Fluctuations in the optimal matching problem

Martin Huesmann

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Mathematics Münster
Cluster of Excellence

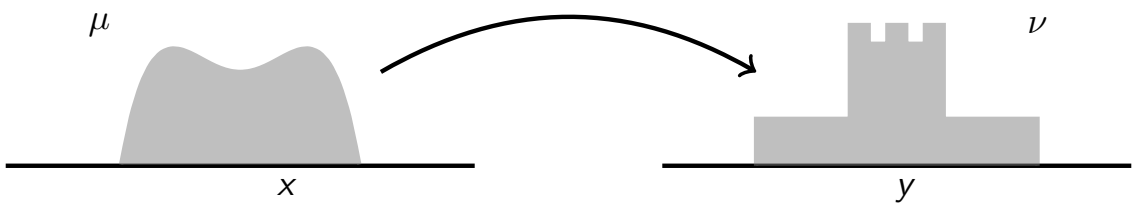
joint work with Michael Goldman

The optimal transport problem

Given $\mu, \nu \in \mathcal{P}(\mathsf{X})$, a cost function $c : \mathsf{X} \times \mathsf{X} \to \mathbb{R}$ the optimal transport problem is

$$P_c(\mu, \nu) := \inf_{q \in \mathsf{Cpl}(\mu,\nu)} \int c(x, y) \, dq(x, y),$$

where $\mathsf{Cpl}(\mu, \nu) = \{q \in \mathcal{P}(\mathsf{X} \times \mathsf{X}) : q(A \times \mathsf{X}) = \mu(A), q(\mathsf{X} \times A) = \nu(A)\}$.



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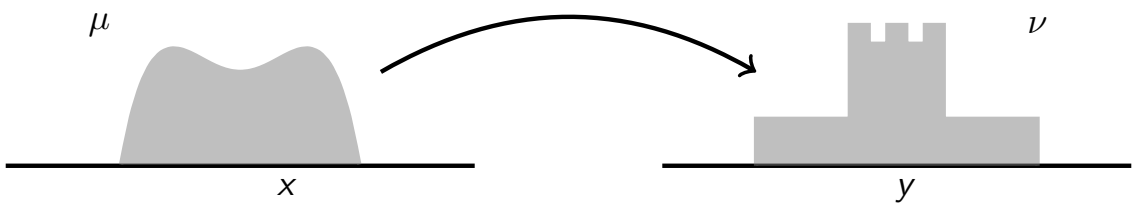

For (X, d) metric space, $c = d^p$, $p \geq 1$,

$$W_p^p(\mu, \nu) := P_c(\mu, \nu)$$

is called the L^p (Kantorovich)-Wasserstein distance.

The optimal matching problem...

... is the optimal transport problem on $\mathbb{R}^d$, with $c(x,y) = |x-y|^p$, between $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$ and μ, where $(X_i)_{i=1}^n$ are iid with law μ.

We concentrate on the model case: $\mu = \mathrm{Leb}_{|[0,1]^d}$.

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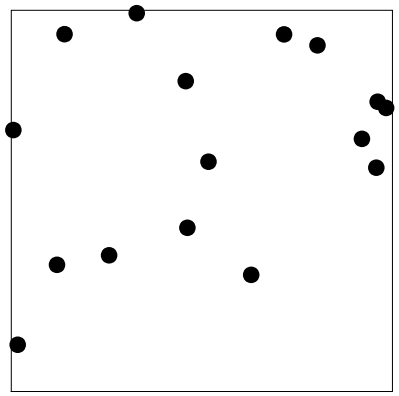

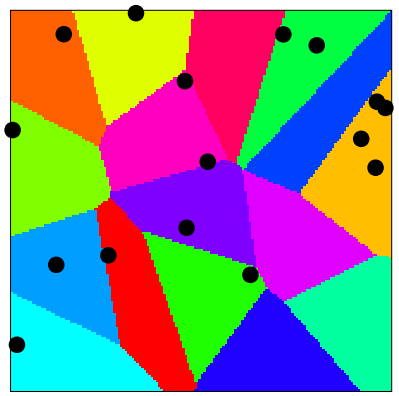

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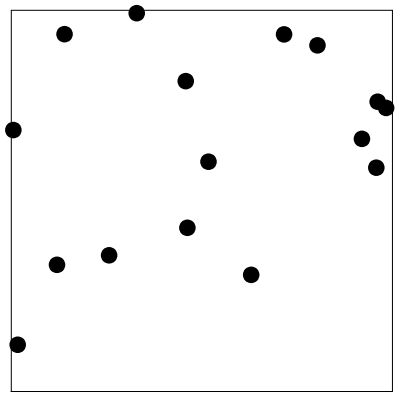
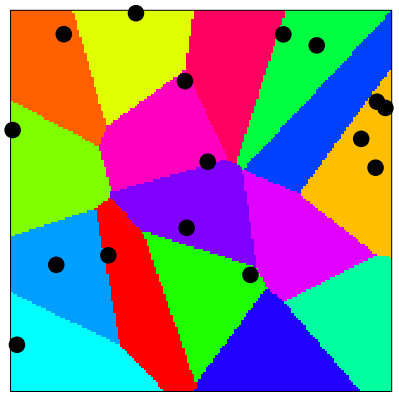
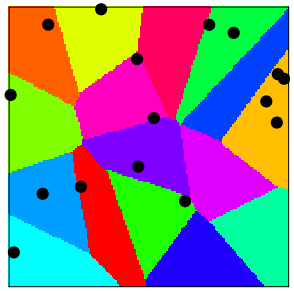




- wide interest e.g. Computer science (Ajtai et al. '84), probability (Talagrand since '92), analysis (Ambrosio since '16), physics (Parisi since '14), statistics and many (!) more
- various variants: bipartite matching, Markov chains, occupation measures, entropic optimal transport...

Macroscopic behaviour – rate of convergence

For model case $\mu = \mathrm{Leb}_{|[0,1]^d}$ and $p \geq 1$:



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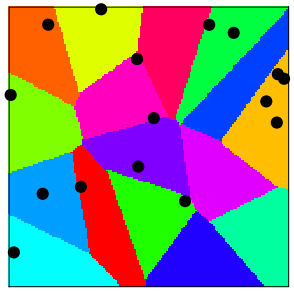

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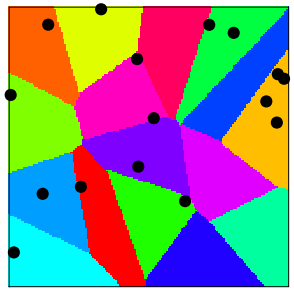

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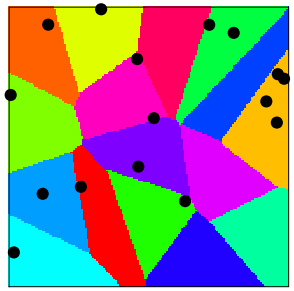

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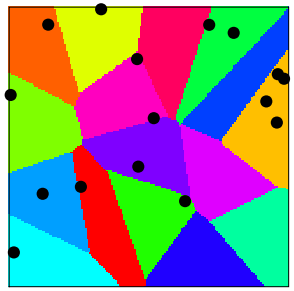

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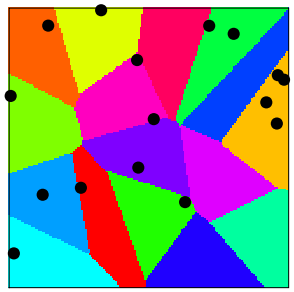

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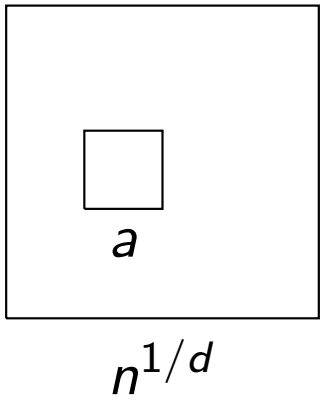
fix cube of sidelenght a, # pts: $a^d \pm a^{d/2}$, surface area: a^{d-1}



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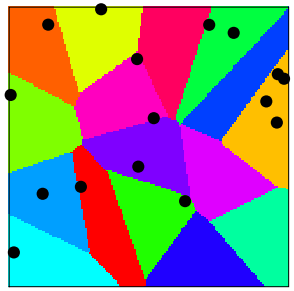

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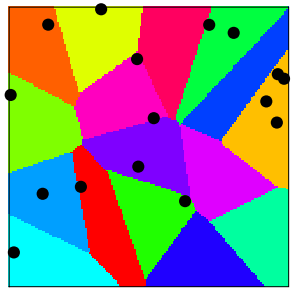

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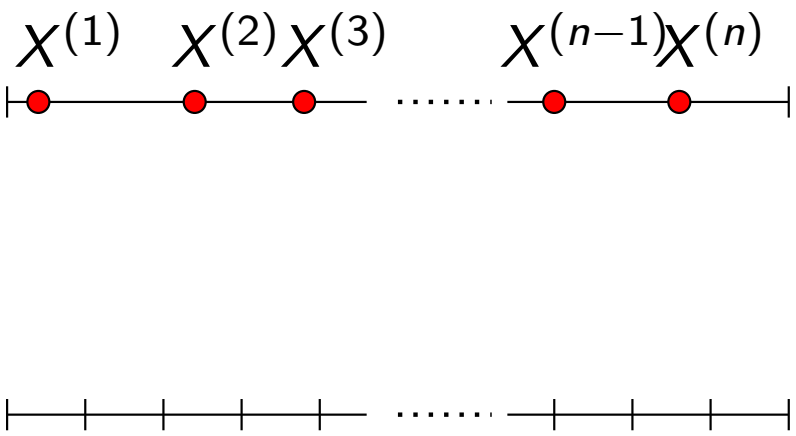
Refined statements:

- Convergence of rescaled cost?
- Mesoscopic behaviour?
- Thermodynamic limit?
- ...

$d = 1$ and $p > 1$



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optimal coupling: $X^{(i)} \mapsto \mu_{|\left[\frac{(i-1)}{n}, \frac{i}{n}\right)}$

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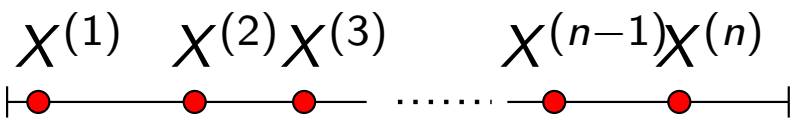

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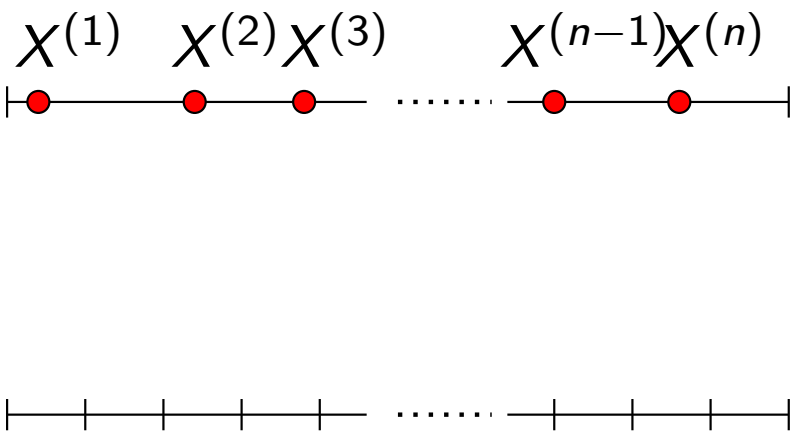

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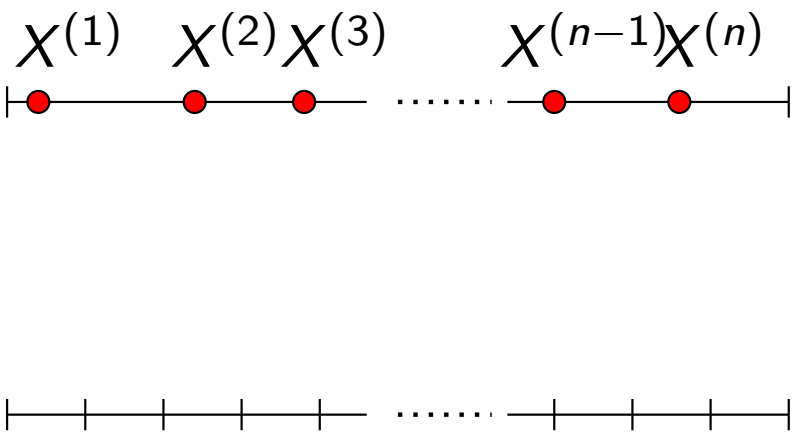

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Previous results for $d \geq 2$

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- Rescaled cost: $\lim_{n\to\infty} n^{p/d}\mathbb{E}W_p^p\left(\frac{1}{n}\sum_{i=1}^n \delta_{X_i}, \mu\right)$ exists (for $2p < d$ Barthe & Bordenave '13, Dereich & Scheutzow & Schottstedt '13 , for $p \geq 1$ Goldman & Trevisan '20)
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Main result – convergence on mesoscopic scales

For $p = 2$ and $d \in \{2, 3\}$ on all mesoscopic scales the averaged displacement of the optimal matching converges to a curl-free GFF.

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$X_1, X_2, \ldots$ iid uniform on torus $Q_L = [-L/2, L/2]^d$

$$\mu^{R,L} = \frac{1}{R^d} \sum_{i=1}^{(RL)^d} \delta_{X_i}, \quad \pi^{R,L} : L^2 \text{ opt. cpl. } \mu^{R,L} \leftrightarrow \mathrm{Leb}_{|\mathrm{Q_L}}$$

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$$Z^{R,L}(f) = R^{\frac{d}{2}} \int f(x) \cdot (y - x) \pi^{R,L}(dx, dy).$$

If T is optimal map, and $A_i = T^{-1}(X_i)$ then

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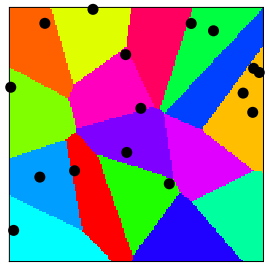

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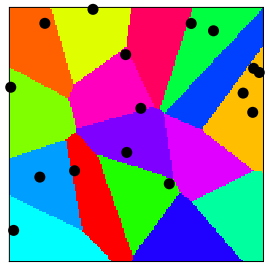

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Let W be white noise and (formally) define $\nabla \Psi$ via

$$\Delta \Psi = W.$$

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Theorem (Goldman & H. '21)

For $d = 3$, and any sequences $R, L \to \infty$

$$Z^{R,L} \to \nabla\Psi$$

weakly in some negative Sobolev space.

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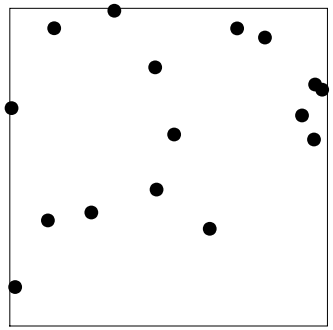
macro scale: $L = 1, R \to \infty$
micro scale: $L \to \infty, R = 1$
meso scale: $L, R \to \infty$



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$W^{R,L} = R^{\frac{d}{2}}(\mu^{R,L} - 1)$ let $u^{R,L}$ be Q_L-periodic solution to

$$\Delta u = W^{R,L}, \quad \int_{Q_L} u = 0.$$

Put $\nabla u_1^{R,L} = \int \eta \nabla u^{R,L}$, η smooth cutoff.

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Theorem (GH '21)

For $d = 2$, and any sequences $R, L \to \infty$

$$Z^{R,L} - \mu^{R,L} \nabla u_1^{R,L} \to \nabla\Psi - \nabla\Psi_1$$

weakly in some negative Sobolev space. Moreover, for any $p \geq 2$

$$W_p\left(\mathrm{Law}(\nabla u_1^{R,L}), \mathcal{N}(0, \sigma^2 \mathrm{Id})\right) \lesssim \frac{1}{R \log^{\frac{1}{p}} L},$$

with $\sigma^2 \sim \log L$.

Idea of the proof:

$$\mu_n \det \nabla^2 \psi = 1 \quad \overset{\text{pathwise}}{\approx} \quad \Delta u = 1 - \mu_n \quad \overset{\text{in law}}{\approx} \quad \Delta \Psi = W$$

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Step 1: Linear part.

Recall: $W^{R,L} = R^{\frac{d}{2}}(\mu^{R,L} - 1), \quad \Delta u^{R,L} = W^{R,L}$
W white noise, $\Delta \Psi = W$

Show: $\nabla u^{R,L} \to \nabla \Psi$ in law

Step 2: Quantitative linearization.

Show: average displacement under $\pi^{R,L}$ is close to $\nabla u^{R,L}$

Approach: a) deterministic estimate b) stochastic input to check assumption of analytic estimate

Main analytical input – quant. lin. of Monge-Ampère eq.

Let $\beta(R) = \log(R+1)$ or $\beta(R) = R^\alpha, \alpha \in [0,1)$.

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Let $\bar{R} > 1$ and μ be positive measure s.t. $\mu(B_{\bar{R}}) = |B_{\bar{R}}|$. Assume that

$$\frac{1}{|B_R|} W_2^2(\mu_{|B_R}, \kappa_R \mathrm{Leb}_{|B_R}) \leq \beta(R), \quad \forall\ R \in [1, \bar{R}].$$

Define ϕ via

$$\begin{cases} \Delta\phi = 1 - \mu & \text{in } B_{\bar{R}} \\ \nu \cdot \nabla\phi = 0 & \text{on } \partial B_{\bar{R}} \end{cases}.$$

If q is L^2 optimal coupling between $\mu_{|B_{\bar{R}}}$ and $\mathrm{Leb}_{|\bar{R}}$, then for $\eta \in C_c^\infty(B_1), \eta_R(x) := \frac{1}{R^d}\eta(x/R)$ there holds

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Thanks for your attention

Rigorous definition of $\nabla\Psi$

For $d = 3$, $\nabla\Psi$ is a random distribution such that for $f \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$,

$$\mathrm{Law}(\nabla\Psi(f)) = \mathcal{N}(0, \int \phi^2),$$

where ϕ is the unique $L^2(\mathbb{R}^d)$ solution to

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Formally:

$$\int \nabla\Psi(f) = -\int \Psi(\nabla \cdot f) = \int \Psi\Delta\phi = \int \Delta\Psi\phi = \int W\phi.$$

Relation of $\nabla\Psi$ to GFF

Let $W_{i,j}$ be independent copies of white noise, $W_i = (W_{i,1}, \ldots, W_{i,d})$ and define the vector valued GFF $h = (h_1, \ldots, h_d)$ via

$$\Delta h_i = \nabla \cdot W_i, \quad 1 \leq i \leq d$$

Then we have

$$\Delta\Psi = \nabla \cdot h$$

so that $\nabla\Psi$ corresponds to the curl-free part of h.