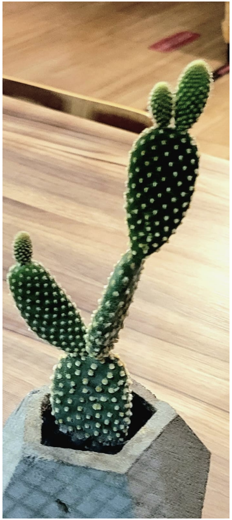


Eigenstate thermalization hypothesis and free probability

Laura Foini

IPhT, CNRS

January 25, 2023

with Silvia Pappalardi and Jorge Kurchan

Phys. Rev. E 99, 042139 (2019); Phys. Rev. Lett. 129, 170603 (2022)

Outline

- ETH and correlations of matrix elements
- From ETH to random matrices (short intermezzo)
- ... and back to ETH

Motivations

ETH wanted to explain why statistical mechanics applies starting from an out-of-equilibrium condition

In this talk: need to characterise better ETH ansatz, all about dynamics *at equilibrium*

Make statistical assumptions which allows to unveil structure of correlations (link with free probability)

General setting

Consider "generic" many-body systems (no conserved quantities, sufficiently high T\energy). E.g.

$$H = \sum_{i=1}^{N} \sigma_i^x \sigma_{i+1}^x + h_z \sum_{i=1}^{N} \sigma_i^z + h_x \sum_{i=1}^{N} \sigma_i^x$$

Physical observable e.g.:

$$A = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^z$$

H and A matrices of size $\mathcal{N} = 2^N$
Interested in the large N (thermodynamic) limit

Properties of many-body Hamiltonians

$$H|E_\alpha\rangle = E_\alpha|E_\alpha\rangle$$

- Spectrum $-Ne_{min} \leq E_\alpha \leq Ne_{max}$
- $\rho(E = Ne) = \sum_{\alpha=1}^{\mathcal{N}} \delta(E - E_\alpha) \propto e^{S(E)} \simeq e^{Ns(e)}$
- Level spacing $E_\alpha - E_{\alpha+1}$ exponentially small in N

Quantum statistical mechanics

$$\langle A \rangle_\beta = \mathrm{Tr}\left[\rho_\beta A\right]$$

Density matrix

$$\rho_\beta = \frac{e^{-\beta H}}{Z} \qquad Z = \sum_{\alpha=1}^{\mathcal{N}} e^{-\beta E_\alpha}$$

- $\rho \geq 0$
- $\mathrm{Tr}\rho = 1$
- Peaked function at some characteristic energy e_β fixed by β

$$\tilde{\rho}(E = Ne) = \sum_{\alpha=1}^{\mathcal{N}} \delta\left(Ne - E_\alpha\right) e^{-\beta E_\alpha} \propto e^{N(s(e)-\beta e)}$$

Dynamics

Heisenberg picture (evolution of the operators):

$$A(t) = e^{iHt} A e^{-iHt} = \sum_{\alpha\beta} e^{i(E_\alpha - E_\beta)t} A_{\alpha\beta} |E_\alpha\rangle\langle E_\beta|$$

$$A_{\alpha\beta} = \langle E_\alpha | A | E_\beta \rangle$$

Dynamical correlation functions:

$$\mathrm{Tr}\left[\rho_\beta A(t) A(0)\right] \underbrace{=}_{\text{Time translation invariance}} \mathrm{Tr}\left[\rho_\beta A(t+\tau) A(\tau)\right]$$

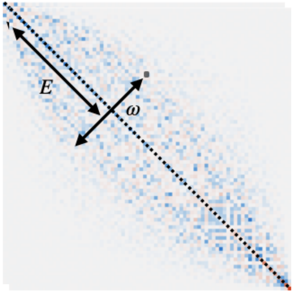
Eigenstate thermalization

Single eigenstates provide equilibrium statistical averages
$\langle E_\alpha | A | E_\alpha \rangle$ varies smoothly with the energy E_α

For dynamics necessary off-diagonal matrix elements

J. Deutsch (1991), M. Srednicki (1994)
Review: D'Alessio, Kafri, Polkovnikov, Rigol (2016)
Mathematical literature on Quantum Unique Ergodicity

Eigenstate thermalization ansatz



$$A_{\alpha\beta} = \mathscr{A}(e)\delta_{\alpha\beta} + e^{-Ns(e)/2} f_e(\omega) R_{\alpha\beta}$$

$$E = (E_\alpha + E_\beta)/2 \quad e = E/N \quad \omega = E_\alpha - E_\beta$$

$R_{\alpha\beta}$ (pseudo)-random numbers

$$\overline{R_{\alpha\beta}} = 0 \quad \overline{R^2_{\alpha\beta}} = 1$$

M. Srednicki (1999)

Fictitious ensemble

$$A_{\alpha\beta} \to \text{random matrix element}$$

Ensemble ?

- Small energy windows
- Perturb with "reasonable" small Hamiltonian $H \to H + \epsilon V$ (Deutsch (1991)). Nearby eigenvectors extremely sensitive even to small perturbations. Physics unchanged

One-time correlation functions

$$\langle A \rangle_\beta = \sum_\alpha \frac{e^{-\beta E_\alpha}}{Z} A_{\alpha\alpha} \simeq N \int_{-e_{min}}^{e_{max}} \mathrm{d}e \, \underbrace{\frac{e^{N(s(e)-\beta e)}}{Z}}_{\text{peaked}} \quad \underbrace{\mathscr{A}(e)}_{\substack{\text{neglect exponentially} \\ \text{small fluctuations}}}$$

$$\sum_\alpha \to \int \mathrm{d}E \rho(E)$$

Two-time correlation functions

$$\langle A(t)A(0)\rangle_\beta - \langle A\rangle_\beta^2 = \frac{1}{Z}\sum_{\alpha\neq\beta} e^{-\beta E_\alpha} e^{i(E_\alpha - E_\beta)t} |A_{\alpha\beta}|^2$$

$$\xrightarrow[N\to\infty]{} \int \mathrm{d}\omega \; e^{-\beta\omega/2} e^{i\omega t} |f_{e_\beta}(\omega)|^2$$

$|A_{\alpha\beta}|^2$ averaged over the ensemble

Two-time correlation functions

$$\langle A(t)A(0)\rangle_\beta - \langle A\rangle_\beta^2 = \frac{1}{Z}\sum_{\alpha\neq\beta} e^{-\beta E_\alpha} e^{i(E_\alpha - E_\beta)t} |A_{\alpha\beta}|^2$$

$$\xrightarrow[N\to\infty]{} \int \mathrm{d}\omega \; e^{-\beta\omega/2} e^{i\omega t} |f_{e_\beta}(\omega)|^2$$

$|A_{\alpha\beta}|^2$ averaged over the ensemble

Two-point function independent of correlations between different matrix elements. Sensitive only to their variance

Multi-point correlation functions

$$C_4^\beta(t_1, t_2, t_3) = \mathrm{Tr}\left[\rho_\beta A(t_1) A(t_2) A(t_3) A(0)\right]$$

$C_4^\beta(t, 0, t)$ Out-of-Time-Order Correlator

"quantum Lyapunov exponent"

Larkin and Ovchinikov (1969)

Maldacena, Shenker and Stanford (2016)

Multi-point correlation functions

In the energy eigenbasis

$$C_p^\beta(t_1,\ldots,t_{p-1}) = \sum_{\alpha_1,\ldots,\alpha_p}\left[\frac{e^{-\beta E_\alpha}}{Z}A_{\alpha_1\alpha_2}(t_1)A_{\alpha_2\alpha_3}(t_2)\ldots A_{\alpha_p\alpha_1}(0)\right]$$

For any $p>2$ products of different matrix elements!

Argument for correlations

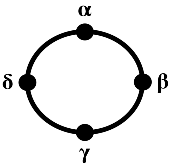
$|f_e(\omega)|^2$ Fourier transform of $C_2^\beta(t)$

$A_{\alpha\beta}$ independent variables $\rightarrow$ all multi-point functions determined solely by $f_e(\omega)$, i.e. by $C_2^\beta(t)$

Unreasonable in general

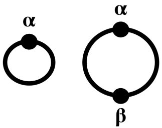
Beyond independent matrix elements

One should consider multipoint functions



$$\overline{A_{\alpha\beta}A_{\beta\gamma}A_{\gamma\delta}A_{\delta\alpha}}$$

$$\propto f_e^{(4)}(\omega_1,\omega_2,\omega_3) \to C_4^{\beta}(t_1,t_2,t_3)$$



In the same spirit as usual ETH

$$f_e^{(1)} = \mathscr{A}(e) \qquad f_e^{(2)}(\omega) = |f_e(\omega)|^2$$

Generalized ETH

$$\overline{A_{\alpha_1\alpha_2}A_{\alpha_2\alpha_3}\dots A_{\alpha_n\alpha_1}} = e^{-(n-1)Ns(e)} f_e^{(n)}(\omega_1,\dots,\omega_{n-1})$$

for $\alpha_1 \neq \alpha_2 \dots \neq \alpha_n$

$$e = \frac{1}{n}\sum_{i=1}^{n} e_{\alpha_i} \qquad \omega_i = E_{\alpha_i} - E_{\alpha_{i+1}}$$

+ other assumptions discussed later

Foini and Kurchan (2019)

Outline

- ETH and correlations of matrix elements
- From ETH to random matrices (short intermezzo)
- ... and back to ETH

Random matrix behavior “on small energy scales"

A small perturbation, large compared to the level spacing will mix nearby vectors. No change in physics

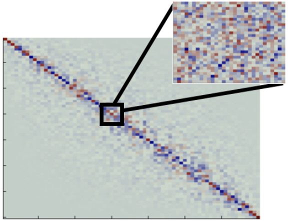
J. Deutsch (1991)

Invariance under change of basis (on small scales)

Invariance under small rotations

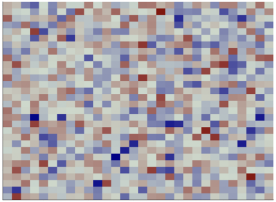
$O_{\alpha\beta} = \langle E'_\alpha | E_\beta \rangle$ → Banded random unitary

U =



Let's go simpler → full rotational invariance $\mathscr{P}(A) = \mathscr{P}(U^\dagger A U)$

U =



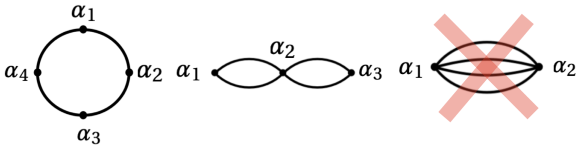
Free cumulants

$$N^3 \overline{A_{\alpha_1\alpha_2} A_{\alpha_2\alpha_3} A_{\alpha_3\alpha_4} A_{\alpha_4\alpha_1}} = \kappa_4$$

for $\alpha_1 \neq \alpha_2 \neq \alpha_3 \neq \alpha_4$

B. Collins, J. A. Mingo, P. Śniady, R. Speicher, arXiv:math/0606431 (2006)
Maillard et al, J. Stat Mech. (2019)

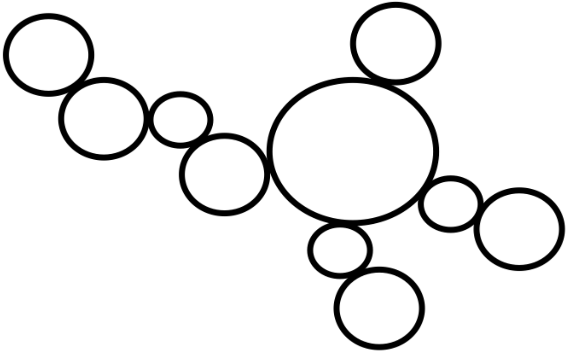
Diagrams



$$E(G) = N^{V-1}\overline{A_{\alpha_1\alpha_2}A_{\alpha_2\alpha_3}A_{\alpha_3\alpha_4}A_{\alpha_4\alpha_1}}$$

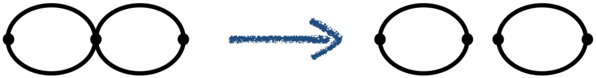
To compute moments only cactus matter

A cactus



(of high order ...)

Cactus diagrams



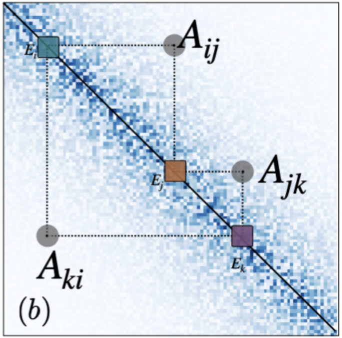
$$E(G_{\text{Cactus}}) = \prod_{i=1}^{\#loops} \kappa_{n_i}$$

Maillard et al, J. Stat Mech. (2019)

Outline

- ETH and correlations of matrix elements
- From ETH to random matrices (short intermezzo)
- ... and back to ETH

Invariance under small rotations



- Only diagrams which contain cycles survive (phases have to cancel)
- Structure in energy

Generalized ETH

$$\overline{A_{\alpha_1\alpha_2}A_{\alpha_2\alpha_3}\dots A_{\alpha_n\alpha_1}} = e^{-(n-1)Ns(e)} f_e^{(n)}(\omega_1,\dots,\omega_{n-1})$$

for $\alpha_1 \neq \alpha_2 \dots \neq \alpha_n$

+ other assumptions. Which ones?

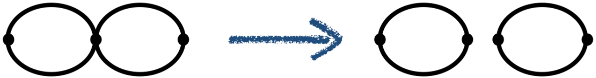
Foini and Kurchan (2019)

Counting

Some assumptions on scaling of correlations

→ Keep only cactus diagrams as before and split them

$$\overline{A_{\alpha_1\alpha_2}\dots A_{\alpha_{k-1}\alpha_1}A_{\alpha_1\alpha_{k+1}}\dots A_{\alpha_n\alpha_1}} = \overline{A_{\alpha_1\alpha_2}\dots A_{\alpha_{k-1}\alpha_1}}\;\overline{A_{\alpha_1\alpha_{k+1}}\dots A_{\alpha_n\alpha_1}}$$



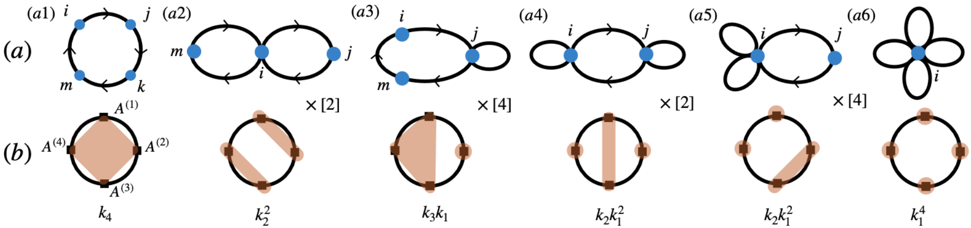
Multi-point correlation functions

In the energy eigenbasis

$$C_4^{\beta}(t_1, \ldots, t_3) = \mathrm{Tr}\left[\frac{e^{-\beta H}}{Z} A(t_1) A(t_2) A(t_3) A(0)\right]$$

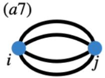
$$= \sum_{\alpha_1, \ldots, \alpha_4} \left[\frac{e^{-\beta E_{\alpha_1}}}{Z} A(t_1)_{\alpha_1 \alpha_2} A(t_2)_{\alpha_2 \alpha_3} A(t_3)_{\alpha_3 \alpha_4} A(0)_{\alpha_4 \alpha_1}\right]$$

Cactus diagrams and non-crossing partitions



- Each edge carries an index of time (as different matrices)
- ! No cyclic invariance

Neglect



Moments and cumulants

$$C_n^\beta(t_1,\ldots,t_{n-1},0) = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H}A(t_1)\ldots A(t_{n-1})A(0)\right]$$

$$\kappa_n^\beta(t_1,\ldots,t_{n-1},0) = \frac{1}{Z}\sum_{\alpha_1\neq\alpha_2\neq\ldots\neq\alpha_n} e^{-\beta E_{\alpha_1}}A(t_1)_{\alpha_1\alpha_2}\ldots A(t_{n-1})_{\alpha_{n-1}\alpha_n}A(0)_{\alpha_n\alpha_1}$$

An example

From FP it is immediate to find the following decomposition:

$$\langle A(t_1)A(t_2)A(t_3)A(0)\rangle_\beta = \kappa_4^\beta(t_1,t_2,t_3) + \kappa_2^\beta(t_1-t_2)\,\kappa_2^\beta(t_3) + \kappa_2^\beta(t_2-t_3)\,\kappa_2^\beta(t_1)$$

(assuming $\langle A\rangle_\beta = 0$)

Free cumulants and ETH

$$
\begin{aligned}
\kappa_n^\beta(t_1,\dots,t_{n-1},0) &= \frac{1}{Z}\sum_{\alpha_1\neq\alpha_2\neq\ldots\neq\alpha_n} e^{-\beta E_{\alpha_1}} A(t_1)_{\alpha_1\alpha_2}\dots A(0)_{\alpha_n\alpha_1} \\
&= \int \mathrm{d}\omega_1\ldots\mathrm{d}\omega_{n-1} e^{i\vec{\omega}\cdot\vec{t}} e^{-\beta\vec{\omega}\cdot\vec{l}_n} f_{\epsilon_\beta}^{(n)}(\omega_1,\dots,\omega_{n-1})
\end{aligned}
$$

$$\vec{l}_n = \left(\tfrac{n-1}{n},\dots,\tfrac{1}{n}\right)$$

Conclusions

- Propose a (simple) ansatz able to account for correlations between matrix elements. Relevant for multi-point functions
- Recognise importance of free probability in connection with quantum correlation functions (see Denis's talk)

Conclusions

- Propose a (simple) ansatz able to account for correlations between matrix elements. Relevant for multi-point functions
- Recognise importance of free probability in connection with quantum correlation functions (see Denis's talk)

Thank you!