Swarming rigid bodies: geometry and topology

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\[PD, Frouvelle, Merino-Aceituno, M3AS 27 (2017)]
\[PD, Frouvelle, Merino-Aceituno, Trescases, MMS 16 (2018)]
\[PD, Diez, Na, SIADS 21 (2022)]
\[PD, Diez, Frouvelle, arXiv. 2111.05614]
\[PD, Frouvelle, EJAM (to appear)]
Summary

1. Introduction

2. Particle dynamics and mean-field approximation

3. Hydrodynamic model

4. Topological solutions

5. Conclusion
1. Introduction
Collective dynamics

Questions:
- Link between micro-scale geometry and large-scale structures
- Topology of collective structures

Object of study: body orientation dynamics

Methodology: dual use of microscopic models and their macroscopic counterparts
Velocity vs body-attitude alignment

Velocity direction alignment: Vicsek model (Phys. Rev. Lett. 95):
Self-propelled agents are polar rods which align single axis

Body-attitude alignment: self-propelled agents
are solid bodies which align their $n$ principle axes (in dim $= n$)

Numerical simulations: A. Frouvelle / A. Diez (using GPU)
Why body-attitude alignment?

In 3D: - **Bird or fish dynamics** [Hemelrijk et al, 2010-2012]

- **Volume exclusion interaction** of complex shape objects
  example: spermatozoa

In any D: - **Flow of data** structured as $n$-dimensional rotations

- **Gives structural information** useful for 3D

2. Particle dynamics and mean-field approximation
Particle dynamics

$X_k(t)$: position of Particle “$k$” ($k = 1, \ldots, N$)

$A_k(t)$: body frame of “$k$” $\equiv$ rotation $A_k(t)$

$\omega_1^k(t) = A_k(t)e_1$: self-propulsion direction

**Vicsek-like:** self-propelled particles (speed 1)
align their body attitude + noise

\[
dX_k(t) = A_k(t)e_1 \, dt
\]

\[
dA_k(t) = P_{T_{A_k(t)}} \circ \left[ \kappa \mathcal{P}(J_k(t)) \, dt + \sqrt{2} dB^k_t \right]
\]

\[
J_k(t) = \sum_{j \text{ s.t. } |X_j(t) - X_k(t)| \leq R} A_j(t)
\]

$\mathcal{P}(J) = (JJ^T)^{-1/2}J$ = projects on rotations

$\kappa = \text{alignment rate}$
Mean-field model (large $N$)

\[ f(x, A, t) = \text{distribution function: at position } x, \text{ rotation } A, \text{ time } t \]

\[ = \text{number of particles in small volume around } (x, A) \text{ at } t \]

\[ \partial_t f + (Ae_1) \cdot \nabla_x f = \frac{1}{\varepsilon} \nabla_A \cdot \left[ -\kappa P_{TA}(P(J_f)) f + \nabla_A f \right] \]

\[ J_f(x, t) = \int f(x, A, t) A \, dA \]

\( \varepsilon \ll 1 \): alignment & noise happen fast, with rate \( \frac{1}{\varepsilon} \)

\( R = \mathcal{O}(\varepsilon) \) so interactions become local

Hydrodynamic model obtained in the limit \( \varepsilon \to 0 \)
Collision operator (Fokker-Planck)

$$\partial_t f + (Ae_1) \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f)$$

$$Q(f) = \nabla_A \cdot \left[ -\kappa P_{TA}(\mathcal{P}(J_f)) f + \nabla_A f \right] , \quad J_f = \int f A dA$$

$$Q(f) = 0 \iff f(A) = \rho M_{\Gamma}(A)$$

$$M_{\Gamma}(A) \sim \exp (\kappa \Gamma \cdot A)$$

where $\Gamma$ can be any rotation and $\rho$ any positive number
Alternate collision operator (BGK)

\[ Q(f) = \rho_f M_{\Gamma_f} - f \]

with \( \rho_f = \int f \, dA, \quad \Gamma_f = \mathcal{P}(J_f) \)

Corresponds to jump process:
3. Hydrodynamic model
\[ f(x, A, t) \xrightarrow{\varepsilon \to 0} \rho(x, t) M_{\Gamma(x, t)}(A) \] where \((\rho, \Gamma)\) satisfy

\[
\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega_1) = 0
\]

\[
\rho (\partial_t + c_2 \Omega_1 \cdot \nabla_x) \Gamma = \mathcal{W} \Gamma
\]

with

\[
\Omega_j := \Gamma e_j
\]

\[
\mathcal{W} = -c_3 \nabla_x \rho \wedge \Omega_1 - c_4 \rho \left[ (\Gamma (\nabla_x \cdot \Gamma)) \wedge \Omega_1 + \nabla_x \wedge \Omega_1 \right]
\]

and

\[
(F \wedge G)_{ij} = F_i G_j - F_j G_i
\]

\[
(\nabla_x \wedge F)_{ij} = \partial_{x_i} F_j - \partial_{x_j} F_i
\]
Derivation from kinetic model: $\varepsilon \to 0$

Write: $\mathcal{T} f = \frac{1}{\varepsilon} Q(f)$, $\mathcal{T} = \partial_t + (Ae_1) \cdot \nabla_x$

$$Q(f) = \varepsilon \mathcal{T} f \xrightarrow[\varepsilon \to 0]{} 0$$

hence

$$f \xrightarrow[\varepsilon \to 0]{} \rho \mathcal{M}_\Gamma$$

for $\rho(x,t)$, $\Gamma(x,t)$ to be determined

$\rho$-equation (continuity eq.):

$$\int \mathcal{T} f \, dA = \frac{1}{\varepsilon} \int Q(f) \, dA \equiv 0$$

$$\xrightarrow[\varepsilon \to 0]{} \int \mathcal{T}(\rho \mathcal{M}_\Theta) \, dA = 0$$
There exists $\mu(A)$ antisymmetric matrix s.t.

$$\int Q(f) \mu(\Gamma_f^T A) \, dA = 0$$

Implies:

$$\varepsilon \to 0 \int T(\rho M_\Theta) \mu(\Gamma^T A) \, dA = 0$$

BGK case, any dimension: $\mu(A) = A - A^T$
Eq. for $\Gamma$, Fokker-Planck case

Fokker-Planck case, $n = 3$:

$$\mu(A) = (A - A^T)\psi(\text{Tr}(A)/2)$$

where $\psi$ is such that

$$\alpha(\theta) = \sin \theta \psi(\cos \theta + 1/2)$$

solves:

$$-\frac{1}{m} \frac{\partial}{\partial \theta} \left( m \frac{\partial \alpha}{\partial \theta} \right) + \frac{\alpha}{1 - \cos \theta} = \sin \theta$$

with

$$m(\theta) = \sin^2(\theta/2) \exp \left( \kappa(\cos \theta + 1/2) \right)$$

Fokker-Planck case, $n \geq 4$: much more complicated

Requires advanced knowledge of rotation group

Maximal torus, Cartan subalgebra, Weyl group
Eq. for $\Gamma$: final steps

Simplification of
\[ \int \mathcal{T}(\rho \mathcal{M}_\Theta) \mu(\Gamma^T A) \, dA = 0 \]
requires Lie-groups representation theory

Explicit formulas for coefficients $c_i$:

\begin{align*}
\text{Fokker-Planck case (}\, n = 3\,): \quad c_1 &= \frac{2}{3} \left\langle \frac{1}{2} + \cos \theta \right\rangle_m \\
&\quad - c_2 = \frac{1}{5} \left\langle 3 - 2 \cos \theta \right\rangle_m \alpha \sin \theta \\
&\quad - c_3 = \frac{1}{\kappa} \\
&\quad - c_4 = \frac{1}{5} \left\langle 1 - \cos \theta \right\rangle_m \alpha \sin \theta
\end{align*}

where $\left\langle f \right\rangle_g = \left( \int_0^\pi f(\theta) \, g(\theta) \, d\theta \right) / \left( \int_0^\pi g(\theta) \, d\theta \right)$

\text{BGK case (}\, n = 3\,): make $\alpha = \sin \theta$
4. Topological solutions
Chiral edge states for Toner & Tu model [Bartolo et al 2017]

Our goal: explore properties of topological states in the body orientation model in the bulk (not edge states) on both the macro and micro levels

Motivation: explore if topological protection could explain living systems robustness in spite of stochasticity
Milling solution

Exact solution of body orientation Hydrodynamic model

Has $\rho = \rho_0 = \text{Constant}$ and $\Gamma$ given by:
Topological phase transition?

Mill goes to flocking state: topological phase transition?
Polarization curve

Within a given slice

Polarization vector of the slice

Average

Polarization curve
Topological phase transitions

- Topological phase transition

  passage through *disordered state*

  where *winding number is not defined*
5. Conclusion
Summary

New collective dynamics model relying on full body-alignment
Derivation of macro model via Lie group theory
Possesses solutions with non-trivial topology
Microscopic model exhibits topological phase transitions

Perspectives

Existence and uniqueness for the hydrodynamic model
Rigorous proof of convergence from micro to macro
Numerical simulations of macroscopic model
Stability of topological solutions

Further directions

Models involving other complex geometric structures
or other classes of topological states