb{E}_{ij} + \mathbf{I}(t\text{-}1)\, \mathbb{F}_{ij} + \text{noise}$$

- …and RMT again to the rescue: the significant eigenvalues of $\mathbb{F}$ determine which factors influence correlations

RA, PAR, JPB

- ➢ Free Random Matrices results for Stieltjes transforms can be extended to the full resolvant matrix → access to overlaps
- ➢ Large dimension « miracles »:
- The Oracle estimator can be estimated
- The hypothesis that large matrices are generated from the same underlying matrix $\mathbb{C}$ can be tested without knowing $\mathbb{C}$

Conclusions/Extensions

- Overlaps: a true statistical test at large N ?
- RIE for cross-correlation SVDs (*with* F Benaych & M Potters)
- Overlaps for covariances matrices computed on overlapping periods?
- Beyond RIE? Prior on eigenvectors?
- Other uses of RMT in economics/finance: firm networks (and ecology), complex games, cone-wise linear dynamics….

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Letter

Non-self-averaging Lyapunov exponent in random conewise linear systems

Théo Dessertaine[1,2] and Jean-Philippe Bouchaud[2,3]
[1] *LadHyX UMR CNRS 7646, Ecole polytechnique, 91128 Palaiseau Cedex, France*
[2] *Chair of Econophysics & Complex Systems, Ecole polytechnique, 91128 Palaiseau Cedex, France*
[3] *Capital Fund Management, 23 Rue de l'Université, 75007 Paris, France*

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We consider a simple model for multidimensional conewise linear dynamics around cusplike equilibria. We assume that the local linear evolution is either $\mathbf{v}' = \mathbb{A}\mathbf{v}$ or $\mathbb{B}\mathbf{v}$ (with $\mathbb{A}, \mathbb{B}$ independently drawn from a rotationally invariant ensemble of symmetric $N \times N$ matrices) depending on the sign of the first component of $\mathbf{v}$. We establish strong connections with the random diffusion persistence problem. When $N \to \infty$, we find that the Lyapunov exponent is non-self-averaging, i.e., one can observe apparent stability and apparent instability for the same system, depending on time and initial conditions. Finite N effects are also discussed and lead to cone trapping phenomena.

Note: related to the 3d diffusion persistence