Percolation of random fields excursions

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Excursions

- Let $f : \mathbb{R}^2 \to \mathbb{R}$ a stationary random field (law invariant under translations).
- For $\ell \in \mathbb{R}$, define

$$\mathcal{E}_{\ell} = \mathcal{E}_{\ell}(f) = \{x \in \mathbb{R}^2 : f(x) \ge \ell\}$$

Figure – White : excursion of a shot noise field (Credit : *PhD Thesis, Antoine Lerbet*)



Percolation

We are interested in the following questions :

- **O** Does \mathcal{E}_{ℓ} have (a unique) unbounded connected component(s)?
- 2 Is there a critical value ℓ_c ?
- Behaviour of

 $\mathbb{P}(\mathcal{E}_{\ell} \text{ crosses large rectangles})$

for $\ell = \ell_c$ or $\ell \neq \ell_c$?

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Poisson shot noise fields

- Let $\mathcal{P} = \{x_i; i \in \mathbb{N}\}$ be a homogeneous Poisson process on \mathbb{R}^2 .
- Let $g: \mathbb{R}^2 \to \mathbb{R}^2$ integrable
- Poisson shot noise field with kernel g :

$$f(x) := \sum_{i \in \mathbb{N}} g(x - x_i); x \in \mathbb{R}^2.$$

Continuous limit



Applications

• Telecommunications, image analysis/synthesis, cognitive sciences...



Figure - Optical illusion - Credit : PNAS, 110 (17) 7080-7085 , https://www.youtube.com/watch?v=QslBXBHvReg

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Gaussian Random Fields

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The same questions have been thouroughly investigated for stationary continuous centred **Gaussian fields**, i.e. random functions of the form

$$\begin{aligned} &(x) = (g \star \underbrace{\mathcal{W}_{\mathcal{G}}}_{\substack{\text{Gaussian} \\ \text{white noise}}})(x) \\ &= \lim_{\varepsilon \to 0} \sum_{i \in \varepsilon \mathbb{Z}^d} \left[g(i-x) \times \underbrace{\mathcal{G}_{i,\varepsilon}}_{\mathcal{N}(0,\varepsilon^2) \text{ Independent}} \right] \end{aligned}$$

where g is smooth and integrable

• Formally, a Gaussian centred field is a random function f such that,

 $\forall x_1, \ldots, x_n, (f(x_1), \ldots, f(x_n))$ is a Gaussian vector

• The law of f is characterised by the covariance function

$$C(x,y) = \operatorname{Cov}(f(x),f(y))$$

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Covariance property of White Noise $\ensuremath{\mathcal{W}}$

• For A, B with finite measure,

$$\operatorname{Cov}(\mathcal{W}(A),\mathcal{W}(B)) = \mathcal{L}^d(A\cap B) = \langle \mathbf{1}_{\{A\}},\mathbf{1}_{\{B\}}\rangle_{L^2(\mathbb{R}^d)}$$

• For all $g_1,g_2\in L^2(\mathbb{R}^d)$

$$\mathsf{Cov}\left(\int g_1 d\mathcal{W}, \int g_2 d\mathcal{W}\right) = \langle g_1, g_2 \rangle = \int g_1 g_2.$$

• In particular, the covariance function of f satisfies

$$C(x - y) = \operatorname{Cov}(f(x), f(y)) = \langle g(x - \cdot), g(y - \cdot), \rangle$$
$$= \int g(x - y)g(x - y - z)dz$$
$$= g\tilde{\star}g(x - y)$$

• Some SDP functions with singular spectral measures cannot be built this way (e.g. Gaussian Random Planar Wave with C = Bessel Function)

Percolation of Gaussian excursions

Figure – Credit : D. Beliaev



FIGURE 1. A simulation of the excursion set \mathcal{E}_{ℓ} of the Bargmann-Fock field restricted to a large square (in grey) at (i) the zero level $\ell = 0$ (leff figure), at (ii) the level $\ell = 0.1$ (right figure), with the connected component of greatest area distinguished (in black). The Bargmann-Fock field is the stationary, centred Gaussian field with covariance kernel $\kappa(x) = e^{-|x|^2/2}$. Credit: Dmitry Beliaev.

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Sharp phase transition (Gaussian case)

Theorem (Beffara & Gayet, Vanneuville, Muirhead, Rivera; 17' - 21') *Assume*

$$f = g \star \mathcal{W}_{\mathcal{G}}$$

is a "nice" Gaussian field. Then $\{\mathcal{E}_{\ell}, \ell \in \mathbb{R}\}$ behaves like **Bernoulli** percolation around the critical value :

($\ell < 0$: \mathcal{E}_{ℓ} has a unique unbounded component a.s. and

for Q a rectangle : $\mathbb{P}(\mathcal{E}_{\ell} \text{ crosses } rQ) > 1 - Ce^{-cr}, r > 0$

- **2** $\ell > 0$: \mathcal{E}_{ℓ} has bounded components a.s.
- $\textcircled{0} \ \ell = 0: \ \mathcal{E}_\ell \ has \ bounded \ components \ and$

$$\mathbb{P}(\mathcal{E}_0 \text{ crosses from } \partial B(0, r) \text{ to } \partial B(0, R)) \leq c \left(\frac{r}{R}\right)^{\beta_{arm}}, r > 0.$$

$$0 < \inf_r \mathbb{P}(\mathcal{E}_0 \text{ crosses } rQ) \leq \sup_r \mathbb{P}(\mathcal{E}_0 \text{ crosses } rQ) < 1.$$

Assumptions ("nice" Gaussian field)

Assumption	Field f	Kernel g
Regularity	\mathcal{C}^3	\mathcal{C}^3
Symmetry	D^4 (Axis reflections, $\frac{\pi}{2}$ – rotations)	D^4
Positive Association	for A,B increasing events $\mathbb{P}(A \cap B) \geqslant \mathbb{P}(A)\mathbb{P}(B)$	$g \geqslant 0$
Asymptotic Independence	for A, B "far away" $\mathbb{P}(A \cap B) pprox \mathbb{P}(A)\mathbb{P}(B)$	for some $\beta > 2$ $g(x) \leqslant c(1 + \ x\)^{-\beta}$

Increasing event

$$\mathcal{A}=\mathcal{A}(f):\mathbf{1}_{\{\mathcal{A}(f)\}}\leqslant\mathbf{1}_{\{\mathcal{A}(g)\}}$$
 for $f\leqslant g$



Forces: 1. A simulation of the constaint set ξ_i of the Bargmann-Field field matching to a large squares (as give) at (i) the new large $l \in -0$ (lifel, figure), at (ii) the level $\ell = 1$ (right give), with the momental comparate of genation area distinguished (ii) black). The Bargmann-Field, field is the restionary, control Gaussian field with constances theread util $\alpha = -2\pi^{2/3}$. Check: Dubit: Philary Beller.

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Example : $A(f) = \{\mathcal{E}_{\ell}(f) \text{ crosses } Q\}$ for some $Q \subset \mathbb{R}^2$

• Symmetry $f \stackrel{(d)}{=} -f$ entails self-duality

 $\mathcal{E}_0 \stackrel{(d)}{=} \mathcal{E}_0^c$ (up to the boundary) \Rightarrow It is natural to expect $\ell_c = 0$.

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- Muirhead, Vanneuville '20 : Optimal condition β > 2 on decay of g, sharp phase transition [MV20]
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Poisson case

• Problem : the field is not symmetric

$$f(x) = \sum_{y \in \mathcal{P}} g(y - x)$$

- Let $Y_i, i \in \mathbb{N}$ iid symmetric integrable variables with law μ .
- Symmetric Poisson shot noise field with kernel g and mark distribution μ :

$$f(x) = \sum_{i \in \mathbb{N}} Y_i g(x - x_i); x \in \mathbb{R}^2.$$

- Other problem : does f(x) have a density?
 - f(x) has no density if g has compact support
 - f(x) has no density if $g(x) = e^{-||x||^2}$.

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Assumptions (Poisson case)

Regularity	\mathcal{C}^3	\mathcal{C}^3
Symmetry	D^4 (Axis reflections, $\frac{\pi}{2}$ – rotations)	D^4
Positive Association	for A,B increasing events $\mathbb{P}(A \cap B) \geqslant \mathbb{P}(A)\mathbb{P}(B)$	$g \geqslant 0$
Asymptotic Independence	for A,B "far away" $\mathbb{P}(A\cap B)pprox \mathbb{P}(A)\mathbb{P}(B)$	for some $\beta > 3$, for $ k \leq 3$ $\partial^k g(x) \leq c(1 + x)^{-\beta - k }$
Self-Duality	$\mathcal{E}_0 \stackrel{(d)}{=} \overline{\mathcal{E}_0^c}$	$Y_i \stackrel{(d)}{=} -Y_i$
Density	$(f(0), \nabla f(0))$ has bounded joint density	$g(x) = c \exp(- x ^{\alpha}), \ \alpha \in (0,1)$ or $g(x) = c(1+ x)^{-\beta}, \ \beta > d$
Concentration +	Use of OSSS inequality	Law of Y_i log-concave

Theorem (Lr, Muirhead 2022 [LRM22])

There is **Bernoulli-like percolation** for **Poisson shot noise fields** satisfying these assumptions.

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Non-symmetric case : high intensity asymptotics

• Let $\lambda > 0$, $\mathcal{P}_{\lambda} \stackrel{(d)}{=} \lambda^{-1/d} \mathcal{P}$ a Poisson homogeneous process with intensity $\lambda \to \infty \Leftrightarrow$ the number of points locally goes to ∞ . We consider

$$f_{\lambda}(x) = g \star \mathcal{W}_{\mathcal{P}_{\lambda}}(x) = \sum_{y \in \mathcal{P}_{\lambda}} g(y - x).$$

• Under mild assumptions, there is a finite critical density

 $\ell_{c}(f_{\lambda}) = \sup\{\ell: \mathbb{P}(\mathcal{E}_{\ell} \text{ has unbounded component}) > 0\} < \infty$



Figure – Shot noise field of intensity $\lambda \to \infty$

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Limit Gaussian field

Elementary Central Limit Theorem

$$ilde{f}_{\lambda}(x) := rac{f_{\lambda}(x) - \lambda \int g}{\sqrt{\lambda}} \xrightarrow[\lambda o \infty]{} G(x)$$

- Multivariate CLT (Heinrich, Schmidt '85) : Convergence of FDD
- G(x) is Gaussian centred with same covariance

$$\operatorname{Cov}(\widetilde{f}_{\lambda}(x),\widetilde{f}_{\lambda}(y)) = \operatorname{Cov}(G(x),G(y)) = g\widetilde{\star}g(x-y).$$

• Question :

$$\ell_c(\tilde{f}_\lambda) \to \ell_c(G) = 0?$$

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Assumptions (Non-symmetric Poisson case)

Assumption	Field f	Kernel g
Regularity	\mathcal{C}^4	\mathcal{C}^4
Symmetry	lsotropy (invariance to rotations)	lsotropy
(Positive Association)	for A,B increasing events $\mathbb{P}(A \cap B) \geqslant \mathbb{P}(A)\mathbb{P}(B)$	$g \geqslant 0$
Asymptotic Independence	for A,B "far away" $\mathbb{P}(A\cap B)pprox \mathbb{P}(A)\mathbb{P}(B)$	for some $\beta > 2$, for $ k \leq 3$ $\partial^k g(x) \leq c(1 + x)^{-\beta}$
Density	$(f_{\lambda}(0), \nabla f_{\lambda}(0))$ has bounded density	$g(x) = c \exp(-\ x\ ^{\alpha}), \alpha \in (0,1)$ or $g(x) = c(1+\ x\)^{-\beta}, \beta > d$

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Critical value approximation

Theorem (Lr,Muirhead 21+[LRM21]) *Recall*

$$\ell_c(\tilde{f}_{\lambda}) = \lambda^{-1/2} (\ell_c(f_{\lambda}) - \lambda \int g)$$

Assume the previous hypotheses, except positive association. Then • without positive association,

$$\ell_c(\widetilde{f}_\lambda) o 0$$

• with positive association ($\Leftrightarrow g \ge 0$),

$$\ell_c(\tilde{f}_{\lambda}) = O(\lambda^{-1/2}\log(\lambda)^{3/2})$$

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Strong Invariance principles

- Proof based on the construction of a coupling (f_{λ}, g) , for each $\lambda > 0$.
- Historical result : **Komlos, Major, Tusnady 85'**, coupling of X_i , i.i.d Rademacher variables with i.i.d Gaussian variables G_1, \ldots, G_n such that

$$\mathbb{P}(\sup_{0\leqslant k\leqslant n}\left|\sum_{i=1}^{k}X_{i}-\sum_{i=1}^{k}G_{i}\right|\geqslant c\ln(n)+t)\leqslant Ce^{-ct}$$

and the order ln(n) is optimal.

• "Random measure" point of view

$$\sum_{i=1}^{k} X_{i} = (\sum_{i=1}^{n} \delta_{X_{i}})(\mathbf{1}_{[1,...,k]}), 1 \leq k \leq n$$

Similarly $f_{\lambda}(\mathbf{k}) = \mathcal{W}_{\mathcal{P}_{\lambda}}(g(\mathbf{k} - \cdot)), \mathbf{k} \in \mathbb{Z}^{d}$
 $G(\mathbf{k}) = \mathcal{W}_{\mathcal{G}}(g(\mathbf{k} - \cdot)), \mathbf{k} \in \mathbb{Z}^{d}$

Strong invariance principle for shot noise fields

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Theorem (Lr, Muirhead 21+)
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For multi-indexes $|\alpha|\leqslant 4$

$$\mathbb{P}\left(\sup_{x\in B(0,R)}|\partial_{\alpha}\tilde{f}_{\lambda}(x)-\partial_{\alpha}G(x)|>\lambda^{-1/2}\ln(\lambda)^{1/2}t\right)< CR^{d}\lambda^{c}\exp(-ct)$$

• Optimal up to the power of $ln(\lambda)$ (see also Berry-Esseen inequality)

• Based on Koltchinski 94' : There is a coupling of \mathcal{P}_{λ} and $\mathcal{W}_{\mathcal{G}}$ such that for any $\mathbf{k} \in \mathbb{Z}^d$,

$$\mathbb{P}(| ilde{f}_{\lambda}(\mathbf{k}) - G(\mathbf{k})| \geqslant t\lambda^{-1/2}\ln(\lambda)) \leqslant Ce^{-ct}$$

• For $x \in \mathbb{R}^d \setminus \mathbb{Z}^d$, approximate f(x) by $f([x]) + \nabla f(\xi) \cdot (x - [x])$.

• There is a coupling of $N \sim \text{Pois}(\lambda)$ and $Z \sim \mathcal{N}(0,1)$ such that

$$\mathbb{P}(|\mathsf{N}-\lambda-\sqrt{\lambda}Z|>t)\leqslant Ce^{-ct}$$

Elements of proof for the symmetric case

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- Box crossing estimates (RSW) stem from the work of Tassion '16 because we have :
 - Positive association of the discretised field (FKG inequality on a finite space)
 - \mathcal{E}_0 is invariant in law under reflections and rotation by $\pi/2$
 - Spatial asymptotic independence (of f, hence of \mathcal{E}_{ℓ})
- One arm decay stems from
 - Positive association of the discretised field (FKG inequality on a finite space)
 - Asymptotic independence
 - Box crossing estimates (RSW)

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Proof of sharp phase transition (bounded Mills ratio case)

- First prove that $\mathbb{P}(\mathrm{Cross}_{\ell}(2R,R)) \to 0$ and then use bootstraping argument
- Proof based on a differential inequality of

$$\theta: h \to \mathbb{P}(f_r^{\varepsilon,h} \in \mathrm{Cross}_{\ell}(2R,R))$$

where $f_r^{\varepsilon,h}$ is obtained from f_r^{ε} by adding *h* to all the marks. We prove

$$\frac{\partial}{\partial h}\theta(h) \ge c \frac{\theta(h)(1-\theta(h))}{\inf_{2r < \rho < R/2} \{2\rho/R + \mathbb{P}(f_r^{\varepsilon} \in \operatorname{Arm}_{\ell}(2r,\rho))\}}$$

Use of the OSSS inequality applied to randomized algorithms; after the ideas of Duminil-Copin, Tassion, Raoufi.

Sharp phase transition

Theorem (Lr & Muirhead 19+)

For $\ell > 0$ there is c > 0 such that

 $\mathbb{P}(\mathrm{Cross}_{\ell}(2R,R)) \leqslant 1 - \exp(-cR), R > 0$

It implies the main result :

• For $\ell \geqslant 0$, \mathcal{E}_{ℓ} has only bounded connected components a.s..

• For $\ell < 0$, \mathcal{E}_{ℓ} has a unique unbounded component a.s..

$$\begin{array}{l} \textbf{Proof:} \bullet \ell \geqslant 0 : \mathbb{P}(\operatorname{Arm}_0(1, R)) \to 0. \\ \bullet \ell < 0 : \text{Borel-Cantelli lemma with} \end{array} \\ \end{array}$$

$$\sum_{k \ge 1} (1 - \mathbb{P}(\operatorname{Cross}_{\ell}(2^{k+1}, 2^k))) < \infty \Rightarrow (\operatorname{Cross}_{\ell}(2^{k+1}, 2^k)) \text{ occurs for } k > k_0$$

and arrange the rectangles so that the connected components overlap.

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OSSS inequality (O'Donnell, Saks, Schramm, Servedio '05) For an event A on a product probability space (E^n, μ^n) and a random

algorithm determining A, $(heta:=\mathbb{P}(A))$

$$\operatorname{Var}(\mathbf{1}_{\{A\}}) = \theta(1-\theta) \leqslant \sum_{i=1}^{n} \delta_{i}^{\mu}(\mathcal{A}) I_{i}^{\mu}(\mathcal{A})$$

where

- $\delta^{\mu}_i(\mathcal{A})$: Probability that coordinate i is revealed by the algorithm
- Influence of coordinate i : I^µ_i(A) = ℙ(1_{A} ≠ 1_{Aⁱ}) where Aⁱ is obtained by resampling coordinate i

For percolation events, typically :

- A is a progressive uncovering of all the connected components touching a random crossing line (in a rectangle) / circle (in a disc)
- δ^μ_i(A) is the probability that a point *i* is "close" to one of these connected components (one-arm decay is useful here)
- $I^{\mu}_i(A)$ is related to $\partial_h heta(h)$ for $h \sim 0$

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Key point

- First remark that crossing events are monotonous in the marks (higher mark = more chances to percolate). Hence for each *i* there is a.s. a random level y_i such that there is percolation for $Y_i \ge y_i$.
- Assume for f^{ε} that mark Y_i is replaced by $Y_i + h_i$ for some parameter $h_i \in \mathbb{R}$. Then

$$\begin{split} \frac{\partial}{\partial h_i} \mathbb{P}(\mathrm{Cross}_{\ell}(2R,R)) &= \frac{\partial}{\partial h_i} \mathbb{P}(Y_i + h_i \ge y_i) \ge u_{\mu_{ac}}(y_i - h_i) \\ I_i &= \mathbb{P}(\mathbf{1}_{\{A\}} \neq \mathbf{1}_{\{A_i\}}) = \mathbb{P}(Y_i + h_i \ge y_i, Y'_i + h_i < Y'_i) \\ &+ \mathbb{P}(Y_i + h_i < y_i, Y'_i + h_i \ge Y'_i) \\ &\leq 2\mathbb{P}(Y_i \ge y_i - h_i) \end{split}$$

$$\stackrel{\mathrm{Mills}}{\leqslant} c u_{\mu_{ac}}(y_i - h_i)$$

• We end up with

$$\frac{\partial}{\partial h}\theta(h) = \sum_{i} \frac{\partial}{\partial h_{i}}\theta(h) \ge c \sum_{i} I_{i} \ge c \frac{\sum_{i} I_{i}\delta_{i}}{\sup_{i}\delta_{i}} \stackrel{OSSS}{=} \frac{\theta(1-\theta)}{\sup_{i}\delta_{i}}$$

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