# Swarming rigid bodies: geometry and topology

Pierre Degond

CNRS & Institut de Mathématiques de Toulouse

pierre.degond@math.univ-toulouse.fr

http://sites.google.com/site/degond/

# Collaborators



Antoine Diez[1], Amic Frouvelle[2],
Sara Merino-Aceituno[3], Mingye Na[4], Ariane Trescases[5]

1: Kyoto U. (Japan)  2: Dauphine & U. Poitiers (France)
3: U. Vienna (Austria)  4: SUSTech Shenzhen (China)  5: CNRS Toulouse (France)


Antoine


Amic


Sara


Mingye


Ariane

[PD, Frouvelle, Merino-Aceituno, M3AS 27 (2017)]
[PD, Frouvelle, Merino-Aceituno, Trescases, MMS 16 (2018)]
[PD, Diez, Na, SIADS 21 (2022)]
[PD, Diez, Frouvelle, arXiv. 2111.05614]
[PD, Frouvelle, EJAM (to appear)]

# Summary

# 1. Introduction

# Collective dynamics

**Micro–scale**

**Individual agents obey simple rules**

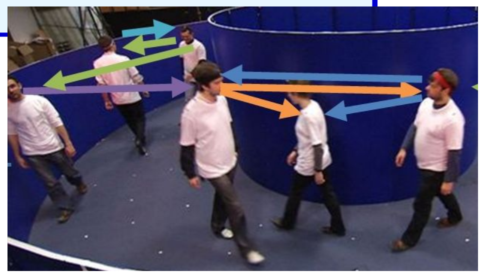
**no leader**

?

**Macro scale**

**Emergence of large–scale coherent structures**

**Not directly encoded in agent's behavior**



## Questions:

Link between micro-scale geometry and large-scale structures

Topology of collective structures

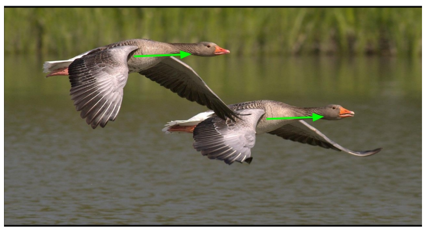
**Object of study:** body orientation dynamics

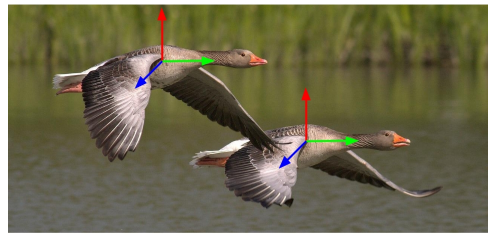
**Methodology:** dual use of microscopic models and their macroscopic counterparts

# Velocity vs body-attitude alignment

Velocity direction alignment: Vicsek model (Phys. Rev. Lett. 95):
Self-propelled agents are polar rods which align single axis

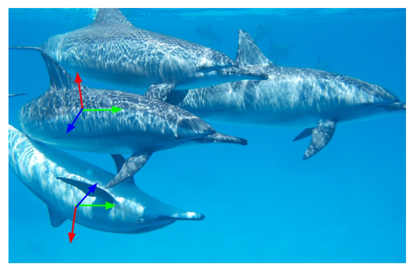
Body-attitude alignment: self-propelled agents
are solid bodies which align their $n$ principle axes (in dim $= n$)





Vicsek model

Body attitude alignment

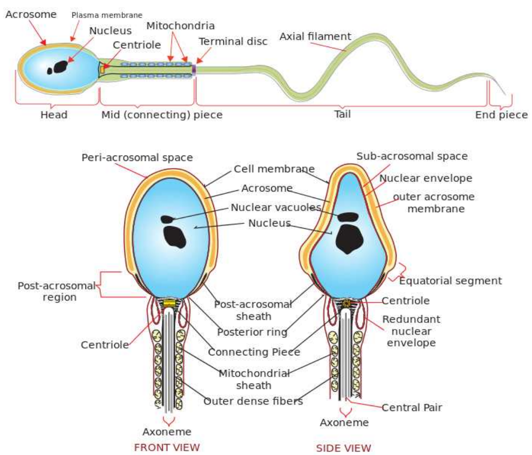


Numerical simulations: A. Frouvelle / A. Diez (using GPU)

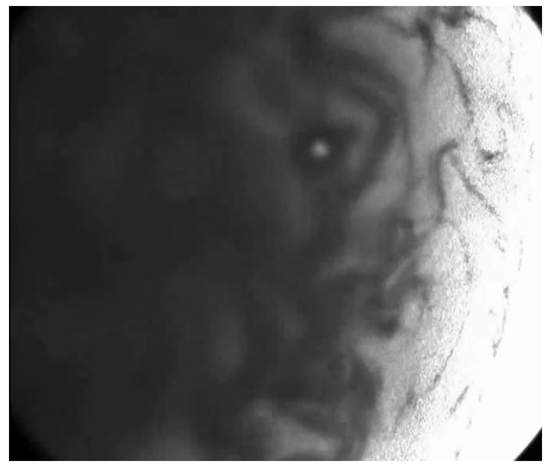
In 3D: - Bird or fish dynamics [Hemelrijk et al, 2010-2012]

- Volume exclusion interaction of complex shape objects

example: spermatozoa




From Wikipedia



[David et al, Animal Reprod. Sci., 2015]

In any D: - Flow of data structured as $n$-dimensional rotations

- Gives structural information useful for 3D

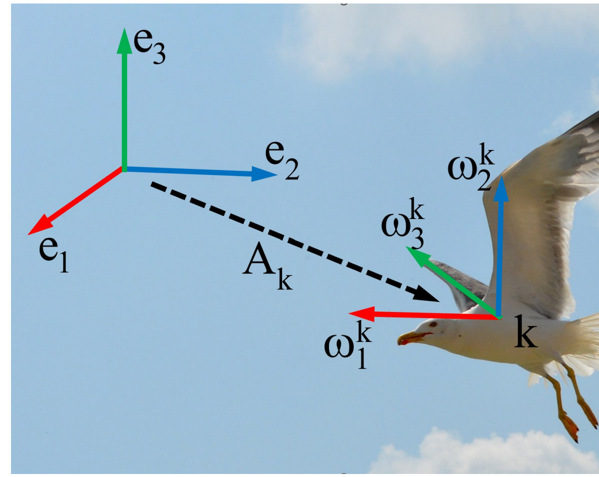
# 2. Particle dynamics and mean-field approximation

# Particle dynamics

$X_k(t)$: position of Particle "$k$" $(k = 1, \dots, N)$

$A_k(t)$: body frame of "$k$" $\equiv$ rotation $A_k(t)$

$\omega_1^k(t) = A_k(t)e_1$: self-propulsion direction

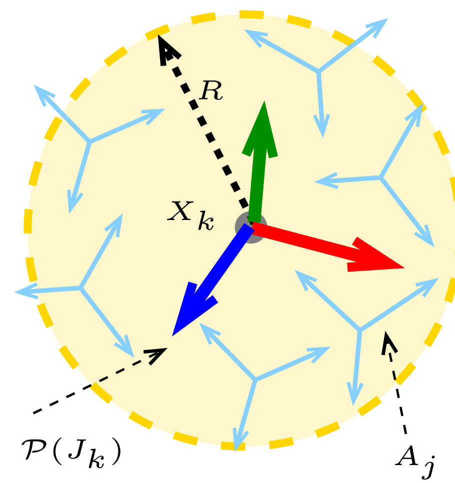



Vicsek-like: self-propelled particles (speed 1)
align their body attitude + noise

$$dX_k(t) = A_k(t)e_1\, dt$$

$$dA_k(t) = P_{T_{A_k(t)}} \circ \left[\kappa \mathcal{P}(J_k(t))dt + \sqrt{2}dB_t^k\right]$$

$$J_k(t) = \sum_{j \text{ s.t. } |X_j(t) - X_k(t)| \leq R} A_j(t)$$




$\mathcal{P}(J) = (JJ^T)^{-1/2}J$ = projects on rotations

$\kappa$ = alignment rate

$f(x, A, t) =$ distribution function: at position $x$, rotation $A$, time $t$

$=$ number of particles in small volume around $(x, A)$ at $t$

$$\partial_t f + (A e_1) \cdot \nabla_x f = \frac{1}{\varepsilon} \nabla_A \cdot \big[ -\kappa P_{T_A}\big(\mathcal{P}(J_f)\big) f + \nabla_A f \big]$$

$$J_f(x, t) = \int f(x, A, t)\, A\, dA$$

$\varepsilon \ll 1$: alignment & noise happen fast, with rate $\frac{1}{\varepsilon}$

$R = \mathcal{O}(\varepsilon)$ so interactions become local

Hydrodynamic model obtained in the limit $\varepsilon \to 0$

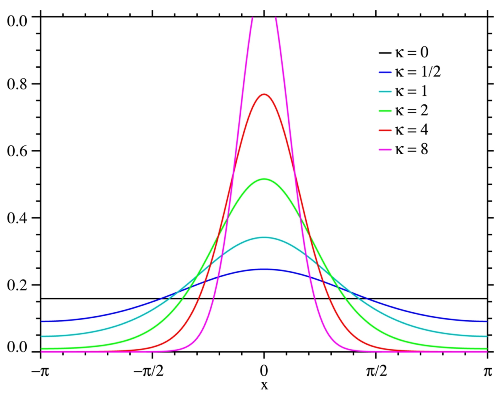
$$\partial_t f + (Ae_1)\cdot\nabla_x f = \frac{1}{\varepsilon} Q(f)$$

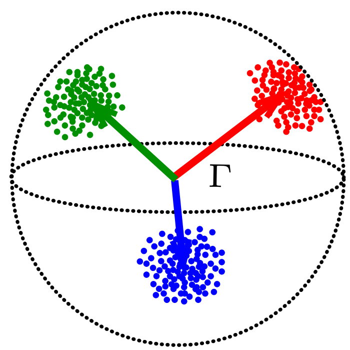
$$Q(f) = \nabla_A \cdot \big[ -\kappa P_{T_A}\big(\mathcal{P}(J_f)\big)\, f + \nabla_A f \big], \quad J_f = \int f\, A\, dA$$

$$Q(f) = 0 \quad \Longleftrightarrow \quad f(A) = \rho M_\Gamma(A)$$

$$M_\Gamma(A) \sim \exp\big(\kappa\, \Gamma \cdot A\big)$$

where $\Gamma$ can be any rotation and $\rho$ any positive number

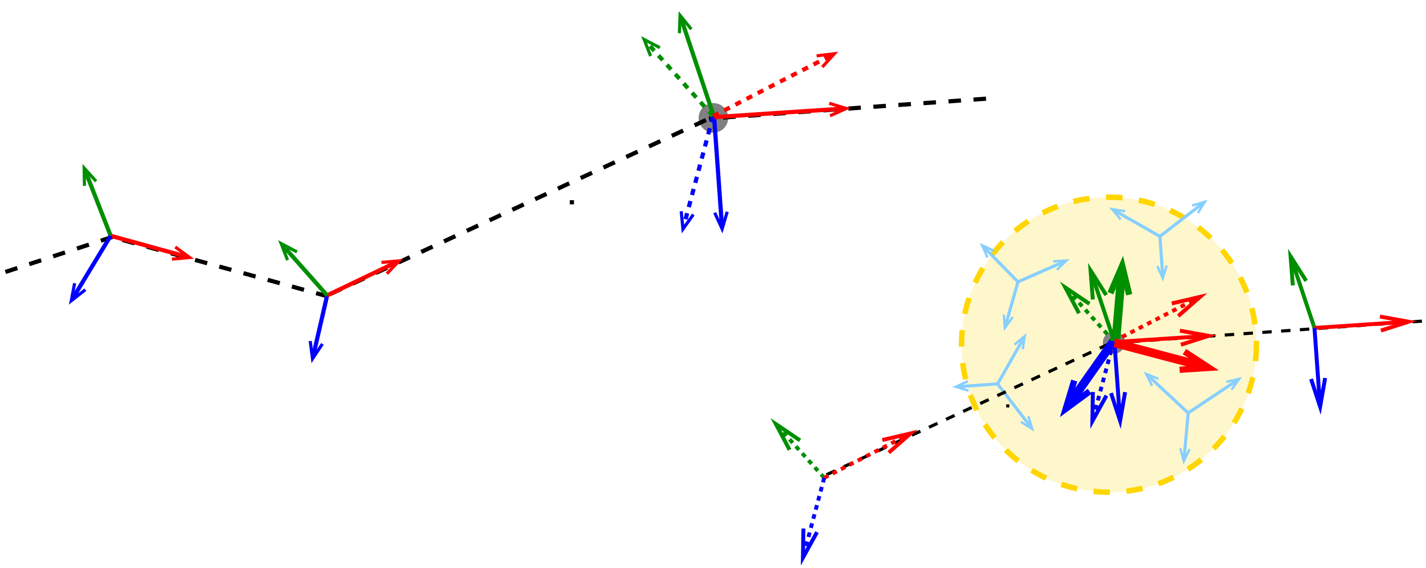


$$Q(f) = \rho_f M_{\Gamma_f} - f$$

$$\text{with} \quad \rho_f = \int f \, dA, \quad \Gamma_f = \mathcal{P}(J_f)$$

Corresponds to jump process:



# 3. Hydrodynamic model

$f(x,A,t) \underset{\varepsilon\to 0}{\longrightarrow} \rho(x,t)\mathcal{M}_{\Gamma(x,t)}(A)$ where $(\rho,\Gamma)$ satisfy

$$\partial_t \rho + \nabla_x \cdot (c_1 \rho \Omega_1) = 0$$

$$\rho\,(\partial_t + c_2 \Omega_1 \cdot \nabla_x)\Gamma = \mathbb{W}\,\Gamma$$

with

$$\Omega_j := \Gamma e_j$$

$$\mathbb{W} = -c_3 \nabla_x \rho \wedge \Omega_1 - c_4 \rho\big[(\Gamma(\nabla_x \cdot \Gamma)) \wedge \Omega_1 + \nabla_x \wedge \Omega_1\big]$$

and

$$(F \wedge G)_{ij} = F_i G_j - F_j G_i$$

$$(\nabla_x \wedge F)_{ij} = \partial_{x_i} F_j - \partial_{x_j} F_i$$

Write: $$\mathcal{T} f = \frac{1}{\varepsilon} Q(f), \qquad \mathcal{T} = \partial_t + (A e_1) \cdot \nabla_x$$

$$Q(f) = \varepsilon \mathcal{T} f \underset{\varepsilon \to 0}{\longrightarrow} 0$$

hence

$$f \underset{\varepsilon \to 0}{\longrightarrow} \rho \mathcal{M}_\Gamma$$

for $\rho(x,t)$, $\Gamma(x,t)$ to be determined

$\rho$-equation (continuity eq.):

$$\int \mathcal{T} f \, dA = \frac{1}{\varepsilon} \int Q(f) \, dA \equiv 0$$

$$\underset{\varepsilon \to 0}{\Longrightarrow} \int \mathcal{T}(\rho \mathcal{M}_\Theta) \, dA = 0$$

# Derivation: Eq. for $\Gamma$

There exists $\mu(A)$ antisymmetric matrix s.t.

$$\int Q(f)\,\mu(\Gamma_f^T A)\,dA = 0$$

$$\underset{\varepsilon\to 0}{\text{Implies:}} \int \mathcal{T}(\rho \mathcal{M}_\Theta)\mu(\Gamma^T A)\,dA = 0$$

BGK case, any dimension: $\mu(A) = A - A^T$

Fokker-Planck case, $n = 3$:

$$\mu(A) = (A - A^T)\psi\big(\mathsf{Tr}(A)/2\big)$$

where $\psi$ is such that

$$\alpha(\theta) = \sin\theta\, \psi(\cos\theta + 1/2)$$

solves:

$$-\frac{1}{m}\frac{\partial}{\partial\theta}\Big(m\frac{\partial\alpha}{\partial\theta}\Big) + \frac{\alpha}{1-\cos\theta} = \sin\theta$$

with

$$m(\theta) = \sin^2(\theta/2)\exp\big(\kappa(\cos\theta + 1/2)\big)$$

Fokker-Planck case, $n \geq 4$: much more complicated

Requires advanced knowledge of rotation group

Maximal torus, Cartan subalgebra, Weyl group

Simplification of $\displaystyle\int \mathcal{T}(\rho \mathcal{M}_\Theta)\mu(\Gamma^T A)\, dA = 0$

requires Lie-groups representation theory

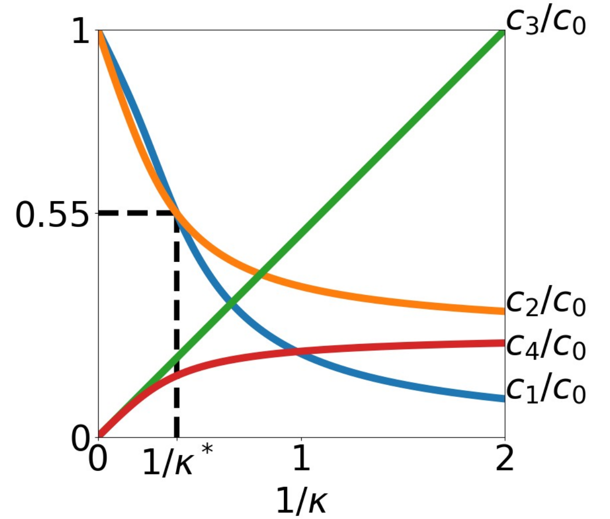
Explicit formulas for coefficients $c_i$:

Fokker-Planck case ($n = 3$):

$c_1 = \frac{2}{3} \left\langle \frac{1}{2} + \cos\theta \right\rangle_m \qquad c_2 = \frac{1}{5} \, \langle 3 - 2\cos\theta \rangle_{m\,\alpha \sin\theta}$

$c_3 = \frac{1}{\kappa} \qquad c_4 = \frac{1}{5} \, \langle 1 - \cos\theta \rangle_{m\,\alpha \sin\theta}$

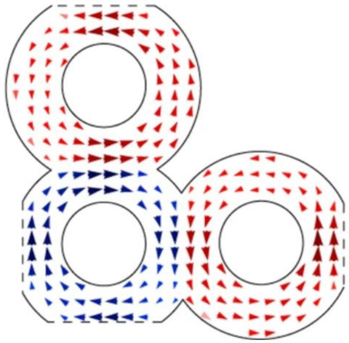
where $\langle f \rangle_g = \Big( \int_0^\pi f(\theta)\, g(\theta)\, \mathrm{d}\theta \Big) \Big/ \Big( \int_0^\pi g(\theta)\, \mathrm{d}\theta \Big)$




BGK case ($n = 3$): make $\alpha = \sin\theta$

# 4. Topological solutions

Chiral edge states for Toner & Tu model [Bartolo et al 2017]



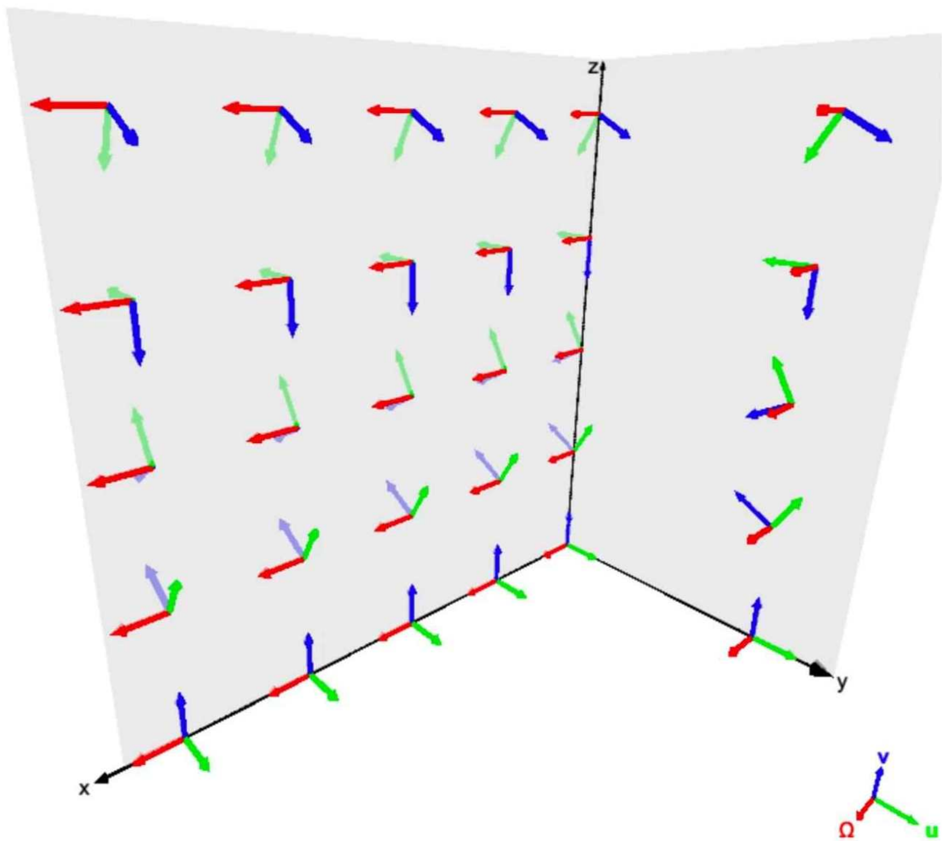
Our goal: explore properties of topological states
in the body orientation model
in the bulk (not edge states)
on both the macro and micro levels

Motivation: explore if topological protection could explain
living systems robustness in spite of stochasticity

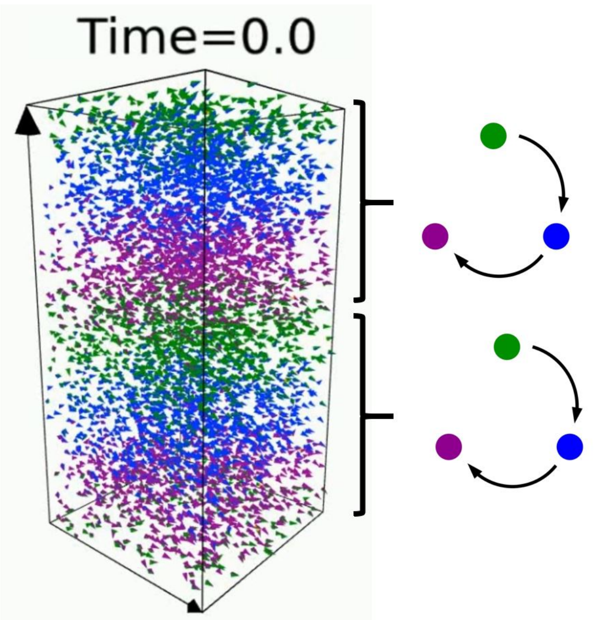
# Milling solution

Exact solution of body orientation Hydrodynamic model

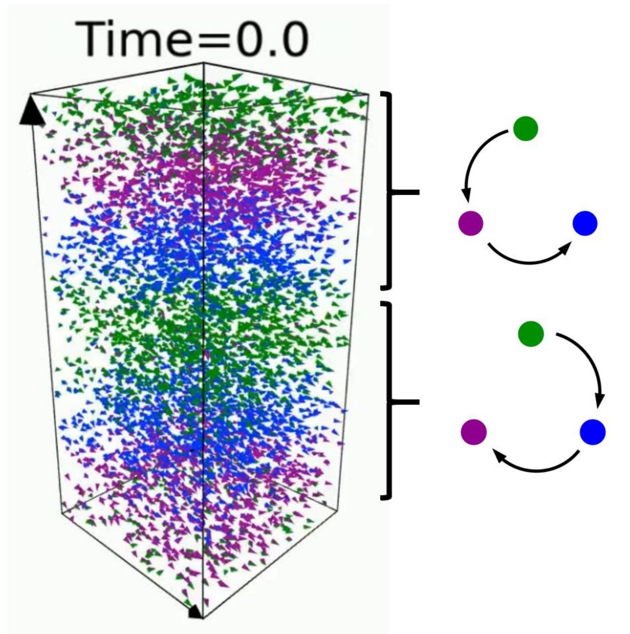
Has $\rho = \rho_0 =$ Constant and $\Gamma$ given by:



# Topological phase transition ?



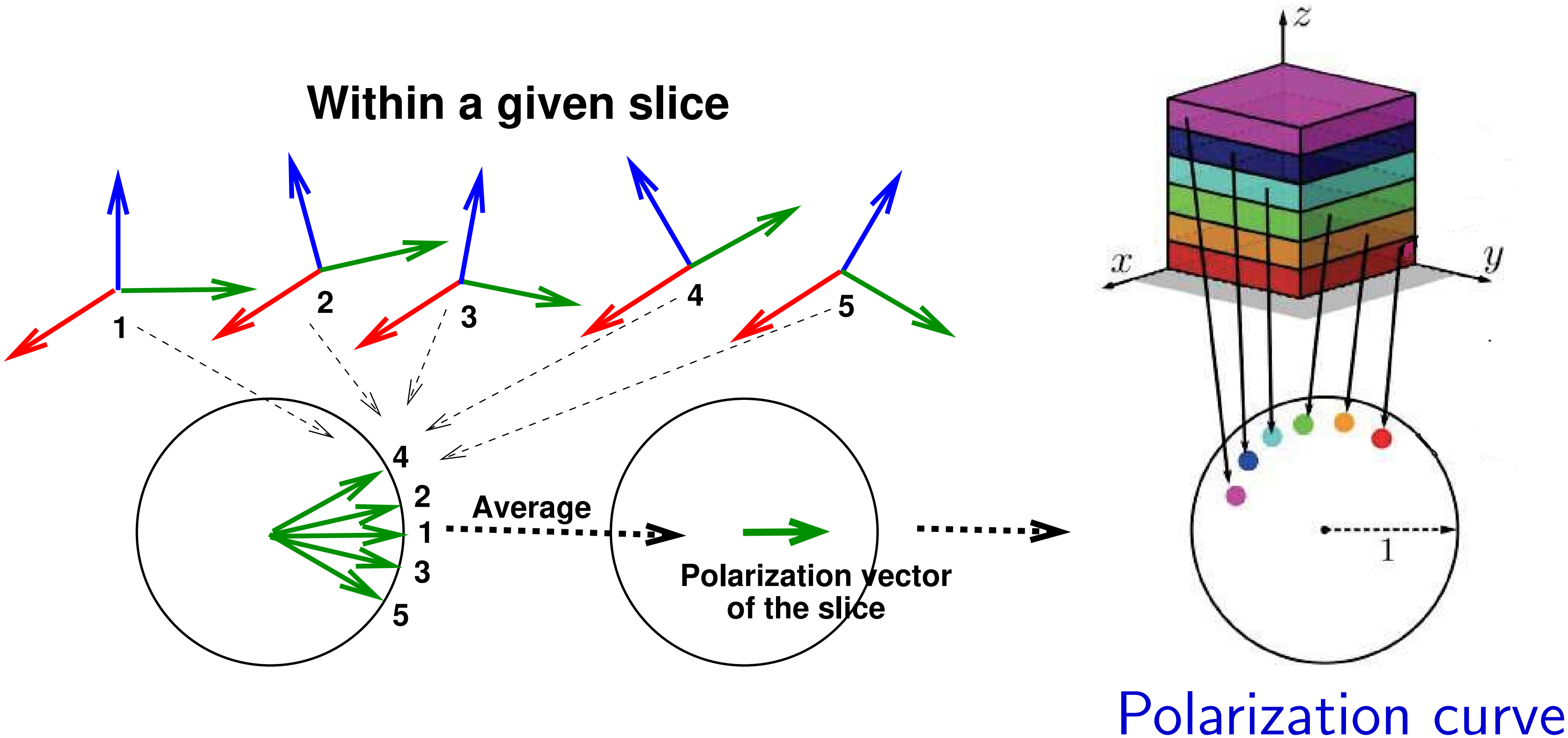

double mill

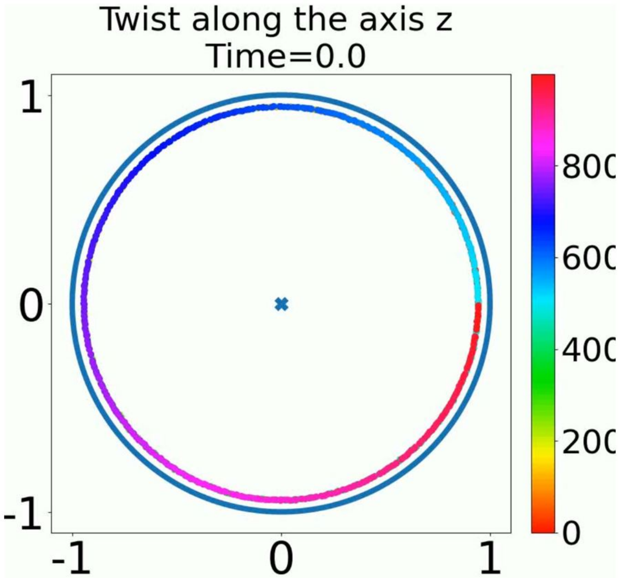



opposite mills

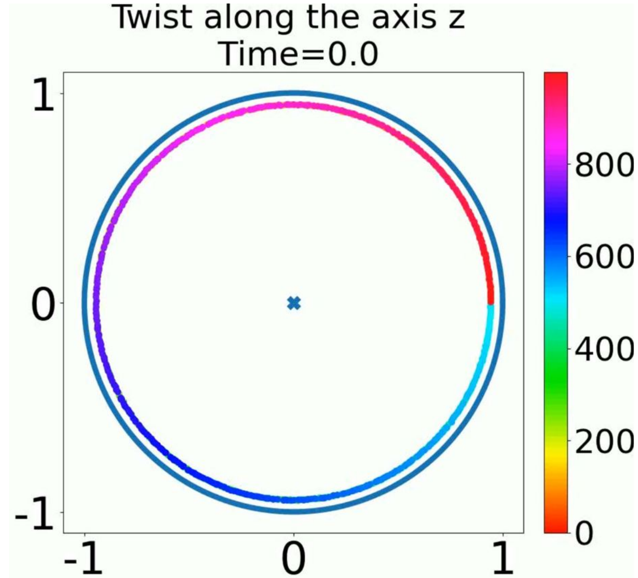
Mill goes to flocking state: topological phase transition ?

# Polarization curve






double mill




Opposite mills

Topological phase transition $\Longrightarrow$

passage through disordered state
where winding number is not defined

# 5. Conclusion

## Summary

New collective dynamics model relying on full body-alignment

Derivation of macro model via Lie group theory

Possesses solutions with non-trivial topology

Microscopic model exhibits topological phase transitions

## Perspectives

Existence and uniqueness for the hydrodynamic model

Rigorous proof of convergence from micro to macro

Numerical simulations of macroscopic model

Stability of topological solutions

## Further directions

Models involving other complex geometric structures

or other classes of topological states